POINT NORMS IN THE CONSTRUCTION OF HARMONIC FORMS

Leo Sario and Mitsuru Nakai
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OF HARMONIC FORMS

Mitsuru Nakai and Leo Sario

Let $V$ be an arbitrary Riemannian $n$-space, and $V_1$, a regular
neighborhood of its ideal boundary. Given a harmonic field
$\sigma$ in $V_1$, necessary and sufficient conditions are known for the
existence in $V$ of a harmonic field $\rho$ which imitates the behavior
of $\sigma$ in $V_1$ in the sense $\int_{V_1} (\rho - \sigma) \wedge * (\rho - \sigma) < \infty$. In the pre-
sent paper we give the solution of the corresponding pro-
blem for harmonic forms in locally flat spaces.

One aspect of our treatment which may have possibilities for
generalization is the use of the point norm defined by $|\varphi|^2 = \varphi \cdot \varphi$. Another approach to generalizations is discussed in [3].

1. Throughout our presentation the symbol $V$ shall stand for a
locally flat Riemannian space. Since the curvature tensor vanishes
in $V$, there exists a covering $\{\tilde{U}_a | a \in V\}$ of $V$ such that $\tilde{U}_a$ is the
carrier of local coordinates $x_a = (x^1_a, \ldots, x^n_a)$ with $x_a(a) = 0$ and

$$|x_a| = \sqrt{|x^1_a|^2 + \cdots + |x^n_a|^2} \leq r_a \quad (0 < r_a < \infty)$$

in $U_a$ with the following property:

$$g_{ij}(x_a) = \delta_{ij} \quad (x_a \in \tilde{U}_a).$$

We moreover require that $V$ is parallel in the sense that the above
$\{U_a\}$ can be chose so as to satisfy

$$x^i_a = x^i_b + c^i_{ab} \quad (i = 1, \ldots, n)$$
in $\tilde{U}_a \cap \tilde{U}_b$ with constants $c^i_{ab}$. We call $\{\tilde{U}_a | a \in V\}$ a parallel coor-
dinate covering and each $U_a$ a distinguished coordinate neighborhood.

2. The space of harmonic $p$-forms $\varphi$, defined by $d\delta\varphi + \delta d\varphi = 0$, will be denoted by $H_p$. For a set $E \subset V$, the notation $\varphi \in H_p(E)$ shall mean that $\varphi$ is a harmonic $p$-form in an open set containing $E$.

Let $\tilde{V}_1$ be the complement in $V$ of a regular subregion [4] of $V$. Suppose $\sigma \in H_p(\tilde{V}_1)$ is given. The problem is to construct a correspond-
ing $\rho \in H_p(V)$, to be called the principal form, characterized by the
existence of a constant $M$ such that

$$|\rho - \sigma| < M < \infty$$
The space $V$ is called hyperbolic or parabolic according as it does or does not possess Green's functions [4].

**Theorem 1.** If $V$ is hyperbolic, then the principal form $\rho$ always exists.

**Theorem 2.** If $V$ is parabolic, then a necessary and sufficient condition for the existence of a principal form $\rho$ is that

$$\int_{\beta} * d < \sigma, c > = 0$$

for every constant form $c$. The principal form is unique up to an additive constant form.

Here $\langle \phi, \psi \rangle = \phi_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p}$, and $\beta$ stands for the ideal boundary of $V$. For constant forms see No. 4 below.

The above theorems will be consequences of the main existence theorem for harmonic forms (No. 7), which we shall first establish.

Theorem 1 is known to be valid without the assumption that $V$ is parallel ([3]).

3. Take a $p$-form $\phi$ on $V$:

$$\phi = \phi_{i_1 \ldots i_p} dx^{i_1}_a \wedge \cdots \wedge dx^{i_p}_a.$$

In $U_a \cap U_b$, $dx^i_a = dx^i_b$ and therefore

$$\phi_{i_1 \ldots i_p} = \phi_{i_1 \ldots i_p}.$$

For this reason there exists a global function $\phi_{i_1 \ldots i_p}$ in $V$ such that

$$\phi_{i_1 \ldots i_p} = \phi_{i_1 \ldots i_p}$$

in $\bar{U}_a$. Conversely, given functions $\phi_{i_1 \ldots i_p}$, there exists a $p$-form $\phi = \phi_{i_1 \ldots i_p} dx^{i_1}_a \wedge \cdots \wedge dx^{i_p}_a$ with $\phi_{i_1 \ldots i_p} \equiv \phi_{i_1 \ldots i_p}$ in each $\bar{U}_a$.

4. We call $\phi$ a constant $p$-form if

$$\Delta \phi = 0,$$

$$|\phi| = \text{const.},$$

and we denote by $K^p$ the class of constant $p$-forms. It is easy to see that

$$d\phi = 0, \delta \phi = 0$$
for $\varphi \in K^p$, i.e., constant forms are harmonic fields. If $\varphi \in H^p(V)$ and $|\varphi|$ is constant in some open set $D \subset V$, then $\varphi \in K^p(V)$. In fact, let

$$\varphi = \varphi_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$  

Then $\Delta \varphi = (\Delta \varphi_{i_1 \ldots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p} = 0$, and we see that each $\varphi_{i_1 \ldots i_p}$ is harmonic. Consequently $(\varphi_{i_1 \ldots i_p})^2$ is subharmonic, and so is

$$|\varphi|^2 = \sum_{i_1 < \cdots < i_p} (\varphi_{i_1 \ldots i_p})^2.$$  

Since $|\varphi|^2 = c$ (const.) in $D$, we have

$$c - (\varphi_{i_1 \ldots i_p})^2 = \sum_{j \neq i} (\varphi_{j_1 \ldots j_p})^2$$

in $D$. The left-hand member is subharmonic and superharmonic and the same is true of $(\varphi_{i_1 \ldots i_p})^2$. But $\Delta (\varphi_{i_1 \ldots i_p})^2 = |\text{grad } \varphi_{i_1 \ldots i_p}|^2$, and for this reason $\varphi_{i_1 \ldots i_p}$ must be constant.

Clearly $K^p$ is an $\binom{n}{p}$-dimensional vector space.

5. Let $L^p$ be the operator in the space of $p$-forms on $\alpha_1 = \partial V_1$ into the space of continuous $p$-forms in $V_1$, harmonic in $V_1$, such that $L^p \varphi |\alpha_1 = \varphi$ and

\begin{align*}
(7) & \quad L^p(\lambda \varphi_1 + \mu \varphi_2) = \lambda L^p \varphi_1 + \mu L^p \varphi_2, \\
(8) & \quad |L^p \varphi| \leq \sup_{\alpha_1} |\varphi|, \\
(9) & \quad \int_{\alpha_1} \ast d < L^p \varphi, c > = 0 \text{ for every } c \in K^p.
\end{align*}

We call $L^p$ a normal operator.

A normal operator $L$ for 0-forms induces one for $p$-forms:

$$L^p \varphi = (L \varphi_{i_1 \ldots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$  

More interesting is the following. Let $i_1 < \cdots < i_p$. We define one for $p$-forms by setting

$$L^p = i_1 \ldots i_p L dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

that is

$$L^p \varphi = (i_1 \ldots i_p L \varphi_{i_1 \ldots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$  

In particular, if $i_1 < \cdots < i_p$ for all $i_1 < \cdots < i_p$, we denote the corresponding $L^p$ by $L_0^p$ or $L_i^p$.

6. Given a compact set $E$ in $V$ let $F^p_E \subset H^p$ be the class of harmonic $p$-forms $\varphi$ in $V$ such that $\langle \varphi, c \rangle$ is not of constant sign in
E except for being identically zero for every \( c \in K^p \). Observe that \( F^p_E \) is closed with respect to uniform convergence in terms of \( | \cdot | \) on compact sets. In fact,

\[
| \langle \varphi_n, c \rangle - \langle \varphi_m, c \rangle | = | \langle \varphi_n - \varphi_m, c \rangle | \leq |c| |\varphi_n - \varphi_m|.
\]

We shall need the following generalization of the \( q \)-lemma for 0-forms [4]:

**Lemma.** There exists a constant \( q_E (0 < q_E < 1) \) such that

\[
\max_E |\varphi| \leq q_E \sup_{\varphi} |\varphi|
\]

for all \( \varphi \in F^p_E \).

We only have to consider forms \( \varphi \) with \( \sup_{\varphi} |\varphi| = 1 \). Suppose there existed a sequence with \( \max_E |\varphi_n| > 1 \). Then since \( \{\varphi \mid \sup_{\varphi} |\varphi| = 1\} \) is a normal family, we would have \( \varphi = \lim \varphi_n \) with \( \max_E |\varphi| = 1 \). By the subharmonicity of \( |\varphi|^2 \), \( \varphi \) would be a constant form \( c \) on \( V \). The contradiction \( \langle \varphi, c \rangle = \langle \varphi, \varphi \rangle = 1 \) completes the proof.

7. With the scene so set for \( p \geq 0 \), we can state the following generalization to \( p \)-forms of the main existence theorem known thus far for 0-forms only [4]:

**Theorem 3.** The principal form \( \rho \in H_p(V) \) characterized by

\[
L(\rho - \sigma) = \rho - \sigma
\]

exists if and only if

\[
\int_\beta *d\langle \sigma, c \rangle = 0
\]

for all \( c \in K^p \). The principal form is unique up to an additive constant form.

The proof is analogous to that for 0-forms [4] and we can restrict ourselves to a brief outline.

Let \( V_0 \subset V \) be a regular region with \( \partial V_0 \subset V_1 \) and \( \partial V_1 \subset V_0 \). Denote by \( L' \) the Dirichlet operator for \( V_0 \). We only have to establish the convergence of \( \varphi = \sum_{n=0}^\infty (LL')^n \sigma_n \), where \( \sigma_0 = \sigma - L \sigma \) and \( L = L^p \).

Observe that condition (11) means that \( \int_\alpha *d\langle \sigma, c \rangle = 0 \) for every \( \alpha \) homologous to \( \alpha_i \), since \( \langle \sigma, c \rangle \) is a harmonic function. We conclude that

\[
\int_{\partial V_1} \langle L'(LL')^n \sigma_0, c \rangle *dh = 0,
\]
where $h$ is the harmonic measure of $\partial V_0$ in $\overline{V}_0 \cap \overline{V}_1$. For this reason $L'(LL')^n\sigma_0 \in F_{\partial V_1}(V_0)$, the lemma applies in $V_0$, and we have the convergence.

Theorem 2 is a consequence of Theorem 3.

8. To prove Theorem 1 suppose $V$ is hyperbolic. The form $\sigma \in H^p(\overline{V}_1)$ may or may not satisfy (11). We set

$$\psi = \sum \left[ \int_{\partial V_1} d\sigma_{i_1, \ldots, i_p} / \int_{\partial V_1} d\omega \right] \omega dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where $\sigma = \sigma_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is the global expression in $\overline{V}_1$ and $\omega$ is the harmonic measure of the ideal boundary $\beta$ of $V$ with respect to $V_1$. Clearly $|\psi|$ is bounded in $V_1$. Consequently, $\tilde{\sigma} = \sigma + \psi$ satisfies (11) and the solution $\rho$ satisfies

$$\rho - \sigma = L^p(\rho - \tilde{\sigma}) + \psi$$
on $V_1$. We infer that $|\rho - \sigma|$ is bounded in $V_1$.

**BIBLIOGRAPHY**


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