NONCOMMUTATIVE RINGS WHOSE CYCLIC MODULES HAVE CYCLIC INJECTIVE HULLS

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A ring $R$ is called hypercyclic if every cyclic $R$-module has cyclic injective hull. If $R$ is hypercyclic and $R/J$ artinian, then $R$ is a ring direct sum of matrix rings over local hypercyclic rings. The structure of local hypercyclic rings is studied.

In [1] and [2] Caldwell has characterized rings $R$ (with 1) such that

(i) $R$ is commutative, and

(ii) Every cyclic $R$-module has cyclic injective hull. In particular, every such ring has the property

(iii) $R/J$ is artinian, where $J$ is the Jacobson radical of $R$.

In § 1, the commutativity condition (i) is dropped, and it is shown that rings satisfying (ii) and (iii) are direct sums of matrix rings over local rings satisfying (ii). In § 2 local rings satisfying (ii) are studied. These are, with one possible exception, almost commutative in the sense that $xR = Rx$ for all $x \in R$. For such rings, Caldwell’s description in the commutative case goes through. The possible exception would imply the existence of a simple radical ring (without 1 of course) which is not nil and whose right ideals and left ideals are linearly ordered. This is why the word “possible” is inserted.

In this paper, $R$ will denote a ring with 1, and all modules will be unital right $R$-modules. If $M_R$ is a module, $E(M_R)$ will denote the injective hull of $M$. $M_R$ is an essential extension of $N_R$ will be denoted $M' \supseteq N$ or $N \subseteq M$. Following Caldwell’s terminology in [1] and [2], $R$ will be called hypercyclic if $R$ satisfies (ii), that is, $E(R/I)$ is cyclic for every right ideal $I$. The socle of $M$ will be denoted $S(M)$, and $R_n$ will denote the ring of $n \times n$ matrices over $R$. $R$ is called regular if every finitely generated right ideal is generated by an idempotent.

1. Hypercyclic rings with chain conditions on $R/J$. In this section we study hypercyclic rings such that $R/J$ is artinian. Such rings will be called restricted hypercyclic. We do not actually use the full force of $R/J$ artinian; it is equivalent to assume that $R$ is hypercyclic and $R/I$ has ascending chain condition on direct summands for all right ideals $I$. Thus if $R$ is hypercyclic, the ascending chain condition on $(R/J)_R$ will imply $R$ is restricted hypercyclic also. We start by quoting several known results.
**Lemma 1.1.** Let $M$ be a finitely generated $R$-module. Then $MJ = M^*M = 0$.

*Proof.* See Jacobson [6], Theorem 10. I suspect the name Nakayama usually associated with this lemma comes from [11], where Nakayama reformulates the statement of Jacobson’s theorem by throwing out extraneous hypotheses.

**Proposition 1.2.** Let $R = I_1 \oplus \cdots \oplus I_n$, where $\{I_j \mid 1 \leq j \leq n\}$ are isomorphic right ideals. Let $R' = \text{Hom}_R(I_1, I_1)$. Then $R \approx (R')^n$.

*Proof.* See Jacobson [8], p. 52.

**Proposition 1.3.** Let $R_R$ be injective. Then $R/J$ is regular, and for each set $\{e_i \mid 1 \leq i \leq n\}$ of orthogonal idempotents in $R/J$ such that $\sum_{i=1}^n e_i = 1 + J$, there exists $\{e_i \mid 1 \leq i \leq n\}$ orthogonal idempotents in $R$ with $e_i + J = e_i$ and $\sum_{i=1}^n e_i = 1$.

*Proof.* See Faith and Utumi [5].

**Proposition 1.4.** Let $e = e^2, f = f^2 \in R$. Then $eR \approx fR \approx eR/eJ \approx fR/fJ$.

*Proof.* See [8], p. 53.

**Proposition 1.5.** Let $F = \sum_{i=1}^n x_iR$ be a free $R$-module with free basis $\{x_i \mid 1 \leq i \leq n\}$. Then $M_R \to \text{Hom}_R(F, M)$ is a category isomorphism between the category of right $R$-modules and the category of right $R_n$-modules with inverse $N_{R_n} \to N \otimes_{R_n} F$.

*Proof.* See Morita [10], Theorem 3.4.

**Proposition 1.6.** Let $R$ be restricted hypercyclic. Then $R' \supseteq S(R)$.

*Proof.* See Caldwell [1], Theorem 3.5. $R/J = \sum_{i=1}^n \oplus S_i$, $S_i$ simple, and every simple $R$-module is isomorphic to some $S_i$. Hence $E(R/J) = \sum \oplus E(S_i)$ is faithful, and $E(S_i) = x_iR$ for some $x_i$. Let $D_i = \{r \in R \mid x_i r \in S_i\}$. Then $D_i \subseteq R$ since $E(S_i)' \supseteq S_i$ and $\bigcap_{i=1}^n D_i \subseteq R$. Then $E(R/J) \cdot \bigcap_{i=1}^n D_i \subseteq R/J$ so $E(R/J) \cdot (\bigcap_{i=1}^n D_i) J = 0$. We conclude $(\bigcap_{i=1}^n D_i) J = 0$, so $\bigcap_{i=1}^n D_i \subseteq S(R) \subseteq R$.

We now come to a basic lemma on restricted hypercyclic rings, extending a result in Faith [3], who proved it in the case that $R$
was perfect and hypercyclic.

**Lemma 1.7.** Let \( R \) be a hypercyclic ring such that every homomorphic image of \( R \) has ascending chain condition on direct summands. Then \( R \) is injective and \( R \) is restricted hypercyclic.

**Proof.** Since \( R \) is hypercyclic, \( E(R) \cong R/I \) for some right ideal \( I \). Let \( f \) embed \( R \) in \( R/I, f(1) = x + I \). Since \( f \) is one-to-one, \( xR \cap I = 0 \), so \( E(R) = E(xR) \oplus E(I) \oplus M \). Since \( xR \cong R, E(xR) \cong E(R) \oplus E(I) \oplus M \) so \( E(R) \cong E(R) \oplus E(I) \oplus M \oplus E(I) \oplus M \cong \cdots \cong E(R) \oplus \sum_{i=1}^{n} [E(I) \oplus M]_i \). Since \( R/I \) has ascending chain condition on direct summands, \( E(I) \oplus M = 0 \), so \( I = 0 \) and \( R \) is injective. Since \( R/J \) is regular by 1.3, the chain condition on \( R/J \) implies \( R/J \) artinian.

**Lemma 1.8.** Let \( R/J \) be artinian, \( I \) a right ideal of \( R, R/I = \sum_{i=1}^{n} M_i \). Then \( m \leq \) the composition length of \( R/J \). Moreover, if \( M_i/MJ \) is simple for each \( i \), the number of \( M_i/MJ \) isomorphic to a given simple module \( M \leq \) the number of factor modules \( \cong M \) in a composition series for \( R/J \).

**Proof.** \( \sum_{i=1}^{n} M_i/MJ \cong (R/I)/(R/I)J \cong R/I + J \cong (R/J)/(I + J)/J \). The lemma then follows from the Jordan-Holder-Schreier theorem. See Jacobson [7], p. 141.

**Corollary 1.9.** Let \( R \) be restricted hypercyclic, \( e = e^2 \in R, length eR/eJ = m \). Then any independent set of submodules of a quotient of \( eR \) has at most \( m \) elements.

**Proof.** Let \( \{M_i \mid 1 \leq i \leq k\} \) be an independent family of submodules of \( eR/eI \). Then \( R/eI \cong (1-e)R \oplus \sum_{i=1}^{k} M_i \), so \( E(R/eI) \) contains a direct sum of \( (length R/J - m + k) \) terms. Hence \( k \leq m \) by 1.8.

**Corollary 1.10.** Let \( R \) be restricted hypercyclic, \( e = e^2 \in R, eR/eJ \) simple. Then the submodules of \( eR \) are linearly ordered.

**Proof.** Let \( A, B \subseteq eR \). By 1.9, \( A/A \cap B = 0 \) or \( B/A \cap B = 0 \), so \( B \subseteq A \cap B \) or \( B \subseteq A \cap B \).

**Corollary 1.11.** Let \( R \) be restricted hypercyclic. Then \( R \) is injective.

**Proof.** Apply 1.8 and 1.7.
LEMMA 1.12. Let $R$ be restricted hypercyclic, $M$ a simple $R$-module. Then $E(M) \cong eR$ for some $e = e^2 e \ldots$ need only show each $e_i$ is central. Assume not. Then there exist $i \neq j$ such that $e_i e_j \neq 0$. Let $e_i r e_j \neq 0$, $i \neq j$. Consider

Proof. By 1.11, $R_R$ is injective. Let $R/J = \sum_{i=1}^n \varepsilon_i e_i R/J$, $\{\varepsilon_i | 1 \leq i \leq m\}$ orthogonal primitive idempotents. By 1.3, there exist $\{\varepsilon_i | 1 \leq i \leq m\}$ orthogonal idempotents of $R$ such that $e_i + J = \varepsilon_i$ and $1 = \sum_{i=1}^m e_i$. By 1.6, $e_i R \cap S(R) \neq 0$. Since $e_i R/e_i J$ is indecomposable, so is $e_i R = E(e_i R \cap S(R))$. Hence $e_i R \cap S(R) = S_i$ is simple. By 1.4, $e_i R/e_i J \cong e_j R/e_j J \Rightarrow S_i \cong S_j$. Hence the correspondence $S_i \mapsto E(S_i)/E(S_i) J$ is one-to-one between isomorphism classes of simple modules containing a representative $S_i \subseteq S(R)$ and isomorphism classes of modules of the form $e_i R/e_i J$. Since every simple $R$-module is isomorphic to some $e_i R/e_i J$ and the set of isomorphism classes of simples is finite, every isomorphism class of simples contains some $S_i$. If $M \cong S_i$, $E(M) \cong e_i R$.

LEMMA 1.13. Let $R$ be hypercyclic, $R/J \cong \Lambda_n$ for some division ring $\Lambda$. Then $R \cong (R')_n$, where $R'/R' J(R') \cong \Lambda$, so $R'$ is local.

Proof. Let $R/J = \sum_{i=1}^n \varepsilon_i e_i R/J$, $\{\varepsilon_i | 1 \leq i \leq m\}$ primitive orthogonal idempotents. Then $R = \sum_{i=1}^n \varepsilon_i e_i R$, $\{\varepsilon_i | 1 \leq i \leq m\}$ orthogonal idempotents with $e_i + J = \varepsilon_i$ by 1.11 and 1.3. By 1.4, $e_i R \approx e_i R$ for all $i$ and $j$ since all simple $\Lambda_n$ modules are isomorphic. By 1.2, $R \approx (\text{Hom}_R(e_i R, e_i R))_n$. But $R' = \text{Hom}_R(e_i R, e_i R) \approx e_i R e_i$, and $R'/R' J(R') \approx e_i R e_i / e_i J e_i \approx \text{Hom}_R(e_i \Lambda_n, e_i \Lambda_n) \approx \Lambda$.

LEMMA 1.14. Let $R = \sum_{i=1}^n \bigoplus R_i$ be a ring direct sum. Then $R$ is hypercyclic $\iff$ each $R_i$ is.

Proof. If $e_i$ is the identity of $R_i$, then $\{e_i | 1 \leq i \leq n\}$ are orthogonal central idempotents of $R$. Let $I$ be an ideal of $R$. Then $I = \sum \bigoplus e_i I$, and one easily verifies that $E(R/I) \cong R/K \Rightarrow E(e_i R/e_i I) \cong e_i R/e_i K$.

LEMMA 1.15. Let $R$ be restricted hypercyclic. Then $R$ is a ring direct sum of matrix rings over local rings.

Proof. Let $R/J = \sum_{i=1}^m \bigoplus (\Lambda_i)_{n_i}$, $\Lambda_i$ a division ring, $n_1 \geq n_2 \geq \cdots \geq n_m$, $(\Lambda_i)_{n_i} = \varepsilon_i R/J$ where $\varepsilon_i$ is a central idempotent of $R/J$. Let $\varepsilon_i = e_i + J$, $\{e_i | 1 \leq i \leq m\}$ orthogonal idempotents of $R$ such that $\sum_{i=1}^m e_i = 1$. By Lemmas 1.13 and 1.14, we need only show each $e_i$ is central. Assume not. Then there exist $i \neq j$ such that $e_i R e_j \neq 0$. Let $e_i r e_j \neq 0$, $i \neq j$. Consider
$$M = R/(\sum_{i=1}^{n_i} e_i R + e_j J + e_i r e_j J) \approx e_j R/e_j J \bigoplus e_i R/e_i r e_j J.$$  

This contains a direct sum of at least $n_j + 1$ copies of the unique simple $(A_i)_{n_j}$ module $S$. Then $E(M) \cong \sum_{i=1}^{n_{i+1}} E(S) \cong R/I$ for some $I$. Since $E(S)/E(S)J$ is simple by the proof of 1.12, lemma 1.8 shows $E(S)/E(S)J$ is the simple $(A_i)_{n_i}$ module for some $l$ with $n_i > n_j$. Now $E(R/J) \cong R/K$ for some $K$, and by 1.12, since $R/J$ is a finite direct sum of simple modules, $E(R/J)$ is projective. Hence $R \cong E(R/J) \bigoplus K$, where the length of $E(R/J)/E(R/J)J = \text{length } R/J$. By 1.8, $K = 0$, so $S(R) \cong R/J$. But then the number of composition factors of $S(R) \cong S$. Thus $n_i = n_j$, a contradiction.

Lemmas 1.14 and 1.15 reduce the study of restricted hypercyclic rings to hypercyclic matrix rings over local rings $R'$. We will show that, for $R'$ local, $(R')_n$ is hypercyclic $\iff R'$ is hypercyclic.

**Lemma 1.16.** Let $R$ be hypercyclic, $R/J$ a simple $R$-module. Let $F = \sum R_i, K \subseteq F$. Let $\{N_i \mid 1 \leq i \leq k\}$ be a family of nonzero independent submodules of $F/K$. Then $k \leq n$.

**Proof.** Let $x_1, \ldots, x_{n+1}$ be any $n + 1$ elements of $F$, $x_i = (x_{1i}, \ldots, x_{ni})$. We will show that $\{x_i \mid 1 \leq i \leq n + 1\}$ is not independent modulo $K$. If some $x_i \in K$, this is immediate, so we may assume $x_i \notin K$ for all $i$. By 1.10, the right ideals of $R$, $\{r_i\}$, has a maximum element, that is an $r_{i_0}$ such that $r_{i_0} R \supseteq r_i R$ for all $i$. Clearly we may permute the $x_i$ and the order of the summands $R_i$ in $F$ without losing generality. Hence we may assume $x_{i_0} = \max \{x_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n\}$. Let $x_{i_0} = x_{i_0} r_{i_0}, 2 \leq i \leq n + 1$, and consider the elements $x_{i} - x_{i_0} r_{i_0}$. These all have zeros in the first component. Then some $x_{ij} - x_{i_0} r_{i_0} = \max \{x_{ij} - x_{i_0} r_{i_0} \mid 2 \leq i \leq n + 1, 2 \leq j \leq n\}$, say $i = j = 2$. Then there exist elements $r_{i_0}, 3 \leq i \leq n + 1$ such that $\{(x_{i} - x_{i_0} r_{i_0}) - (x_{2} - x_{2} r_{2}) r_{i_0} \mid 2 \leq i \leq n + 1\}$ all have zeros in the first two components. Continuing in this manner, and permuting so the largest coefficient is in the $k$, $k$ position, we get $n + 1 - k$ elements of the form $x_m - \sum_{i=1}^{k} x_{i} s_{ik}$ which have zeros in the first $k$ positions. When $k = n$, we get $x_{n+1} - \sum_{i=1}^{n} x_{i} s_{in} = 0$, and since $x_{n+1} \notin K$, this gives a nontrivial dependence of $\{x_i \mid 1 \leq i \leq n + 1\}$ modulo $K$.

**Lemma 1.17.** Let $R'$ be a local ring. Then $R'$ is hypercyclic if and only if $(R')_n$ is hypercyclic for some $n$.
Proof. $\Rightarrow$ Let $R'$ be hypercyclic, $e = e^2 \in (R')_n = R$, $e$ a primitive idempotent. Identify $R'$ with $eRe$. Then $Re_{R'} \approx \sum_{i=1}^n \oplus R'_i$. Since the category isomorphism of 1.5 takes $R' \to \text{Hom}_{R'}(Re, R') \approx eR$, every quotient of $eR$ has injective hull a quotient of $eR$. Let $I$ be a right ideal of $R$. Then

$$R \to R/I \to 0$$

is exact, so

$$R \otimes_R Re \to R/I \otimes_R Re \to 0$$

is exact. Since $R \otimes_R Re_{R'} \approx Re_{R'}, R/I \otimes_R Re$ is an $R'$-quotient of $Re$. By 1.16, $R/I \otimes_R Re$ is an essential extension of at most $n$ cyclic $R'$-modules. By the category isomorphism, $R/I$ is an essential extension of a sum of at most $n$ quotients of $eR$. Hence its injective hull is a direct sum of at most $n$ quotients of $eR$, and thus a quotient of $R$.

$\Leftarrow$. Let $R = (R')_n$ be hypercyclic, $e$ a primitive idempotent, $A \subseteq e(R')_n$. Then $E(R/A) = (1 - e)R \oplus E(eR/A)$, so $E(eR/A)/E(eR/A)J$ is simple, and $E(eR/A)$ is a quotient of $eR$. By the category isomorphism 1.5, every quotient of $R'$ has injective hull also a quotient of $R'$.

Putting 1.17, 1.15 and 1.14 together we get

**Theorem 1.18.** $R$ is a restricted hypercyclic ring $\Leftrightarrow R$ is a ring direct sum of matrix rings over local hypercyclic rings.

2. Local hypercyclic rings. By Theorem 1.18, hypercyclic rings $R$ such that $R/J$ is semi-simple Artin are ring direct sums of matrix rings over local hypercyclic rings. In this section we study local hypercyclic rings. These turn out to be, with one possible exception, the rings studied by Caldwell in [2], with $R$ commutative replaced by $xR = Rx$ for all $x \in R$.

By 1.11, a local hypercyclic ring is right self injective; by 1.10 its right ideals are linearly ordered.

We have the well-known

**Proposition 2.1.** Let $M_R$ contain a copy of the injective hull of every simple $R$-module. Then every right ideal of $R$ is the annihilator of some subset of $M$.

**Proposition 2.2.** Let $R_R$ be injective. Then any finitely generated left ideal is the annihilator of some right ideal, and the right socle of $R \subseteq$ the left socle of $R$.

Let $X \subseteq R$. Define $X^r = \{r \in R \mid Xr = 0\}$, $X^l = \{r \in R \mid rX = 0\}$.
COROLLARY 2.3. Let R be a local hypercyclic ring. Then the left ideals of R are linearly ordered.

Proof. Let \( x_1, x_2 \in R \). Then \( x_1 \subseteq x_2 \) or \( x_2 \subseteq x_1 \), so by 2.2, \( x_1^1 = Rx_1 \subseteq x_2^1 = Rx_2 \) or \( Rx_1 \supseteq Rx_2 \). Now let \( I_1, I_2 \) be two left ideals, \( I_1 \not\subseteq I_2 \). Let \( x \in I_1 - I_2 \). For \( y \in I_2, Ry \nsubseteq Rx \) since \( x \in I_2 \). Hence \( Rx \subseteq Ry \) so \( I_2 \subseteq I_1 \).

LEMMA 2.4. Let \( x, z \in R, 0 \neq xz \in S(R) \). Then \( (Rx)^r = zJ, (Jx)^r = zR, (zJ)^i = Rx, (zR)^i = Jx \).

Proof. Since \( 0 \neq xz \in S, xzJ = 0, xzR \neq 0 \). By the linear ordering on right ideals, \( zJ \subseteq (Rx)^r \subseteq zR \). Since \( zR/zJ \) is simple, \( zJ = (Rx)^r \). Since left ideals are linearly ordered by 2.3, \( Jx = (zR)^i \) by symmetry. By 2.2, principal left ideals are annihilators so \( Rx = (zJ)^i \). By 2.1, \( zR \) is an annihilator so \( (Jx)^r = zR \).

We note that given \( x \neq 0 \) or \( z \neq 0 \), by the linear ordering on right and left ideals, we can always find the other such that \( 0 \neq xz \in S(R) \).

COROLLARY 2.5. Let R be a local hypercyclic ring. Then every right ideal and every left ideal is an annihilator ideal.

Proof. Since \( R \) is injective and contains a copy of the unique simple \( R \)-module, 2.1 states every right ideal is a right annihilator.

Now let \( I' \) be a left ideal, and let \( Z = \bigcap_{x \in I'} Rx \). Then \( Z = (\sum_{x \in I'} x^i) \) since \( (x^i)^i = Rx \). If \( I' \neq Z \), let \( y \in Z - I' \). Then \( Ry \supseteq I' \), and \( Rz \supseteq I' \Rightarrow Rz \supseteq Ry \). Hence \( Ry/I' \) is a simple left module. By Nakayama's lemma, \( Jy \neq Ry \). Hence \( Jy \subseteq I' \). Since \( Ry/Jy \) is also simple, \( Jy = I' \). By 2.4, \( I' \) is an annihilator left ideal.

LEMMA 2.6. Let I be a right ideal of the local hypercyclic ring \( R, R \supseteq I \supseteq S(R) \). Then \( R/I \) is not injective.

Proof. Since \( I \neq R, I \subseteq J \). Let \( x \in I, x \in S(R) \). Since the left ideals of \( R \) are linearly ordered and \( S(R) \) is a two sided ideal, there exists \( y \in R \) such that \( 0 \neq yx \in S(R) \). Then \( xJ = (Ry)^r \) by 2.4. Since \( x \in S(R), y \) is not a unit, so \( y \in J \). We now proceed as in Caldwell [2]; the map \( f: yR \rightarrow R/I \) given by \( f(yr) = r + I \) is well defined since \( y^i = xJ \subseteq I \). \( R/I \) injective implies there exists \( m \in R \) such that \( my + I = 1 + I \). Hence \( 1 - my \in I \), and since \( y \in J, (1 - my)(1 - my)^{-1} = 1 \in I \), a contradiction.

LEMMA 2.7. Let \( R \) be local hypercyclic. Then every right (left)
ideal is of the form $xR$ or $xJ$ ($Rx$ or $Jx$).

Proof. Let $I$ be a right ideal. Since the injective hull of $R/I$ is cyclic, by 2.6 $R/I$ embeds either in $R$ or in $R/S(R)$. Assume $f$ embeds $R/I$ in $R$. Let $0 \neq f(x + I) \in S(R)$. Then $xJ \subseteq I \subseteq xR$, so $I = xJ$. If $g$ embeds $R/I$ in $R/S(R)$, let $g(1) = m + S(R)$. Let $0 \neq my \in S(R)$. Then $g(y) = 0$ so $y \in I$. Let $x \in I$. Then $mx \in S(R)$. If $mx = 0$, $x \in yJ = m^*$. If $mx \neq 0$, $m^* = xJ = yJ$, so by the linear ordering $xR = yR$. Hence $x \in y^R$, and $I = yR$.

Since every left ideal $I$ is an annihilator by 2.5, it is of the form $(xR)$ or $(xJ)$ for some $x$. If $x \neq 0$, select $y$ such that $0 \neq py \in S(I)$. Then $y^R = y^2R$, where $yz'R = yzR$.

PROPOSITION 2.8. Let $R$ be local hypercyclic, $y \in J$. Either $y$ is nilpotent or $0 \neq \bigcap_{n=0}^\infty y^nR = z'R$, where $yz'R = z'R$.

Proof. (See Caldwell [2] Theorem 2.20) Since $R' \supseteq S(R)$, $y^nR \supseteq S(R)$ or $y^n = 0$. Thus $I = \bigcap_{n=0}^\infty y^nR = 0 \iff y^n = 0$ for some $n$.

Let $y^n \in y^{n+1}R$. Then for some $r \in R$, $y^n = y^{n+1}r$, so $y^n(1 - yr) = 0$, and since $yr \in J$, $1 - yr$ is invertible. Hence $y^n = 0$. Thus $y$ not nilpotent implies $yR \supset y^2R \supset \cdots$ is a strictly descending chain of right ideals. Let $(y^iR)^i = Jz_i$. Then $I = (\bigcup_{i=0}^\infty Jz_i)^i$, where $\bigcup_{i=0}^\infty Jz_i = K$ is the union of a strictly ascending chain of left ideals if $y$ is not nilpotent. Then $K$ cannot be finitely generated so $K = Jz$ for some $z \in R$ by 2.7, and $I = K^r = z'R$ for some $z'$. Since $y(y^nR) \subseteq y^nR$ for all $n$, $yI \subseteq I$. Now $z' \in I$, so $z' = yr$ for some $r \in R$. Assume $r \in I$. Then there is an $n$ such that $r \in y^nR$, so $rR \supset y^nR$. Let $rs = y^s$. Then $z's = yrs = y^{s+1} \in I$. If $y$ is not nilpotent, this cannot occur, so $r \in I$ and $z' \in yI$. Hence $yI = I$.

COROLLARY 2.9. Let $yR = Ry$ for $y \in J$. Then $y$ is nilpotent.

Proof. Assume not. Then $0 \neq I = \bigcap_{n=0}^\infty y^nR = \bigcap_{n=0}^\infty Ry^n$. As in the proof of 2.8, $(\bigcap_{n=0}^\infty Ry^n)y = \bigcap_{n=0}^\infty Ry^n$. By 2.8, for some $z'$, $I = z'R$. Then $I = z'Ry$, so there exists $r \in R$ with $z'ry = z'$, $z'(ry - 1) = 0$. Since $y \in J$, $ry - 1$ is invertible, so $z' = 0$, a contradiction.

THEOREM 2.10. Let $R$ be local hypercyclic. Then $J$ is nil $\iff yR = Ry$ for all $y \in R$.

Proof. $\Leftarrow$. This is just 2.9. $\Rightarrow$. Assume $J$ is nil, and let $0 \neq y, r \in R, yr \in Ry$. Then $Ryr \supseteq Ry$, so $y = xyr$ for some $x \in J$. Then $y = xyr = x^2yrs = \cdots = x^nyr^n = 0$, a contradiction. Hence $yR \supseteq Ry$.

By symmetry, $Ry \subseteq yR$.

Thus if $J$ is nil, every one sided ideal of $R$ is two sided, and all
of Caldwell’s arguments in [2] may be carried over to this case almost verbatim. The reader is referred to Caldwell for further discussion of this case.

Whether a local hypercyclic ring can have nonnil radical is unknown. We show that it implies the existence of a very elusive type of ring, namely a simple radical ring (without 1 of course), which has linear ordering on both right and left ideals.

**Lemma 2.11.** Let $R$ be a hypercyclic local ring, $J$ not nil. Then $J^2 = J$ and for all $y \in J$, $JyJ \neq J \Rightarrow \exists z \in J, zR = Rz$ and $y \in zR$.

**Proof.** If $J^2 \neq J$, by the linear ordering on one sided ideals, for any $x \in J - J^2, J = xR = Rx$. Hence $x$ is nilpotent by 2.9, so $J^* = (xR)^* = x^nR = 0$ for some $n$, a contradiction. Hence $J = J^2$. Let $y \in J$. Then $(JyJ)J = JyJ = J(JyJ)$, so $JyJ$ is not a finitely generated right or left ideal. Hence $JyJ = zJ = Jz'$ for some $z, z' \in R$ by 2.7. Let $\nu$ denote the natural map from $R$ onto $R/JyJ$. Then $\nu(z)R = S(\nu(R)_R), R\nu(z') = S(\nu(R))$ since the right and left ideals of $\nu(R)$ are linearly ordered and $\nu(z)R$ and $R\nu(z')$ are simple. Moreover, $\nu(z')R \supseteq \nu(z)R$. Since $S(\nu(R))$ is a two sided ideal of $\nu(R), S(\nu(R)) = R\nu(z') \subseteq \nu(z)R = S(\nu(R)_R)$. By symmetry, $S(\nu(R)_R) \subseteq S(\nu(R))$, so $R\nu(z') = \nu(z)R = \nu(z')R = R\nu(z)$ since it is a simple $R$-module on both right and left. Taking $\nu^{-1}$ of both sides we get $z'R = Rz' = zR = Rz$. If $JyJ \neq J, S(\nu(R)_R) \neq R/J$, so $z \in J$.

Moreover $zR \supseteq JyJ = zR \supseteq JyJ$ for all $j \in J = zR \supseteq JyR$ for all $j \in J$ by the linear ordering $= zR \supseteq JyR$. Similarly $zR = Rz \supseteq RyR$.

**Theorem 2.12.** Let $R$ be a local hypercyclic ring, $J$ not nil. Then there exists a nilpotent ideal $zR \subseteq J$ such that $zR$ is a maximal proper two sided ideal of $J$ (so $J/zR$ is a simple radical ring.)

**Proof.** Let $I$ be the union of all the nil two-sided ideals of $R$. Then $I$ is a nil two-sided ideal since ideals are linearly ordered. Moreover, $I \neq 0$ since $S$ is a nilpotent two-sided ideal, and $I \neq J$ since $J$ is not nil.

Let $K = JI$. Then $JK = KJ = K$, so as in 2.11, $K = zJ = Jz$ where $zR = Rz = \nu^{-1}(S(R/K))$ and $x \in Rz$ for all $x \in I$. Since $[\nu^{-1}(S(R/K))]^* \subseteq K$ and $K \subseteq I, zR$ is nil. Hence $zR = Rz = I$. If $z^* = 0$, then $I^* = (zR)^* = z^nR = 0$, so $I$ is nilpotent.

Now let $y \in J - I$. If $JyJ \neq J$, by 2.11, $y$ belongs to some nilpotent ideal of $R$ and hence to $I$. Thus $JyJ = J$, and $J/zR$ is a
simple ring. By [8], p. 10, $J$ is a radical ring, and hence so is $J/zR$.

REFERENCES


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