REPRODUCING KERNELS IN SEPARABLE HILBERT SPACES

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A theorem on the existence of a reproducing kernel in a separable Hilbert space of functions is proved. As an application of this theorem, a method of interpolation of the functions in a separable Hilbert space with a reproducing kernel is given. This method is used to construct the elements of the Hilbert space generated by a second order stochastic process, in case this space is separable.

Theorems 2, 3 and 4 of this paper, which were motivated by Parzen’s work [2], [3], were originally proved in somewhat different form in collaboration with J. Ricatte [4]. In this paper it will be shown that these three theorems are the consequences of a more general statement given in what follows as Theorem 1.

1. Preliminaries. Let $\mathcal{H}$ be a Hilbert space of real or complex functions defined on an arbitrary set $T$. The scalar product of any ordered pair of functions $f, g$ in $\mathcal{H}$ will be denoted by $\langle f, g \rangle$ and the norm of a function $f \in \mathcal{H}$ by $\| f \|$. A two variable function $K$ defined on the product set $T \times T = T^2$ is the reproducing kernel of $\mathcal{H}$, if it satisfies the following two conditions:

(A) $K(t, \cdot) \in \mathcal{H}, \forall t \in T$.

(B) $\langle f, K(t, \cdot) \rangle = f(t), \forall t \in T$ and $\forall f \in \mathcal{H}$.

The last property is called reproduction property of $K$.

$K$ is self-reproducing, i.e. $K(t, \tau) = \langle K(t, \cdot), K(t, \tau) \rangle$. It is positive-semi-definite, i.e.

$$\sum_{i,j \in I} \lambda_i \overline{\lambda_j} K(t_i, t_j) = \left\| \sum_{i \in I} K(t_i, \cdot) \right\|^2 > 0, \lambda_i \in C, \forall i \in I \subset N.$$  

(where $C$ is the set of complex numbers, $I$ an arbitrary finite subset of the set $N$ of positive integers and $\overline{\lambda_j}$ the conjugate of $\lambda_j$). In particular, $K$ has the Hermitian symmetry ($K(t, \tau) = \overline{K(t, \tau)}, \forall t, \tau \in T$) and

$$0 \leq \| K(t, \cdot) \|^2 = K(t, t) < \infty, \forall t \in T.$$  

If $\mathcal{H}$ has a reproducing kernel, this kernel is always unique, for if $K$ and $K'$ were two distinct reproducing kernels of $\mathcal{H}$, their reproduction property would imply

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1 For a more general and detailed presentation of the Theory of Reproducing Kernels, see the article by Aronzajn [1].
\[ K(t, \tau) = \langle K(t, \cdot), K'(\tau, \cdot) \rangle = \langle K'(\tau, \cdot), K(t, \cdot) \rangle = K'(\tau, t) = K'(t, \tau). \]

The weak convergence (consequently the strong convergence) of a sequence \( \{f_n\} \subset H \) to a function \( f \in H \) implies its pointwise convergence to the same function \( f \), for

\[
\lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \langle f_n, K(t, \cdot) \rangle = \langle f, K(t, \cdot) \rangle = f(t).
\]

If a topology is defined on \( T \), then the continuity of \( K \) with respect to the product topology on \( T^2 \) implies the continuity of each function in \( \mathcal{D} \). This is the consequence of the Schwarz inequality applied to \((B)\):

\[
|f(t) - f(t_0)|^2 = |\langle f, K(t, \cdot) - K(t_0, \cdot) \rangle|^2 \leq ||f||^2[K(t, t) - K(t, t_0) - K(t_0, t) + K(t_0, t_0)].
\]

Given a finite and positive-semi-definite function \( K \) on \( T^2 \), there exists a uniquely defined Hilbert space of functions on \( T \), whose reproducing kernel is \( K \) (Moore’s Theorem). This space is obtained in the following way: Let \( L_K \) be the linear set generated by \( \{K(t, \cdot), t \in T,\} \) i.e. the set of all finite linear combinations

\[
\sum_i \lambda_i K(t_i, \cdot), \lambda_i \in \mathcal{C},
\]

Let a scalar product of any ordered pair of elements \( f, g \in L_K \) be defined by

\[
\langle f, g \rangle = \sum_{i,j} \lambda_i \mu_j K(t_i, t_j)
\]

where

\[
f = \sum_i \lambda_i K(t_i, \cdot), \ g = \sum_j \mu_j K(t_j, \cdot).
\]

This scalar product induces a norm on \( L_K \), so that \( L_K \) is a pre-Hilbert space. Obviously

\[
f(t) = \langle f, K(t, \cdot) \rangle, \ \forall t \in T \text{ and } \forall f \in L_K.
\]

If \( \{f_n\} \) is a Cauchy sequence in \( L_K \), then \( \{f_n\} \) converges everywhere to a function \( f \), for

\[
|f_m(t) - f_n(t)|^2 \leq ||f_m - f_n||^2 K(t, t).
\]

If the norm of \( f \) is defined by \( ||f|| = \lim_{n \to \infty} ||f_n|| \), the space obtained by the adjunction to \( L_K \) of pointwise limits of Cauchy sequences in \( L_K \) is a Hilbert space and \( K \) reproduces all functions of this space. The space generated by \( \{K(t, \cdot), t \in T\} \) will be denoted by \( \mathcal{D}_K \).

Let \( \mathcal{D} \) be any Hilbert space whose reproducing kernel is \( K \). Then
the class \( \{K(t, \cdot), t \in T\} \) is a basis for \( S \), so that \( S \) coincides with \( S_K \). Consequently, if a closed subspace \( S \) of a Hilbert space \( H \) of functions on \( T \) has a reproducing kernel \( K \), then for any function \( h \in H \), the scalar product \( \langle h, K(t, \cdot) \rangle \) gives the projection of \( h \) onto \( S \). Also, if \( S \) is a closed subspace of \( S_K \), then the reproducing kernel of \( S \) is the projection \( \tilde{K}(t, \cdot) \) of \( K(t, \cdot) \) onto \( S \).

2. The case of separable Hilbert spaces. The following theorem gives a necessary and sufficient condition for a separable Hilbert space of functions to have a reproducing kernel.

**Theorem 1.** Let \( S \) be a separable Hilbert space of functions defined on \( T \) and let \( \{e_i\} \) be a countable class of linearly independent functions in \( S \) forming a basis for \( S \). Let \( \{K_n\} \) be the sequence defined by

\[
K_n(t, \tau) = \sum_{i,j=1}^{n} \gamma_{ij} \alpha_i(t) \alpha_j(\tau)
\]

where \( (\gamma_{ij})_{1 \leq i,j \leq n} \) is the inverse of the matrix \( (\langle e_i, e_j \rangle)_{1 \leq i,j \leq n} \).

(C1) If \( \forall t \in T, \{K_n(t, t)\} \) converges as \( n \to \infty \), then any Cauchy sequence \( \{\sum_{i=1}^{n} \alpha_n, e_i\} \subset S \) converges everywhere on \( T \).

(C2) If, moreover, pointwise limits of such Cauchy sequences coincide with their limits in norm,

then \( K(t, \tau) = \lim_{n \to \infty} K_n(t, \tau) \), which exists \( \forall t, \tau \in T \), is the reproducing kernel of \( S \).

Conversely, if \( S \) has a reproducing kernel \( K \), then the conditions \( C_1 \) and \( C_2 \) are fulfilled and \( \forall t, \tau \in T, K(t, \tau) = \lim_{n \to \infty} K_n(t, \tau) \).

**Proof.** To avoid all trivialities, \( S \) can be supposed to be infinite dimensional.

**Sufficiency of \( C_1 \) and \( C_2 \).** Consequences of \( C_1 \). Let \( \bar{S}_n \) be the subspace generated by \( \{e_i, 1 \leq i \leq n\} \). \( K_n(t, \cdot) \) is obviously an element of \( \bar{S}_n \) and it reproduces all functions in \( S \). Moreover, \( \bar{S}_n \subset \bar{S}_m \) for \( m > n \). Then \( K_n(t, \cdot) \) is the projection of \( K_m(t, \cdot) \) onto \( \bar{S}_n \). Consequently, the relations

\[
\langle K_m(t, \cdot), K_n(\tau, \cdot) \rangle = K_n(t, \tau), m > n
\]

(3) \[ \| K_m(t, \cdot) - K_n(t, \cdot) \|^2 = K_m(t, t) - K_n(t, t), m > n \]

hold. By the last relation, it can be seen that \( \{K_n(t, t)\} \) is an increasing sequence which converges by hypothesis, so that \( \{K_n(t, \cdot)\} \) is a Cauchy
sequence in $\Phi$ for every $t \in T$. Let $K(t, \cdot)$ be the limit of this sequence.

For a given function $f \in \Phi$, the function $f_n$ defined by

\begin{equation}
\hat{f}_n(t) = \langle f, K_n(t, \cdot) \rangle = \sum_{i,j=1}^{n} \beta_i \gamma_{ij} e_j(t), \quad \beta_i = \langle f, e_i \rangle
\end{equation}

is the projection of $f$ onto $\Phi_n$. Thus, the relations

\begin{align}
&\text{(4)} \quad \| f - \hat{f}_n \|^2 = \| f_n \|^2 - \| \hat{f}_n \|^2 \\
&\text{(5)} \quad \| \hat{f}_m - \hat{f}_n \|^2 = \| \hat{f}_m \|^2 - \| \hat{f}_n \|^2, \quad m > n \\
&\text{(6)} \quad \| \hat{f}_m \| \leq \| \hat{f}_m \| \leq \| f \|, \quad m > n.
\end{align}

hold. Consequently, $\{\| \hat{f} \| \}$ is a nondecreasing sequence bounded by $\| f \|$, therefore it converges. Then, according to (6), $\{f_n\}$ is a Cauchy sequence in $\Phi$.

Let us suppose that

\begin{equation}
\hat{f}_n = \sum_{k=1}^{n} \alpha_{nk} e_k
\end{equation}

is a sequence converging to $f$. Since $f_n \in \Phi_n$, the relation

\begin{equation}
\langle f, f_n \rangle = \langle f - \hat{f}_n + \hat{f}_n, f_n \rangle = \langle \hat{f}_n, f_n \rangle
\end{equation}

holds. Then, $\lim_{n \to \infty} \langle \hat{f}_n, f_n \rangle = \lim_{n \to \infty} \langle f, f_n \rangle = \| f \|^2$, and according to (7),

\begin{equation}
0 \leq \lim_{n \to \infty} \| \hat{f}_n - f_n \|^2 = \lim_{n \to \infty} (\| \hat{f}_n \|^2 - \langle \hat{f}_n, f_n \rangle - \langle f_n, \hat{f}_n \rangle + \| f_n \|^2)
\end{equation}

\begin{equation}
= \lim_{n \to \infty} \| \hat{f}_n \|^2 - \| f \|^2 \leq 0.
\end{equation}

Consequently, $\lim_{n \to \infty} \| \hat{f}_n \| = \| f \|$. Then the relation (5) shows that $\{\hat{f}_n\}$ converges to $f$ in norm.

Since the strong convergence of $\{\hat{f}_n\}$ implies its weak convergence, one has

\begin{equation}
\lim_{n \to \infty} \hat{f}_n(t) = \lim_{n \to \infty} \langle \hat{f}_n, K(t, \cdot) \rangle
\end{equation}

\begin{equation}
= \langle f, K(t, \cdot) \rangle = g(t).
\end{equation}

Thus, $\{\hat{f}_n\}$ converges everywhere. From this, it is easy to see that any Cauchy sequence of the type (8) also converges everywhere. In fact,

\begin{equation}
\hat{f}_n(t) - f_n(t) = \langle f - f_n, K_n(t, \cdot) \rangle.
\end{equation}

By applying the Schwarz inequality and taking into account the fact
that \( K_n(t, t) < K(t, t) \), one can write

\[
|\hat{f}_n(t) - f_n(t)|^2 \leq \|f - f_n\|^2 K_n(t, t) \leq \|f - f_n\|^2 K(t, t).
\]

Since \( \{f_n\} \) converges to \( f \) in norm, it is seen that \( \lim_{n \to \infty} |\hat{f}_n(t) - f_n(t)| = 0 \).

Finally, the inequality

\[
|g(t) - f_n(t)| \leq |g(t) - \hat{f}_n(t)| + |\hat{f}_n(t) - f_n(t)|
\]

shows that \( \{f_n(t)\} \) converges to the same limit \( g(t) \) as \( \{\hat{f}_n(t)\} \).

**Consequences of \( C_2 \).** In case the pointwise limit and the limit in norm of Cauchy sequences of the type (8) coincide, then by (9) the reproduction property \( g(t) = f(t) = \langle f, K(t, \cdot) \rangle \) is obtained. Also, the sequence \( \{K_n(t, \tau)\} \) converges to \( K(t, \tau), \forall t, \tau \in T \). Hence, \( K(t, \tau) = \lim_{n \to \infty} K_n(t, \tau) \) is the reproducing kernel of \( \mathfrak{S} \).

**Necessity of \( C_1 \) and \( C_2 \).** Suppose that \( \mathfrak{S} \) possesses a reproducing kernel \( K \). The relation (3) which is still valid, together with the relation

\[
\|K(t, \cdot) - K_n(t, \cdot)\|^2 = K(t, t) - K_n(t, t),
\]

obtained from (5) by replacing \( f(\cdot) \) by \( K(t, \cdot) \), imply that

\[
K_n(t, t) < K_m(t, t) < K(t, t) \quad \text{for} \quad m > n.
\]

Thus, \( \{K_n(t, t)\} \) is an increasing sequence bounded by \( K(t, t) < \infty \). Hence, it converges, so that the condition \( C_1 \) is fulfilled. On the other hand, since \( \mathfrak{S} \) possesses a reproducing kernel, the condition \( C_2 \) is automatically fulfilled.

Consequently, \( \lim_{n \to \infty} K_n(t, \tau) \) is a reproducing kernel of \( \mathfrak{S} \). Reproducing kernel being always unique, one has \( K(t, \tau) = \lim_{n \to \infty} K_n(t, \tau) \).

**Remark.** If only the condition \( C_1 \) holds, then the space \( \mathfrak{S} \) can be made isomorphic to a Hilbert space whose reproducing kernel is \( \Gamma(t, \tau) = \langle K(t, \cdot), K(\tau, \cdot) \rangle \) with \( K(t, \cdot) \) as the strong limit of \( \{K_n(t, \cdot)\} \) in \( \mathfrak{S} \). In fact, any Cauchy sequence of the type (8) converging to \( f \in \mathfrak{S} \) converges everywhere in \( T \) to a function \( g \). As in the theorem of Moore, if the set of all linear combinations of the functions \( \{e_i\} \) is completed by the adjunction of pointwise limits of Cauchy sequences of this set with respect to the topology of \( \mathfrak{S} \), and if the limit of the norms for each sequence is assigned as the norm of the pointwise limit of the sequence, then a Hilbert space \( \mathfrak{S}_r \) is obtained. The reproducing kernel of \( \mathfrak{S}_r \) turns out to be \( \Gamma \). This latter space is obviously isomorphic to \( \mathfrak{S} \). This isomorphism can be represented by
\[ g(t) = \langle f, K(t, \cdot) \rangle, \forall t \in T, f \in \mathcal{H} \text{ and } g \in \mathcal{H}_r. \]

It can be proved also, that the class of functions \( \{K(t, \cdot), t \in T\} \) generates \( \mathcal{H} \), in the sense that it is a basis for \( \mathcal{H} \), that is, any function \( f \in \mathcal{H} \) for which \( \langle f, K(t, \cdot) \rangle = 0 \) for all \( t \in T \), has its norm equal to zero. In fact, let \( f \) be such a function. Then the function \( g \in \mathcal{H}_r \) corresponding to \( f \) in the isomorphism between \( \mathcal{H} \) and \( \mathcal{H}_r \) is the null function in \( \mathcal{H}_r \). Consequently, its norm and the norm of \( f \) equal zero.

It is worth mentioning that in view of this remark and the following theorem, there exists a countable subset \( S \) of \( T \) such that \( K = \Gamma \) on both \( S \times T \) and \( T \times S \).

In what follows, a separable Hilbert space \( \mathcal{H}_K \) of functions on \( T \), with reproducing kernel \( K \), will be considered. Since the class \( \{K(t, \cdot), t \in T\} \) generates \( \mathcal{H}_K \), there exists a countable subset \( S \) of \( T \) such that \( \{K(t_i, \cdot), t_i \in S, i \in \mathbb{N}\} \) is a class of linearly independent functions forming a basis for \( \mathcal{H}_K \). The matrix \( (\gamma_{ij})_{i \leq i, j \leq n} \) will denote the inverse of the matrix \( (K(t_i, t_j))_{i \leq i, j \leq n} \) and \( S_n \) will denote \( \{t_1, t_2, \ldots, t_n\} \subset S \).

**Theorem 2.** For any function \( f \in \mathcal{H}_K \), the sequence of functions defined by

\[ \hat{f}_n(\cdot) = \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} K(t_j, \cdot) \]

converges to \( f \), as \( n \to \infty \), (both in norm and everywhere).

**Proof.** To prove the theorem, it suffices to replace \( e_i \) by \( K(t_i, \cdot) \) in the preceding theorem. Then \( K_n(t, \tau) \) becomes

\[ K_n(t, \tau) = \sum_{i,j=1}^{n} K(t, t_i) \gamma_{ij} K(t_j, \tau) \]

and the function (4) reduces to (10).

Notice that \( K_n \) coincides with \( K \) on \( S_n \times T \) and \( T \times S_n \), and consequently, \( \hat{f}_n = f \) on \( S_n \). According to the second part of Theorem 1, \( K_n(t, \cdot) \) converges to \( K(t, \cdot) \) in norm and everywhere, and the first part of the proof of the same theorem shows that the sequence (10) converges to \( f \) in norm and everywhere.

So, it appears that \( \hat{f}_n \) gives an approximation of \( f \) in norm and everywhere in terms of the values taken by \( f \) on the finite subset \( S_n \) of \( S \).

**Corollary.** The scalar product of any pair of functions \( f, g \in \mathcal{H}_K \) is given by
\[ \langle f, g \rangle = \lim_{n \to \infty} \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} \bar{g}(t_j). \]

Consequently, the norm of any function \( f \in \mathcal{K} \) is given by
\[ ||f||^2 = \lim_{n \to \infty} \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} \bar{f}(t_j). \]

**Theorem 3.** Let \( f \) be an arbitrary function defined on \( T \), such that
\[ \lim_{n \to \infty} \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} \bar{f}(t_j) < \infty, \ t_i, t_j \in S, \forall i, j \in N. \]

Then the sequence of functions defined by
\[ f_n(\cdot) = \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} K(t_j, \cdot) \]
is a Cauchy sequence in \( \mathcal{K} \), whose limit \( f' \) coincides with \( f \) on \( S \).

**Proof.** The relation
\[ ||f_m - f_n||^2 = ||f_m||^2 - ||f_n||^2, \quad m > n \]
holds for the sequence (15), with
\[ ||f_n||^2 = \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} \bar{f}(t_j). \]

It is then seen that \( ||f_n||^2 \) is a nondecreasing sequence converging to (14), so that \( \{f_n\} \) is a Cauchy sequence in \( \mathcal{K} \). Let \( f' \) be its limit. Since \( f' \in \mathcal{K} \), according to Theorem 2, the sequence
\[ \hat{f}'(t_i) = \sum_{i,j=1}^{n} f'(t_i) \gamma_{ij} K(t_j, \cdot) \]
is also a Cauchy sequence converging to \( f' \) and therefore \( \{f_n - \hat{f}'_n\} \) converges to the null function in \( \mathcal{K} \). Since the relation
\[ ||f_m - \hat{f}'_m||^2 - ||f_m - \hat{f}'_m||^2 = ||f_m - \hat{f}'_m||^2 - ||f_n - \hat{f}'_n||^2, \quad m > n \]
holds, one has
\[ 0 \leq ||f_n - \hat{f}'_n|| \leq \lim_{m \to \infty} ||f_m - \hat{f}'_m|| = 0 \]
so that \( \forall n \in N, ||f_n - \hat{f}'_n|| = 0 \). Consequently \( \forall t \in T \) and \( \forall n \in N, f_n(t) = \hat{f}'_n(t) \). In particular \( \forall i \leq n, f(t_i) = f_n(t_i) = \hat{f}'_n(t_i) = f'(t_i) \). Thus, \( f(t) = f'(t), \ all \ t \in S. \)

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1 This extension was suggested to the author by Professor H. L. Royden.
It follows from the last theorem that the set $\mathcal{F}$ of all functions satisfying the condition (14) is a Hilbert space in which the scalar product of $f$ by $g$ is given by

\begin{equation}
\lim_{n \to \infty} \sum_{i,j=1}^{n} f(t_i) \gamma_{ij} g(t_j), \quad t_i, t_j \in S, \forall i, j \in N.
\end{equation}

In this space all the functions coinciding on $S$ belong to the same equivalence class defined by the relation

$$f \sim g \Rightarrow \lim_{n \to \infty} \sum_{i,j=1}^{n} [f(t_i) - g(t_i)] \gamma_{ij} [\bar{f}(t_i) - \bar{g}(t_j)] = 0.$$ 

In particular, the function $f \in \mathcal{F}$ and the function $f' \in \mathcal{D}_K$ corresponding to $f$ as the limit of the sequence (15) are equivalent.

3. Hilbert Space generated by a second order random process.

Let $(\Omega, \Sigma, P)$ be a probability space, where $\Omega$ is a sample space, $\Sigma$ is the $\sigma$-algebra generated by a class of subsets of $\Omega$ and $P$ a probability measure defined on $\Sigma$. Let $\{X_t, t \in T\}$ be a class of complex valued random variables defined on $\Omega$ and measurable with respect to $\Sigma$. The symbol $E$ will denote the mathematical expectation with respect to the probability measure $P$. It will be supposed that $\forall t \in T$, $E(X_t) = 0$ and $E(|X_t|^2) < \infty$. The covariance function $E(X_t \bar{X}_\tau)$ of thus defined second order stochastic process will be denoted by $K(t, \tau)$.

Let $L_x$ be the linear set of all finite linear combinations

$$\sum_i \lambda_i X_{t_i}, \quad t_i \in T, \lambda_i \in C.$$ 

A scalar product on $L_x$ can be defined for any ordered pair of elements

$$Y = \sum_i \lambda_i X_{t_i}, \quad Z = \sum_j \mu_j X_{t_j}$$

by the bilinear form

$$E(Y \bar{Z}) = \sum_{i,j} \lambda_i \mu_j K(t_i, t_j)$$

which induces, for any element $Y \in L_x$, a norm whose square is defined by

$$E(|Y|^2) = \sum_{i,j} \lambda_i \bar{\lambda}_j K(t_i, t_j).$$

The Hilbert space which is the closure of $L_x$ in the topology induced by this norm will be denoted by $\mathcal{D}_x$ and will be said to be generated by the process $\{X_t, t \in T\}$.

The theorem of Moore says that there exists a uniquely defined Hilbert space $\mathcal{D}_K$ of functions on $T$, admitting $K$ as its reproducing
kernel. The construction of $\mathcal{H}_X$ and of $\mathcal{H}_K$ shows that these two spaces are isomorphic if $K$ is the covariance function of $\{X_t, t \in T\}$. Under this isomorphism, the random variable $X_t$ corresponds obviously to $K(t, \cdot)$. Consequently, the two spaces are simultaneously separable and if $\{K(t_i, \cdot), t_i \in S\}$ is a basis for $\mathcal{H}_K$ in the sense given in Theorem 1, then $\{X_{t_i}, t_i \in S\}$ is a basis for $\mathcal{H}_X$.

Given an element $Z$ in $\mathcal{H}_X$, the element $f_Z$ in $\mathcal{H}_K$ corresponding to $Z$ is given by

$$f_Z(t) = \langle f_Z, K(t, \cdot) \rangle = E(ZX_t).$$

For separable $\mathcal{H}_K$ (or equivalently $\mathcal{H}_X$) the following theorem gives a representation of the element of $\mathcal{H}_X$ corresponding to any given function $f$ in $\mathcal{H}_K$. The symbols have exactly the same meaning as in the two preceding theorems.

**Theorem 4.** For any function $f \in \mathcal{H}_X$, the stochastic element $X(f) \in \mathcal{H}_X$ corresponding to $f$ under the isomorphism between $\mathcal{H}_K$ and $\mathcal{H}_X$, is given by the limit in the quadratic mean of

$$X(f_n) = \sum_{i,j=1}^{n} f(t_i)\gamma_{ij}X_{t_j}$$

as $n \to \infty$.

**Proof.** By replacing $X(t_i)$ by $K(t_j, \cdot)$ in (17), it is seen that $X(f_n)$ is the element of $\mathcal{H}_X$ corresponding to (10). Since $\{f_n\}$ is a Cauchy sequence in $\mathcal{H}_X$ converging to $f$. Then $\{X(f_n)\}$ is a Cauchy sequence converging to $X(f)$.

In view of the analogy between (12) and (17), the element $X(f)$ can be represented, following Parzen, as $\langle f(\cdot), \bar{X}(\cdot) \rangle$. But this is not really a scalar product because, almost surely, $X(\cdot)$ does not belong to $\mathcal{H}_K$.

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