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A ring R is called hypercyclic if every cyclic R-module has cyclic injective hull. If R is hypercyclic and R/J artinian, then R is a ring direct sum of matrix rings over local hypercyclic rings. The structure of local hypercyclic rings is studied.

In [1] and [2] Caldwell has characterized rings R (with 1) such that

(i) R is commutative, and

(ii) Every cyclic R-module has cyclic injective hull. In particular, every such ring has the property

(iii) R/J is artinian, where J is the Jacobson radical of R.

In § 1, the commutativity condition (i) is dropped, and it is shown that rings satisfying (ii) and (iii) are direct sums of matrix rings over local rings satisfying (ii). In § 2 local rings satisfying (ii) are studied. These are, with one possible exception, almost commutative in the sense that $xR = Rx$ for all $x \in R$. For such rings, Caldwell's description in the commutative case goes through. The possible exception would imply the existence of a simple radical ring (without 1 of course) which is not nil and whose right ideals and left ideals are linearly ordered. This is why the word "possible" is inserted.

In this paper, R will denote a ring with 1, and all modules will be unital right R-modules. If M_R is a module, $E(M_R)$ will denote the injective hull of M. M_R is an essential extension of N_R will be denoted $M' \supseteq N$ or $N \subseteq' M$. Following Caldwell's terminology in [1] and [2], R will be called hypercyclic if R satisfies (ii), that is, $E(R/I)$ is cyclic for every right ideal I. The socle of M will be denoted $S(M)$, and R_n will denote the ring of $n \times n$ matrices over R. R is called regular if every finitely generated right ideal is generated by an idempotent.

1. Hypercyclic rings with chain conditions on R/J.

In this section we study hypercyclic rings such that R/J is artinian. Such rings will be called restricted hypercyclic. We do not actually use the full force of R/J artinian; it is equivalent to assume that R is hypercyclic and R/I has ascending chain condition on direct summands for all right ideals I. Thus if R is hypercyclic, the ascending chain condition on $(R/J)_R$ will imply R is restricted hypercyclic also. We start by quoting several known results.

LEMMA 1.1. *Let M be a finitely generated R-module. Then* $MJ = M \Leftrightarrow M = 0$.

Proof. See Jacobson [6], Theorem 10. I suspect the name Nakayama usually associated with this lemma comes from [11], where Nakayama reformulates the statement of Jacobson's theorem by throwing out extraneous hypotheses.

PROPOSITION 1.2. Let $R = I_1 \oplus \cdots \oplus I_n$, where $\{I_j \mid 1 \leqq j \leqq n\}$ are isomorphic right ideals. Let $R' = \mathrm{Hom}_R(I_1, I_1)$. Then $R \approx (R')_n$.

Proof. See Jacobson [8], p. 52.

PROPOSITION 1.3. Let R_R be injective. Then R/J is regular, and for each set $\{\varepsilon_i \mid 1 \leqq i \leqq n\}$ of orthogonal idempotents in R/J such that $\sum_{i=1}^n \varepsilon_i = 1 + J$, there exists $\{e_i \mid 1 \leqq i \leqq n\}$ orthogonal idempotents in R with $e_i + J = \varepsilon_i$ and $\sum_{i=1}^n e_i = 1$.

Proof. See Faith and Utumi [5].

PROPOSITION 1.4. Let $e = e^2, f = f^2 \in R$. Then $eR \approx fR \Leftrightarrow eR/eJ \approx fR/fJ$.

Proof. See [8], p. 53.

PROPOSITION 1.5. Let $F = \sum_{i=1}^n x_i R$ be a free R-module with free basis $\{x_i \mid 1 \leqq i \leqq n\}$. Then $M_R \rightarrow \mathrm{Hom}_R(F, M)$ is a category isomorphism between the category of right R-modules and the category of right R_n-modules with inverse $N_{R_n} \rightarrow N \otimes_{R_n} F$.

Proof. See Morita [10], Theorem 3.4.

PROPOSITION 1.6. Let R be restricted hypercyclic. Then $R\ '\supseteqq S(R)$.

Proof. See Caldwell [1], Theorem 3.5. $R/J = \sum_{i=1}^n \oplus S_i$, S_i simple, and every simple R-module is isomorphic to some S_i. Hence $E(R/J) = \sum \oplus E(S_i)$ is faithful, and $E(S_i) = x_i R$ for some x_i. Let $D_i = \{r \in R \mid x_i r \in S_i\}$. Then $D_i \subseteqq' R$ since $E(S_i)' \supseteqq S_i$ and $\bigcap_{i=1}^n D_i \subseteqq' R$. Then $E(R/J) \cdot \bigcap_{i=1}^n D_i \subseteqq R/J$ so $E(R/J) \cdot (\bigcap_{i=1}^n D_i) J = 0$. We conclude $(\bigcap_{i=1}^n D_i) J = 0$, so $\bigcap_{i=1}^n D_i \subseteqq' S(R) \subseteqq' R$.

We now come to a basic lemma on restricted hypercyclic rings, extending a result in Faith [3], who proved it in the case that R

was perfect and hypercyclic.

LEMMA 1.7. *Let R be a hypercyclic ring such that every homomorphic image of R has ascending chain condition on direct summands. Then R_R is injective and R is restricted hypercyclic.*

Proof. Since R is hypercyclic, $E(R) \approx R/I$ for some right ideal I. Let f embed R in $R/I, f(1) = x + I$. Since f is one-to-one, $xR \cap I = 0$, so $E(R) = E(xR) \oplus E(I) \oplus M$. Since $xR \approx R$, $E(xR) \approx E(R) \oplus E(I) \oplus M$ so $E(R) \approx E(R) \oplus E(I) \oplus M \oplus E(I) \oplus M \approx E(R) \oplus E(I) \oplus M \oplus E(I) \oplus M \oplus E(I) \oplus M \approx \cdots \approx E(R) \oplus \sum_{i=1}^{n} [E(I) \oplus M]_i$. Since R/I has ascending chain condition on direct summands, $E(I) \oplus M = 0$, so $I = 0$ and R_R is injective. Since R/J is regular by 1.3, the chain condition on R/J implies R/J artinian.

LEMMA 1.8. *Let R/J be artinian, I a right ideal of $R, R/I = \sum_{i=1}^{m} \oplus M_i$. Then $m \leqq$ the composition length of R/J. Moreover, if M_i/M_iJ is simple for each i, the number of M_i/M_iJ isomorphic to a given simple module $M \leqq$ the number of factor modules $\approx M$ in a composition series for R/J.*

Proof. $\sum_{i=1}^{m} \oplus M_i/M_iJ \approx (R/I)/(R/I)J \approx R/I + J \approx (R/J)/(I + J/J)$. The lemma then follows from the Jordan-Holder-Schreier theorem. See Jacobson [7], p. 141.

COROLLARY 1.9. *Let R be restricted hypercyclic, $e = e^2 \in R$, length $eR/eJ = m$. Then any independent set of submodules of a quotient of eR has at most m elements.*

Proof. Let $\{M_i \mid 1 \leqq i \leqq k\}$ be an independent family of submodules of eR/eI. Then $R/eI \supseteqq (1 - e)R \oplus \sum_{i=1}^{k} \oplus M_i$, so $E(R/eI)$ contains a direct sum of (length $R/J - m + k$) terms. Hence $k \leqq m$ by 1.8.

COROLLARY 1.10. *Let R be restricted hypercyclic, $e = e^2 \in R$, eR/eJ simple. Then the submodules of eR are linearly ordered.*

Proof. Let $A, B \subseteqq eR$. By 1.9, $A/A \cap B = 0$ or $B/A \cap B = 0$, so $B \subseteqq A \cap B$ or $B \subseteqq A \cap B$.

COROLLARY 1.11. *Let R be restricted hypercyclic. Then R_R is injective.*

Proof. Apply 1.8 and 1.7.

LEMMA 1.12. *Let R be restricted hypercyclic, M a simple R-module. Then $E(M) \approx eR$ for some $e = e^2 \in R$.*

Proof. By 1.11, R_R is injective. Let $R/J = \sum_{i=1}^m \varepsilon_i R$, $\{\varepsilon_i \mid 1 \leqq i \leqq m\}$ orthogonal primitive idempotents. By 1.3, there exist $\{e_i \mid 1 \leqq i \leqq m\}$ orthogonal idempotents of R such that $e_i + J = \varepsilon_i$ and $1 = \sum_{i=1}^m e_i$. By 1.6, $e_i R \cap S(R) \neq 0$. Since $e_i R/e_i J$ is indecomposable, so is $e_i R = E(e_i R \cap S(R))$. Hence $e_i R \cap S(R) = S_i$ is simple. By 1.4, $e_i R/e_i J \approx e_j R/e_j J \Leftrightarrow S_i \approx S_j$. Hence the correspondence $S_i \leftrightarrow E(S_i)/E(S_i)J$ is one-to-one between isomorphism classes of simple modules containing a representative $S_i \subseteqq S(R)$ and isomorphism classes of modules of the form $e_i R/e_i J$. Since every simple R-module is isomorphic to some $e_i R/e_i J$ and the set of isomorphism classes of simples is finite, every isomorphism class of simples contains some S_i. If $M \approx S_{i_0}$, $E(M) \approx e_{i_0} R$.

LEMMA 1.13. *Let R be hypercyclic, $R/J \approx \Delta_n$ for some division ring Δ. Then $R \approx (R')_n$, where $R'/R'J(R') \approx \Delta$, so R' is local.*

Proof. Let $R/J = \sum_{i=1}^m \varepsilon_i R/J$, $\{\varepsilon_i \mid 1 \leqq i \leqq m\}$ primitive orthogonal idempotents. Then $R = \sum_{i=1}^m e_i R$, $\{e_i \mid 1 \leqq i \leqq m\}$ orthogonal idempotents with $e_i + J = \varepsilon_i$ by 1.11 and 1.3. By 1.4, $e_i R \approx e_j R$ for all i and j since all simple Δ_n modules are isomorphic. By 1.2, $R \approx (\mathrm{Hom}_R(e_i R, e_i R))_n$. But $R' = \mathrm{Hom}_R(e_i R, e_i R) \approx e_i R e_i$, and $R'/R'J(R') \approx e_i R e_i/e_i J e_i \approx \mathrm{Hom}_{R/J}(\varepsilon_i \Delta_n, \varepsilon_i \Delta_n) \approx \Delta$.

LEMMA 1.14. *Let $R = \sum_{i=1}^n \oplus R_i$ be a ring direct sum. Then R is hypercyclic $\Leftrightarrow$ each R_i is.*

Proof. If e_i is the identity of R_i, then $\{e_i \mid 1 \leqq i \leqq n\}$ are orthogonal central idempotents of R. Let I be an ideal of R. Then $I = \sum \oplus e_i I$, and one easily verifies that $E(R/I) \approx R/K \Leftrightarrow E(e_i R/e_i I) \approx e_i R/e_i K$.

LEMMA 1.15. *Let R be restricted hypercyclic. Then R is a ring direct sum of matrix rings over local rings.*

Proof. Let $R/J = \sum_{i=1}^m \oplus (\Delta_i)_{n_i}$, Δ_i a division ring, $n_1 \geqq n_2 \geqq \cdots \geqq n_m$, $(\Delta_i)_{n_i} = \varepsilon_i R/J$ where ε_i is a central idempotent of R/J. Let $\varepsilon_i = e_i + J$, $\{e_i \mid 1 \leqq i \leqq m\}$ orthogonal idempotents of R such that $\sum_{i=1}^m e_i = 1$. By Lemmas 1.13 and 1.14, we need only show each e_i is central. Assume not. Then there exist $i \neq j$ such that $e_i R e_j \neq 0$. Let $e_i r e_j \neq 0$, $i \neq j$. Consider

$$M = R/(\sum_{l\neq i,j} e_l R + e_j J + e_i r e_j J) \approx e_j R/e_j J \oplus e_i R/e_i r e_j J\ .$$

This contains a direct sum of at least $n_j + 1$ copies of the unique simple $(\Delta_j)_{n_j}$ module S. Then $E(M) \supseteqq \sum_{i=1}^{n_j+1} E(S) \approx R/I$ for some I. Since $E(S)/E(S)J$ is simple by the proof of 1.12, lemma 1.8 shows $E(S)/E(S)J$ is the simple $(\Delta_l)_{n_l}$ module for some l with $n_l > n_j$. Now $E(R/J) \approx R/K$ for some K, and by 1.12, since R/J is a finite direct sum of simple modules, $E(R/J)$ is projective. Hence $R \approx E(R/J) \oplus K$, where the length of $E(R/J)/E(R/J)J = \text{length}\, R/J$. By 1.8, $K = 0$, so $S(R) \approx R/J$. But then the number of composition factors of $S(R) \approx S =$ the number of times $E(S)/E(S)J$ appears as a composition factor of $R/J =$ the number of composition factors of $R/J \approx S$. Thus $n_l = n_j$, a contradiction.

Lemmas 1.14 and 1.15 reduce the study of restricted hypercyclic rings to hypercyclic matrix rings over local rings R'. We will show that, for R' local, $(R')_n$ is hypercyclic $\Leftrightarrow R'$ is hypercyclic.

LEMMA 1.16. *Let R be hypercyclic, R/J a simple R-module. Let $F = \sum_1^n \oplus R_i$, $K \subseteqq F$. Let $\{N_i \mid 1 \leqq i \leqq k\}$ be a family of nonzero independent submodules of F/K. Then $k \leqq n$.*

Proof. Let $x_1, \cdots, x_{n+1}$ be any $n+1$ elements of F, $x_i = (x_{i1}, \cdots, x_{in})$. We will show that $\{x_i \mid 1 \leqq i \leqq n+1\}$ is not independent modulo K. If some $x_i \in K$, this is immediate, so we may assume $x_i \notin K$ for all i. By 1.10, the right ideals of R are linearly ordered. Hence any finite subset of R, $\{r_i\}$, has a maximum element, that is an r_{i_0} such that $r_{i_0}R \supseteqq r_i R$ for all i. Clearly we may permute the x_i and the order of the summands R_i in F without losing generality. Hence we may assume $x_{11} = \max\{x_{ij} \mid 1 \leqq i \leqq n+1, 1 \leqq j \leqq n\}$. Let $x_{i1} = x_{11} r_{1i}$, $2 \leqq i \leqq n+1$, and consider the elements $x_i - x_1 r_{1i}$. These all have zeros in the first component. Then some $x_{ij} - x_{1j} r_{1i} = \max\{x_{ij} - x_{1j} r_{1i} \mid 2 \leqq i \leqq n+1, 2 \leqq j \leqq n\}$, say $i = j = 2$. Then there exist elements r_{2i}, $3 \leqq i \leqq n+1$ such that $\{(x_i - x_1 r_{1i}) - (x_2 - x_1 r_{12}) r_{2i} \mid 3 \leqq i \leqq n+1\}$ all have zeros in the first two components. Continuing in this manner, and permuting so the largest coefficient is in the k, k position, we get $n+1-k$ elements of the form $x_m - \sum_{i=1}^k x_i s_{ik}$ which have zeros in the first k positions. When $k = n$, we get $x_{n+1} - \sum_{i=1}^n x_i s_{in} = 0$, and since $x_{n+1} \notin K$, this gives a nontrivial dependence of $\{x_i \mid 1 \leqq i \leqq n+1\}$ modulo K.

LEMMA 1.17. *Let R' be a local ring. Then R' is hypercyclic $\Leftrightarrow (R')_n$ is hypercyclic for some n.*

Proof. $\Rightarrow$ Let R' be hypercyclic, $e = e^2 \in (R')_n = R$, e a primitive idempotent. Identify R' with eRe. Then $Re_{R'} \approx \sum_{i=1}^{n} \oplus R'_i$. Since the category isomorphism of 1.5 takes $R' \rightarrow \mathrm{Hom}_{R'}(Re, R') \approx eR$, every quotient of eR has injective hull a quotient of eR. Let I be a right ideal of R. Then

$$R \rightarrow R/I \rightarrow 0$$

is exact, so

$$R \otimes_R Re \rightarrow R/I \otimes_R Re \rightarrow 0$$

is exact. Since $R \otimes_R Re_{R'} \approx Re_{R'}$, $R/I \otimes_R Re$ is an R'-quotient of Re. By 1.16, $R/I \otimes_R Re$ is an essential extension of at most n cyclic R'-modules. By the category isomorphism, R/I is an essential extension of a sum of at most n quotients of eR. Hence its injective hull is a direct sum of at most n quotients of eR, and thus a quotient of R.

$\Leftarrow$. Let $R = (R')_n$ be hypercyclic, e a primitive idempotent, $A \subsetneqq e(R')_n$. Then $E(R/A) = (1 - e)R \oplus E(eR/A)$, so $E(eR/A)/E(eR/A)J$ is simple, and $E(eR/A)$ is a quotient of eR. By the category isomorphism 1.5, every quotient of R' has injective hull also a quotient of R'.

Putting 1.17, 1.15 and 1.14 together we get

THEOREM 1.18. *R is a restricted hypercyclic ring $\Leftrightarrow R$ is a ring direct sum of matrix rings over local hypercyclic rings.*

2. **Local hypercyclic rings.** By Theorem 1.18, hypercyclic rings R such that R/J is semi-simple Artin are ring direct sums of matrix rings over local hypercyclic rings. In this section we study local hypercyclic rings. These turn out to be, with one possible exception, the rings studied by Caldwell in [2], with R commutative replaced by $xR = Rx$ for all $x \in R$.

By 1.11, a local hypercyclic ring is right self injective; by 1.10 its right ideals are linearly ordered.

We have the well-known

PROPOSITION 2.1. Let M_R contain a copy of the injective hull of every simple R-module. Then every right ideal of R is the annihilator of some subset of M.

PROPOSITION 2.2. Let R_R be injective. Then any finitely generated left ideal is the annihilator of some right ideal, and the right socle of $R \subsetneqq$ the left socle of R.

Let $X \subsetneqq R$. Define $X^r = \{r \in R \mid Xr = 0\}$, $X^l = \{r \in R \mid rX = 0\}$.

COROLLARY 2.3. *Let R be a local hypercyclic ring. Then the left ideals of R are linearly ordered.*

Proof. Let $x_1, x_2 \in R$. Then $x_1^r \subseteqq x_2^r$ or $x_2^r \subseteqq x_1^r$, so by 2.2, $x_1^{rl} = Rx_1 \subseteqq x_2^{rl} = Rx_2$ or $Rx_1 \supseteqq Rx_2$. Now let I_1, I_2 be two left ideals, $I_1 \not\subseteqq I_2$. Let $x \in I_1 - I_2$. For $y \in I_2$, $Ry \not\supseteqq Rx$ since $x \notin I_2$. Hence $Ry \subseteqq Rx$ so $I_2 \subseteqq I_1$.

LEMMA 2.4. *Let $x, z \in R$, $0 \neq xz \in S(R)$. Then $(Rx)^r = zJ$, $(Jx)^r = zR$, $(zJ)^l = Rx$, $(zR)^l = Jx$.*

Proof. Since $0 \neq xz \in S$, $xzJ = 0$, $xzR \neq 0$. By the linear ordering on right ideals, $zJ \subseteqq (Rx)^r \subsetneqq zR$. Since zR/zJ is simple, $zJ = (Rx)^r$. Since left ideals are linearly ordered by 2.3, $Jx = (zR)^l$ by symmetry. By 2.2, principal left ideals are annihilators so $Rx = (zJ)^l$. By 2.1, zR is an annihilator so $(Jx)^r = zR$.

We note that given $x \neq 0$ or $z \neq 0$, by the linear ordering on right and left ideals, we can always find the other such that $0 \neq xz \in S(R)$.

COROLLARY 2.5. *Let R be a local hypercyclic ring. Then every right ideal and every left ideal is an annihilator ideal.*

Proof. Since R_R is injective and contains a copy of the unique simple R-module, 2.1 states every right ideal is a right annihilator.

Now let I' be a left ideal, and let $Z = \bigcap_{Rx \supseteqq I'} Rx$. Then $Z = (\sum_{Rx \supseteqq I'} x^r)^l$ since $(x^r)^l = Rx$. If $I' \neq Z$, let $y \in Z - I'$. Then $Ry \supseteqq I'$, and $Rz \supseteqq I' \Rightarrow Rz \supseteqq Ry$. Hence Ry/I' is a simple left module. By Nakayama's lemma, $Jy \neq Ry$. Hence $Jy \subseteqq I'$. Since Ry/Jy is also simple, $Jy = I'$. By 2.4, I' is an annihilator left ideal.

LEMMA 2.6. *Let I be a right ideal of the local hypercyclic ring R, $R \supsetneqq I \supsetneqq S(R)$. Then R/I is not injective.*

Proof. Since $I \neq R$, $I \subseteqq J$. Let $x \in I$, $x \notin S(R)$. Since the left ideals of R are linearly ordered and $S(R)$ is a two sided ideal, there exists $y \in R$ such that $0 \neq yx \in S(R)$. Then $xJ = (Ry)^r$ by 2.4. Since $x \notin S(R)$, y is not a unit, so $y \in J$. We now proceed as in Caldwell [2]; the map $f\colon yR \rightarrow R/I$ given by $f(yr) = r + I$ is well defined since $y^r = xJ \subseteqq I$. R/I injective implies there exists $m \in R$ such that $my + I = 1 + I$. Hence $1 - my \in I$, and since $y \in J$, $(1 - my)(1 - my)^{-1} = 1 \in I$, a contradiction.

LEMMA 2.7. *Let R be local hypercyclic. Then every right (left)* *ideal is of the form xR or xJ (Rx or Jx).*

Proof. Let I be a right ideal. Since the injective hull of R/I is cyclic, by 2.6 R/I embeds either in R or in $R/S(R)$. Assume f embeds R/I in R. Let $0 \neq f(x + I) \in S(R)$. Then $xJ \subseteqq I \subsetneqq xR$, so $I = xJ$. If g embeds R/I in $R/S(R)$, let $g(1) = m + S(R)$. Let $0 \neq my \in S(R)$. Then $g(y) = 0$ so $y \in I$. Let $x \in I$. Then $mx \in S(R)$. If $mx = 0, x \in yJ = m^r$. If $mx \neq 0, m^r = xJ = yJ$, so by the linear ordering $xR = yR$. Hence $x \in y^R$, and $I = yR$.

Since every left ideal I is an annihilator by 2.5, it is of the form $(xR)^l$ or $(xJ)^l$ for some x. If $x \neq 0$, select y such that $0 \neq yx \in S(R)$. Then $I = Jy$ or Ry by 2.4.

PROPOSITION 2.8. Let R be local hypercyclic, $y \in J$. Either y is nilpotent or $0 \neq \bigcap_{i=0}^{\infty} y^n R = z'R$, where $yz'R = z'R$.

Proof. (See Caldwell [2] Theorem 2.20) Since $R' \supseteqq S(R)$, $y^n R \supseteqq S(R)$ or $y^n = 0$. Thus $I = \bigcap_{n=0}^{\infty} y^n R = 0 \Leftrightarrow y^n = 0$ for some n.

Let $y^n \in y^{n+1}R$. Then for some $r \in R, y^n = y^{n+1}r$, so $y^n(1 - yr) = 0$, and since $yr \in J, 1 - yr$ is invertible. Hence $y^n = 0$. Thus y not nilpotent implies $yR \supset y^2R \supset \cdots$ is a strictly descending chain of right ideals. Let $(y^iR)^l = Jz_i$. Then $I = (\bigcup_{i=0}^{\infty} Jz_i)^r$, where $\bigcup_{i=0}^{\infty} Jz_i = K$ is the union of a strictly ascending chain of left ideals if y is not nilpotent. Then K cannot be finitely generated so $K = Jz$ for some $z \in R$ by 2.7, and $I = K^r = z'R$ for some z'. Since $y(y^nR) \subseteqq y^nR$ for all $n, yI \subseteqq I$. Now $z' \in I$, so $z' = yr$ for some $r \in R$. Assume $r \notin I$. Then there is an n such that $r \notin y^nR$, so $rR \supset y^nR$. Let $rs = y^n$. Then $z's = yrs = y^{n+1} \in I$. If y is not nilpotent, this cannot occur, so $r \in I$ and $z' \in yI$. Hence $yI = I$.

COROLLARY 2.9. *Let $yR = Ry$ for $y \in J$. Then y is nilpotent.*

Proof. Assume not. Then $0 \neq I = \bigcap_{i=0}^{\infty} y^nR = \bigcap_{i=0}^{\infty} Ry^n$. As in the proof of 2.8, $(\bigcap_{i=0}^{\infty} Ry^n)y = \bigcap_{i=0}^{\infty} Ry^n$. By 2.8, for some $z', I = z'R$. Then $I = z'Ry$, so there exists $r \in R$ with $z'ry = z', z'(ry - 1) = 0$. Since $y \in J, ry - 1$ is invertible, so $z' = 0$, a contradiction.

THEOREM 2.10. *Let R be local hypercyclic. Then J is nil $\Leftrightarrow yR = Ry$ for all $y \in R$.*

Proof. $\Leftarrow$. This is just 2.9. $\Rightarrow$. Assume J is nil, and let $0 \neq y, r \in R, yr \notin Ry$. Then $Ryr \supsetneqq Ry$, so $y = xyr$ for some $x \in J$. Then $y = xyr = x^2yr^2 = \cdots = x^nyr^n = 0$, a contradiction. Hence $yR \subseteqq Ry$. By symmetry, $Ry \subseteqq yR$.

Thus if J is nil, every one sided ideal of R is two sided, and all of Caldwell's arguments in [2] may be carried over to this case almost verbatim. The reader is referred to Caldwell for further discussion of this case.

Whether a local hypercyclic ring can have nonnil radical is unknown. We show that it implies the existence of a very elusive type of ring, namely a simple radical ring (without 1 of course), which has linear ordering on both right and left ideals.

LEMMA 2.11. *Let R be a hypercyclic local ring, J not nil. Then $J^2 = J$ and for all $y \in J$, $JyJ \neq J \Rightarrow \exists z \in J$, $zR = Rz$ and $y \in zR$.*

Proof. If $J^2 \neq J$, by the linear ordering on one sided ideals, for any $x \in J - J^2$, $J = xR = Rx$. Hence x is nilpotent by 2.9, so $J^n = (xR)^n = x^n R = 0$ for some n, a contradiction. Hence $J = J^2$. Let $y \in J$. Then $(JyJ)J = JyJ = J(JyJ)$, so JyJ is not a finitely generated right or left ideal. Hence $JyJ = zJ = Jz'$ for some $z, z' \in R$ by 2.7. Let ν denote the natural map from R onto R/JyJ. Then $\nu(z)R = S(\nu(R)_R)$, $R\nu(z') = S({}_R\nu(R))$ since the right and left ideals of $\nu(R)$ are linearly ordered and $\nu(z)R$ and $R\nu(z')$ are simple. Moreover, $\nu(z')R \supseteq \nu(z)R$. Since $S({}_R\nu(R))$ is a two sided ideal of $\nu(R)$, $S({}_R\nu(R)) = R\nu(z') \subseteq \nu(z)R = S(\nu(R)_R)$. By symmetry, $S(\nu(R)_R) \subseteq S({}_R\nu(R))$, so $R\nu(z') = \nu(z)R = \nu(z')R = R\nu(z)$ since it is a simple R-module on both right and left. Taking ν^{-1} of both sides we get $z'R = Rz' = zR = Rz$. If $JyJ \neq J$, $S(\nu(R)_R) \neq R/J$, so $z \in J$.

Moreover $zR \supsetneqq JyJ \Rightarrow zR \supsetneqq jyJ$ for all $j \in J \Rightarrow zR \supseteq jyR$ for all $j \in J$ by the linear ordering $\Rightarrow zR \supseteq JyR$. Similarly $zR = Rz \supseteq RyR$. Hence $y \in zR$.

THEOREM 2.12. *Let R be a local hypercyclic ring, J not nil. Then there exists a nilpotent ideal $zR \subseteq J$ such that zR is a maximal proper two sided ideal of J (so J/zR is a simple radical ring.)*

Proof. Let I be the union of all the nil two-sided ideals of R. Then I is a nil two-sided ideal since ideals are linearly ordered. Moreover, $I \neq 0$ since S is a nilpotent two-sided ideal, and $I \neq J$ since J is not nil.

Let $K = JIJ$. Then $JK = KJ = K$, so as in 2.11, $K = zJ = Jz$ where $zR = Rz = \nu^{-1}(S(R/K))$ and $x \in Rz$ for all $x \in I$. Since

$$[\nu^{-1}(S(R/K))]^2 \subseteq K$$

and $K \subseteq I$, zR is nil. Hence $zR = Rz = I$. If $z^n = 0$, then $I^n = (zR)^n = z^n R = 0$, so I is nilpotent.

Now let $y \in J - I$. If $JyJ \neq J$, by 2.11, y belongs to some nilpotent ideal of R and hence to I. Thus $JyJ = J$, and J/zR is a simple ring. By [8], p. 10, J is a radical ring, and hence so is J/zR.

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