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**SOME DUAL SERIES EQUATIONS INVOLVING LAGUERRE  
POLYNOMIALS**

JOHN S. LOWNDES

## SOME DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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**In this paper an exact solution is found for the dual series equations**

$$(1) \quad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = f(x), \quad 0 \leq x < d,$$

$$(2) \quad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = g(x), \quad d < x < \infty,$$

**where  $\alpha + \beta > 0, 0 < \beta < 1, L_n(\alpha; x) = L_n^\alpha(x)$  is the Laguerre polynomial and  $f(x)$  and  $g(x)$  are known functions.**

In a recent paper Srivastava [3] has solved the equations

$$(3) \quad \sum_{n=0}^{\infty} \{A_n / \Gamma(\alpha + 1 + n)\} L_n(\alpha; x) = f(x), \quad 0 \leq x < d,$$

$$(4) \quad \sum_{n=0}^{\infty} \{A_n / \Gamma(\alpha + 1/2 + n)\} L_n(\alpha; x) = g(x), \quad d < x < \infty, \alpha > -1/2,$$

by considering separately the equations when (a)  $g(x) \equiv 0$ , (b)  $f(x) \equiv 0$ , and reducing the problem in each case to that of solving an Abel integral equation. Srivastava's equations are a special case of (1) and (2) with  $\beta = 1/2$  and  $A_n = \Gamma(\alpha + 1 + n)\Gamma(\alpha + 1/2 + n)C_n$ .

The solution presented in this paper employs a multiplying factor technique which is more direct than the method given in [3] and is similar to that used by Noble [2] to solve some dual series equations involving Jacobi polynomials.

2. In the course of the analysis we shall use the following results.

From [1, p. 293(5), p. 405(20)] it is readily shown that

$$(5) \quad \int_0^y x^\alpha (y-x)^{\beta-1} L_n(\alpha; x) dx = \frac{\Gamma(\beta)\Gamma(\alpha+1+n)}{\Gamma(\alpha+\beta+1+n)} y^{\alpha+\beta} L_n(\alpha+\beta; y),$$

where  $-1 < \alpha, \beta > 0$ , and

$$(6) \quad \int_y^\infty (x-y)^{-\beta} e^{-x} L_n(\alpha; x) dx = \Gamma(1-\beta) e^{-y} L_n(\alpha+\beta-1; y),$$

where  $1 > \beta, \alpha + \beta > 0$ .

The orthogonality relation for the Laguerre polynomials is

$$(7) \quad \int_0^\infty x^\alpha e^{-x} L_n(\alpha; x) L_m(\alpha; x) dx = \frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)} \delta_{mn}, \quad \alpha > -1,$$

where  $\delta_{mn}$  is the Kronecker delta.

**3. Solution of the problem.** Multiplying equation (1) by  $x^\alpha(y-x)^{\beta-1}$ , equation (2) by  $(x-y)^{-\beta}e^{-x}$  and integrating with respect to  $x$  over  $(0, y)$  and  $(y, \infty)$  respectively we find on using the results (5) and (6)

$$(8) \quad \sum_{n=0}^{\infty} C_n \frac{\Gamma(\alpha+1+n)}{(\alpha+\beta+n)} L_n(\alpha+\beta; y) = \frac{y^{-\alpha-\beta}}{\Gamma(\beta)} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx,$$

where  $0 < y < d, \alpha > -1, \beta > 0$ , and

$$(9) \quad \begin{aligned} & \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha+\beta-1; y) \\ &= \frac{e^y}{\Gamma(1-\beta)} \int_y^\infty (x-y)^{-\beta} e^{-x} g(x) dx, \end{aligned}$$

for  $d < y < \infty, 1 > \beta, \alpha + \beta > 0$ .

If we now multiply equation (8) by  $y^{\alpha+\beta}$ , differentiate with respect to  $y$  and use the formula

$$(10) \quad \frac{d}{dx} \{x^\alpha L_n(\alpha; x)\} = (n+\alpha)x^{\alpha-1} L_n(\alpha-1; x),$$

we find

$$(11) \quad \begin{aligned} & \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha+\beta-1; y) \\ &= \frac{y^{1-\alpha-\beta}}{\Gamma(\beta)} \frac{d}{dy} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx, \end{aligned}$$

where  $0 < y < d, \beta > 0, \alpha > -1$ .

The left hand sides of equations (9) and (11) are now identical and using the orthogonality relation (7) we see that the solution of equations (1) and (2) for  $\alpha + \beta > 0, 0 < \beta < 1$ , is given by

$$(12) \quad C_n = \frac{\Gamma(n+1)}{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+n)} B_n(\alpha, \beta; d),$$

where

$$(13) \quad \begin{aligned} B_n(\alpha, \beta; d) &= \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} L_n(\alpha+\beta-1; y) F(y) dy \\ &+ \frac{1}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} L_n(\alpha+\beta-1; y) G(y) dy, \end{aligned}$$

and

$$(14) \quad F(y) = \frac{d}{dy} \int_0^y x^\alpha(y-x)^{\beta-1} f(x) dx,$$

$$(15) \quad G(y) = \int_y^\infty (x - y)^{-\beta} e^{-x} g(x) dx .$$

To obtain the solution of Srivastava's equations (3) and (4) we write  $\beta = 1/2$ ,  $A_n = \Gamma(\alpha + 1 + n)\Gamma(\alpha + 1/2 + n)C_n$  in (12) and find that

$$(16) \quad A_n = \frac{\Gamma(n + 1)}{\Gamma(1/2)} \left\{ \int_d^d e^{-y} L_n(\alpha - 1/2 ; y) F_1(y) dy + \int_d^\infty y^{\alpha-1/2} L_n(\alpha - 1/2 ; y) G_1(y) dy \right\} ,$$

for  $\alpha > -1/2$ , and where  $F_1(y)$  and  $G_1(y)$  are given by equations (14) and (15) respectively with  $\beta = 1/2$ .

Comparing the above solution with that obtained in [3] it can be seen that they are in agreement except for the form of the function  $G_1(y)$ . The limits on the integrals of equations (4.7) and (4.8) in Srivastava's paper are wrong and should read  $(x, \infty)$  and  $(u, \infty)$  respectively. When these corrections have been made we find that his term corresponding to  $G_1(y)$  can be written in the notation of the present paper as

$$(17) \quad - \frac{d}{dy} \int_y^\infty (x - y)^{-1/2} dx \int_x^\infty e^{-u} g(u) du .$$

After inverting the order of integration, carrying out the integration in  $x$  and performing the differentiation with respect to  $y$  it is found that (17) is equal to  $G_1(y)$ . Hence with this simplification Srivastava's solution reduces to that given by equation (16).

4. It is also possible without computing the coefficients  $C_n$  to find the values of series (1) and (2) in the regions where their values are not specified. We define (1) to have the value  $h(x)$ ,  $d < x < \infty$ , and (2) to have the value  $k(x)$ ,  $0 \leq x < d$ .

(a) *Calculation of  $h(x)$ .* Substituting for  $C_n$  from equation (12) into (1) and interchanging the order of integration and summation we find

$$(18) \quad h(x) = \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} F(y) S_1(x, y) dy + \frac{1}{\Gamma(1 - \beta)} \int_d^\infty y^{\alpha+\beta-1} G(y) S_1(x, y) dy , \quad d < x < \infty ,$$

where

$$(19) \quad S_1(x, y) = \sum_{n=0}^\infty \frac{\Gamma(n + 1)}{\Gamma(\alpha + 1 + n)} L_n(\alpha ; x) L_n(\alpha + \beta - 1 ; y) .$$

Using the results (6) and (7) it is easily shown that

$$(20) \quad S_1(x, y) = \frac{e^y x^{-\alpha}(x-y)^{-\beta}}{\Gamma(1-\beta)} H(x-y),$$

where  $H(x)$  is the Heaviside unit function.

From equations (18) and (20) we see that  $h(x)$  is given by

$$(21) \quad \begin{aligned} \Gamma(1-\beta)x^\alpha h(x) &= \frac{1}{\Gamma(\beta)} \int_0^d (x-y)^{-\beta} F(y) dy \\ &+ \frac{1}{\Gamma(1-\beta)} \int_d^x e^y y^{\alpha+\beta-1} (x-y)^{-\beta} G(y) dy, \end{aligned}$$

for  $d < x < \infty$ , where  $F(y)$  and  $G(y)$  are given by equations (14) and (15).

(b) *Calculation of  $k(x)$ .* Using the differentiation formula

$$(22) \quad e^{-x} L_n(\alpha; x) = -\frac{d}{dx} \{e^{-x} L_n(\alpha-1; x)\},$$

we may write equation (2) as

$$(23) \quad \begin{aligned} \frac{d}{dx} e^{-x} \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha-1; x) \\ = -e^{-x} k(x), \quad 0 \leq x < d. \end{aligned}$$

Substituting for  $C_n$  and interchanging the order of integration and summation we find

$$(24) \quad \begin{aligned} e^{-x} k(x) &= -\frac{d}{dx} e^{-x} \left\{ \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} F(y) S_2(x, y) dy \right. \\ &\left. + \frac{1}{\Gamma(1-\beta)} \int_d^{\infty} y^{\alpha+\beta-1} G(y) S_2(x, y) dy \right\}, \end{aligned}$$

for  $0 \leq x < d$ , and

$$(25) \quad \begin{aligned} S_2(x, y) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(\alpha+\beta+n)} L_n(\alpha-1; x) L_n(\alpha+\beta-1; y) \\ &= \frac{1}{\Gamma(\beta)} e^x (y-x)^{\beta-1} y^{1-\alpha-\beta} H(y-x), \end{aligned}$$

where the series has been summed using the results (6) and (7).

Substituting for  $S_2(x, y)$  in (24) we see that  $k(x)$  is given by

$$(26) \quad \begin{aligned} \Gamma(\beta) e^{-x} k(x) &= -\frac{d}{dx} \left\{ \frac{1}{\Gamma(\beta)} \int_x^d e^{-y} (y-x)^{\beta-1} y^{1-\alpha-\beta} F(y) dy \right. \\ &\left. + \frac{1}{\Gamma(1-\beta)} \int_d^{\infty} (y-x)^{\beta-1} G(y) dy \right\}, \end{aligned}$$

when  $0 \leq x < d$ .

It is perhaps interesting to note that the expressions for the functions  $k(x)$  and  $h(x)$  do not involve Laguerre polynomials.

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