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**SOME DUAL SERIES EQUATIONS INVOLVING LAGUERRE
POLYNOMIALS**

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In this paper an exact solution is found for the dual series equations

$$(1) \quad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = f(x), \quad 0 \leq x < d,$$

$$(2) \quad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = g(x), \quad d < x < \infty,$$

where $\alpha + \beta > 0$, $0 < \beta < 1$, $L_n(\alpha; x) = L_n^\alpha(x)$ is the Laguerre polynomial and $f(x)$ and $g(x)$ are known functions.

In a recent paper Srivastava [3] has solved the equations

$$(3) \quad \sum_{n=0}^{\infty} \{A_n / \Gamma(\alpha + 1 + n)\} L_n(\alpha; x) = f(x), \quad 0 \leq x < d,$$

$$(4) \quad \sum_{n=0}^{\infty} \{A_n / \Gamma(\alpha + 1/2 + n)\} L_n(\alpha; x) = g(x), \quad d < x < \infty, \alpha > -1/2,$$

by considering separately the equations when (a) $g(x) \equiv 0$, (b) $f(x) \equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation. Srivastava's equations are a special case of (1) and (2) with $\beta = 1/2$ and $A_n = \Gamma(\alpha + 1 + n) \Gamma(\alpha + 1/2 + n) C_n$.

The solution presented in this paper employs a multiplying factor technique which is more direct than the method given in [3] and is similar to that used by Noble [2] to solve some dual series equations involving Jacobi polynomials.

2. In the course of the analysis we shall use the following results.

From [1, p. 293(5), p. 405(20)] it is readily shown that

$$(5) \quad \int_0^y x^\alpha (y-x)^{\beta-1} L_n(\alpha; x) dx = \frac{\Gamma(\beta) \Gamma(\alpha + 1 + n)}{\Gamma(\alpha + \beta + 1 + n)} y^{\alpha+\beta} L_n(\alpha + \beta; y),$$

where $-1 < \alpha, \beta > 0$, and

$$(6) \quad \int_y^\infty (x-y)^{-\beta} e^{-x} L_n(\alpha; x) dx = \Gamma(1-\beta) e^{-y} L_n(\alpha + \beta - 1; y),$$

where $1 > \beta, \alpha + \beta > 0$.

The orthogonality relation for the Laguerre polynomials is

$$(7) \quad \int_0^\infty x^\alpha e^{-x} L_n(\alpha; x) L_m(\alpha; x) dx = \frac{\Gamma(\alpha + 1 + n)}{\Gamma(n + 1)} \delta_{mn}, \quad \alpha > -1,$$

where δ_{mn} is the Kronecker delta.

3. Solution of the problem. Multiplying equation (1) by $x^\alpha(y-x)^{\beta-1}$, equation (2) by $(x-y)^{-\beta}e^{-x}$ and integrating with respect to x over $(0, y)$ and (y, ∞) respectively we find on using the results (5) and (6)

$$(8) \quad \sum_{n=0}^{\infty} C_n \frac{\Gamma(\alpha+1+n)}{(\alpha+\beta+n)} L_n(\alpha+\beta; y) = \frac{y^{-\alpha-\beta}}{\Gamma(\beta)} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx,$$

where $0 < y < d$, $\alpha > -1$, $\beta > 0$, and

$$(9) \quad \begin{aligned} & \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha+\beta-1; y) \\ & = \frac{e^y}{\Gamma(1-\beta)} \int_y^{\infty} (x-y)^{-\beta} e^{-x} g(x) dx, \end{aligned}$$

for $d < y < \infty$, $1 > \beta$, $\alpha + \beta > 0$.

If we now multiply equation (8) by $y^{\alpha+\beta}$, differentiate with respect to y and use the formula

$$(10) \quad \frac{d}{dx} \{x^\alpha L_n(\alpha; x)\} = (n+\alpha)x^{\alpha-1} L_n(\alpha-1; x),$$

we find

$$(11) \quad \begin{aligned} & \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha+\beta-1; y) \\ & = \frac{y^{1-\alpha-\beta}}{\Gamma(\beta)} \frac{d}{dy} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx, \end{aligned}$$

where $0 < y < d$, $\beta > 0$, $\alpha > -1$.

The left hand sides of equations (9) and (11) are now identical and using the orthogonality relation (7) we see that the solution of equations (1) and (2) for $\alpha + \beta > 0$, $0 < \beta < 1$, is given by

$$(12) \quad C_n = \frac{\Gamma(n+1)}{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+n)} B_n(\alpha, \beta; d),$$

where

$$(13) \quad \begin{aligned} B_n(\alpha, \beta; d) & = \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} L_n(\alpha+\beta-1; y) F(y) dy \\ & + \frac{1}{\Gamma(1-\beta)} \int_d^{\infty} y^{\alpha+\beta-1} L_n(\alpha+\beta-1; y) G(y) dy, \end{aligned}$$

and

$$(14) \quad F(y) = \frac{d}{dy} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx,$$

$$(15) \quad G(y) = \int_y^\infty (x - y)^{-\beta} e^{-x} g(x) dx .$$

To obtain the solution of Srivastava's equations (3) and (4) we write $\beta = 1/2$, $A_n = \Gamma(\alpha + 1 + n)\Gamma(\alpha + 1/2 + n)C_n$ in (12) and find that

$$(16) \quad A_n = \frac{\Gamma(n + 1)}{\Gamma(1/2)} \left\{ \int_d^d e^{-y} L_n(\alpha - 1/2; y) F_1(y) dy + \int_d^\infty y^{\alpha-1/2} L_n(\alpha - 1/2; y) G_1(y) dy \right\} ,$$

for $\alpha > -1/2$, and where $F_1(y)$ and $G_1(y)$ are given by equations (14) and (15) respectively with $\beta = 1/2$.

Comparing the above solution with that obtained in [3] it can be seen that they are in agreement except for the form of the function $G_1(y)$. The limits on the integrals of equations (4.7) and (4.8) in Srivastava's paper are wrong and should read (x, ∞) and (u, ∞) respectively. When these corrections have been made we find that his term corresponding to $G_1(y)$ can be written in the notation of the present paper as

$$(17) \quad - \frac{d}{dy} \int_y^\infty (x - y)^{-1/2} dx \int_x^\infty e^{-u} g(u) du .$$

After inverting the order of integration, carrying out the integration in x and performing the differentiation with respect to y it is found that (17) is equal to $G_1(y)$. Hence with this simplification Srivastava's solution reduces to that given by equation (16).

4. It is also possible without computing the coefficients C_n to find the values of series (1) and (2) in the regions where their values are not specified. We define (1) to have the value $h(x)$, $d < x < \infty$, and (2) to have the value $k(x)$, $0 \leq x < d$.

(a) *Calculation of $h(x)$.* Substituting for C_n from equation (12) into (1) and interchanging the order of integration and summation we find

$$(18) \quad h(x) = \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} F(y) S_1(x, y) dy + \frac{1}{\Gamma(1 - \beta)} \int_d^\infty y^{\alpha+\beta-1} G(y) S_1(x, y) dy , \quad d < x < \infty ,$$

where

$$(19) \quad S_1(x, y) = \sum_{n=0}^\infty \frac{\Gamma(n + 1)}{\Gamma(\alpha + 1 + n)} L_n(\alpha; x) L_n(\alpha + \beta - 1; y) .$$

Using the results (6) and (7) it is easily shown that

$$(20) \quad S_1(x, y) = \frac{e^y x^{-\alpha} (x - y)^{-\beta}}{\Gamma(1 - \beta)} H(x - y),$$

where $H(x)$ is the Heaviside unit function.

From equations (18) and (20) we see that $h(x)$ is given by

$$(21) \quad \Gamma(1 - \beta)x^\alpha h(x) = \frac{1}{\Gamma(\beta)} \int_0^d (x - y)^{-\beta} F(y) dy \\ + \frac{1}{\Gamma(1 - \beta)} \int_d^x e^y y^{\alpha+\beta-1} (x - y)^{-\beta} G(y) dy,$$

for $d < x < \infty$, where $F(y)$ and $G(y)$ are given by equations (14) and (15).

(b) *Calculation of $k(x)$.* Using the differentiation formula

$$(22) \quad e^{-x} L_n(\alpha; x) = - \frac{d}{dx} \{e^{-x} L_n(\alpha - 1; x)\},$$

we may write equation (2) as

$$(23) \quad \frac{d}{dx} e^{-x} \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n(\alpha - 1; x) \\ = - e^{-x} k(x), \quad 0 \leq x < d.$$

Substituting for C_n and interchanging the order of integration and summation we find

$$(24) \quad e^{-x} k(x) = - \frac{d}{dx} e^{-x} \left\{ \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} F(y) S_2(x, y) dy \right. \\ \left. + \frac{1}{\Gamma(1 - \beta)} \int_d^{\infty} y^{\alpha+\beta-1} G(y) S_2(x, y) dy \right\},$$

for $0 \leq x < d$, and

$$(25) \quad S_2(x, y) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(\alpha + \beta + n)} L_n(\alpha - 1; x) L_n(\alpha + \beta - 1; y) \\ = \frac{1}{\Gamma(\beta)} e^x (y - x)^{\beta-1} y^{1-\alpha-\beta} H(y - x),$$

where the series has been summed using the results (6) and (7).

Substituting for $S_2(x, y)$ in (24) we see that $k(x)$ is given by

$$(26) \quad \Gamma(\beta) e^{-x} k(x) = - \frac{d}{dx} \left\{ \frac{1}{\Gamma(\beta)} \int_x^d e^{-y} (y - x)^{\beta-1} y^{1-\alpha-\beta} F(y) dy \right. \\ \left. + \frac{1}{\Gamma(1 - \beta)} \int_d^{\infty} (y - x)^{\beta-1} G(y) dy \right\},$$

when $0 \leq x < d$.

It is perhaps interesting to note that the expressions for the functions $k(x)$ and $h(x)$ do not involve Laguerre polynomials.

REFERENCES

1. A. Erdélyi et al., *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, 1954.
2. B. Noble, *Some dual series equations involving Jacobi polynomials*, Proc. Camb. Phil. Soc. **59** (1963), 363-372.
3. K. N. Srivastava, *On dual series relations involving Laguerre polynomials*, Pacific J. Math. **19** (1966), 529-533.

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