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The primary purpose of this paper is to establish some implications between normality and pointwise paracompactness in Moore spaces. In particular, it is proved that if either of two conjectures, raised by R. W. Heath and E. E. Grace, is true then each normal Moore space is indeed metrizable.

In [4], Heath and Grace raised questions regarding the substitution of normality for the condition of pointwise paracompactness in several of the theorems proved in that paper. The resulting statements appear below as Conjecture A and Conjecture B. The purpose of this note is to establish that the truth of either of the conjectures implies that each normal separable Moore space is metrizable. Thus, if either Conjecture A or Conjecture B is proved true then the condition that $2^{\aleph_0} < 2^{\aleph_1}$ would be removed from Jones' result [8, Th. 5] on the metrization of normal separable Moore spaces.

For definitions and results related to the question of metrization of normal Moore spaces, refer to [1], [2], [5], [6], [7], [8], [9], [10], [11], [13], [14].

CONJECTURE A. Suppose that S is a connected normal Moore space such that S contains no cut points and it is true that if each of P and Q is a point of S and R is a region containing P then some separable, closed connected subset N of R separates P from Q in S. Then S is separable.

CONJECTURE B. Suppose that S is a connected, locally connected, normal Moore space containing a separable closed set which separates S and each separable closed set which separates S contains two points which are separated by a separable closed set. Then S is separable.

THEOREM 1. *If Conjecture A is true then each normal separable Moore space is metrizable.*

Proof. Suppose that the theorem is false and that (S, Ω) is a normal separable nonmetrizable Moore space. There exist [8, Lemma C], in S, an uncountable set M with no limit point and a countable dense set K of $S - M$ such that each point of M is a limit point of K. The subspace $K + M$, with the relative topology, is normal, separable, nonmetrizable and a Moore space. If (S_1, Ω_1) denotes the subspace $K + M$ of (S, Ω), denote by (S_1, Ω_2) the space whose topology is precisely that of (S_1, Ω_2) of [2, Th. 2]. For purposes of clarity, this definition is included below:

Enumerate $K: A_1, A_2, A_3 \cdots$. For each point x of M denote by $\{n_i(x)\}_{i=1}^{\infty}$ an increasing sequence of positive integers such that $\text{limit}_{i \to \infty} A_{n_i}(x) = x$, according to the topology Ω_1. Now consider the space (S_1, Ω_2), where Ω_2 is the topology induced by the following definition of region:

The point set R is a region if and only if either (1) for some point P of K, R is the degenerate set whose only element is P, or (2) some point x of M and some positive integer i, R is the set to which P belongs if and only if $P = x$ or $P = A_{nj}(x)$ for some j greater than or equal to i.

Using the method employed in [12, Th. 3] (that of inserting copies of open intervals of the real line between "adjacent" points of K which are points of a region containing a point of M) it is seen that it is possible to embed (S_1, Ω_2) in a space (S_2, Ω_3) which is normal, separable, nonmetrizable, arcwise connected, and locally arcwise connected. Indeed, (using precisely the notation of [2, Th. 2]), if A_i is a point of K, at most finitely many intervals "connect" A_i to points of K having subscripts less than i, and at most countably many intervals "connect" A_i to points of K having subscripts greater than i.

Now, denote by Z a space with discrete topology such that $\bar{\bar{Z}} = \bar{\bar{M}}$ and Z does not intersect S_2. There is a reversible transformation T which throws Z onto M.

Consider the topological product space, $Z \times [0, 1)$. It follows that T induces a reversible transformation T^1 from $Z \times \{0\}$ onto M such that if m is a point of M then there exists a point z of Z such that $T^1(z, 0) = m$, where $(z, 0)$ is a point of $z \times [0, 1)$. This essentially attaches mutually exclusive segments to the points of M.

Denote by (S_3, Ω_4) the space in which "point" means point of S_2 or point of $z \times [0, 1)$ for some z of Z and in which Ω_4 is the topology induced by the following definition of region:

The point set R is a region if and only if either.

(i) there is a region g of Ω_3 such that g contains no point of M and $g = R$, or

(ii) there is a region g of Ω_3 such that g contains a point of M, say m, and R is the set to which x belongs if and only if x is a point of g or, for some positive integer n and some point z of A such that $T^1(z, 0) = m$, x is a point of $z \times [0, 1/n)$, or

(iii) there are a point z of Z and a positive integer n such that R is a subsegment of $z \times (0, 1)$ of length less than $1/n$.

It follows that (S_3, Ω_4) is a normal, nonmetrizable, arcwise connected Moore space which contains uncountably many mutually exclusive domains.

Now denote by (S_4, Ω_5) the space which is the topological product space resulting from (S_3, Ω_4) being crossed with $(0, 1)$. It follows from Dowker's result [3, Th. 2] that each normal Moore space is countably paracompact. Dowker also proved [3, Th. 4] that if X is countably paracompact and normal then the topological prodoct $X \times I$ of X with the closed line interval $I = [0, 1]$ is normal. Since the product of a normal Moore space with the interval $[0, 1]$ is again a normal Moore space and each such space is completely normal [8, Th. 6], it is evident that (S_4, Ω_5) is normal. Indeed, (S_4, Ω_5) is an arcwise connected, locally arcwise connected Moore space which contains no cut points and is such that if each of x and y is a point of S_4 and R is an open set containing x then there is a closed, connected, separable subset N of R such that N separates x from y in S. However, (S_4, Ω_5) is definitely not separable since its construction insists upon the existence of uncountably many mutually exclusive domains.

THEOREM 2. *If Conjecture B is true then each normal separable Moore space is metrizable.*

Proof. The example constructed in the proof of Theorem 1 would deny Conjecture B.

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