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In this paper we consider the partial ordering induced on 1-dimensional continua by quasi dimension type. For example, we show that the pseudo-arc precedes every snake-like continuum in this partial ordering. We also obtain sufficient conditions in terms of quasi dimension type for the embedding of Peano continua in the plane. Finally, necessary and sufficient conditions in terms of quasi dimension type are given for a continuum to be tree-like and to be 1-dimensional.

All spaces considered in this paper are metric. The distance between two points x, y will be denoted by $\rho(x, y)$ and the diameter of a set A will be denoted by $\delta(A)$. By "covering" we mean "open covering". We use dX and qX for the Fréchet and quasi dimension types of X respectively. Recall that $dX \leqq dY$ provided there is a homeomorphism of X into Y; $qX \leqq qY$ provided that for each covering α of X there is a closed α-mapping of X into Y. (A continuous function $f: X \to Y$ is an α-map if there is a covering β of Y such that $f^{-1}[\beta]$ refines α and f is closed if it takes closed subsets of X into closed subsets of Y.) A compact connected space of more than one point will be called a *continuum*. A *chain* is a finite collection of open sets $U_1, U_2, \cdots, U_n$ such that U_i intersects U_j if $i = j - 1, j, j + 1$, and otherwise $\rho(U_i, U_j) > 0$. If the links of a chain are of diameter less than ε, the chain is called an *ε-chain*. A continuum X is called *snake-like* if for each $\varepsilon > 0$ it can be covered by an ε-chain (whose links are subsets of X). Finally, $S_{(x,\varepsilon)} = \{y \mid \rho(x, y) < \varepsilon\}$, R^n denotes Euclidean n-space and J denotes the positive integers.

1. Quasi dimension types of snake-like continua.

Bing [1] proved that any two snake-like hereditarily indecomposable continua are homeomorphic. Such continua are called pseudo-arcs and throughout this paper we use ψ to denote a pseudo-arc.

LEMMA 1.1. [22, Th. 6]. *If, for each $\varepsilon > 0$, a continuum can be ε-mapped onto a hereditarily indecomposable continuum, then X is hereditarily indecomposable.*

LEMMA 1.2. *If, for each $\varepsilon > 0$, a continuum X can be ε-mapped onto a snake-like continuum, then X is snake-like.*

THEOREM 1.1. *If X is not totally disconnected and $qX \leqq q\psi$, then $qX = q\psi$.*

Thus we have that no nontotally disconnected space precedes ψ and hence in this sense ψ is minimal for those nontotally disconnected spaces with which it is comparable. We will now consider how large this class of spaces is. Certainly, for any snake-like continuum X which contains an arc or a pseudo-arc, we have $q\psi \leqq qX$. However, this does not exhaust the class of snake-like continua. Bing [3], Lelek [17], and others have shown that ψ can be mapped onto any snake-like continuum. We sharpen this result by proving that, for any $\varepsilon > 0$, ψ can be ε-mapped onto any snake-like continuum. Furthermore, since for any $\varepsilon > 0$, an arc can be ε-mapped onto any Peano space X which contains an open n-cell, $n \geqq 2$, as an open set (see [16]) we may conclude that ψ can be ε-mapped onto X (so that $q\psi \leqq qX$). But since we need only ε-mappings of ψ into a space, we have $q\psi \leqq qX$ if X contains a snake-like continuum. So ψ occupies a position with respect to spaces which have snake-like subcontinua similar to that of the Cantor set with respect to all infinite subsets of the line.

THEOREM 1.2. *If X is a snake-like continuum and $\varepsilon > 0$, then there is an ε-mapping of ψ onto X (and so $q\psi \leqq qX$).*

Proof. To obtain the desired result we modify Bing's proof [3, p. 446] that ψ can be mapped onto X. For any $\varepsilon > 0$, let $D_1, D_2, \cdots$ be a sequence such that D_i is a $(\varepsilon/2i)$-chain covering X and D_{i+1} is a refinement of D_i. It follows from Theorem 7 of [4] that there is a sequence of coverings $E_1, E_2, \cdots$ of ψ such that: (1) E_i is a $(\varepsilon/2i)$-chain covering ψ, (2) E_i has the same number of links as D_i and (3) for the jth link of D_{i+1} there is an integer $n(i, j)$ such that the jth link of D_{i+1} lies in the $n(i, j)$th link of D_i and the jth link of E_{i+1} lies in the $n(i, j)$th link of E_i.

For each point p of ψ let $e(p, i)$ be the union of the links of E_i containing p and $d(p, i)$ be the union of the corresponding links of D_i. Note that $e(p, i + 1) \subset e(p, i)$ and $d(p, i + 1) \subset d(p, i)$. For each point p of ψ let $f_\varepsilon(p)$ be the intersection of the closures of the $d(p, i)$'s. Then f_ε is a mapping of ψ onto X and we need show that it is an ε-mapping.

Suppose $f_\varepsilon(p) = f_\varepsilon(q)$. Then $d(p, 2) \cap d(q, 2) \neq \varnothing$ and this implies that two adjacent links of D_2, say d_j^2 and d_k^2 (where $k = j - 1, j$ or $j + 1$), are such that

(4) $f_\varepsilon(p) \in d_j^2 \cap d_k^2$ (which implies $f_\varepsilon(p) \in d^1_{n(1,j)} \cap d^1_{n(1,k)}$) and

(5) $d_j^2 \subset d(p, 2)$ and $d_k^2 \subset d(q, 2)$.

Therefore, from (4) we get

(6) $e^1_{n(i,j)} \cap e^1_{n(i,k)} \supset e^2_j \cap e^2_k \neq \emptyset$

and from (5) we get

(7) $p \in e^2_j$ and $q \in e^2_k$.

So we have $e(p, 1) \cap e(q, 1) \neq \emptyset$ and it follows that

$$\rho(p, q) \leqq \delta(e(p, 1)) + \delta(e(q, 1)) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

This yields the desired result that f_ε is an ε-mapping.

THEOREM 1.2′. *If X is a space which contains a snake-like continuum, then $q\psi \leqq qX$.*

We now show that there is a planar continuum M (originally described by Whyburn [24] as an example of a continuum each subcontinuum of which separates the plane) such that $q\psi \leqq qM$ is false. By Theorem 1.1 this means that ψ and M are not comparable. What we actually prove is that no subcontinuum of M is the continuous image of ψ.

The continuum M has a rather complicated description but we can obtain our desired result by using only some of its properties. First note that M arises in a condensation process from a ray spiralling on a circle. Letting A denote the latter continuum it is not difficult to see that A is the continuous image of M. Therefore ψ cannot be mapped onto M; otherwise, we could map ψ onto A and this is impossible (see [17, Example 2]). Furthermore, Whyburn shows that (1) every subcontinuum of M contains a continuum which is homeomorphic to M and (2) if any subcontinuum N of M contains a point of a subcontinuum W of M which is homeomorphic to M, then N contains W or W contains N. We use $C(X)$ to denote the space of all nonvacuous subcontinua of a continuum X metrized by the Hausdorff metric (see [14]). If Y is a closed subset of X, then $C(X, Y)$ denotes the subset of $C(X)$ consisting of continua which contain Y.

THEOREM 1.3. *The continuum M has no subcontinuum which is the continuous image of pseudo-arc ψ.*

Proof. Suppose M properly contains a continuum X which is the continuous image of ψ. Say f is a mapping of ψ onto X. Then X contains a continuum W which is homeomorphic to M. Suppose $x \in W$. Then there is a $y \in \psi$ such that $f(y) = x$. Let $B = \{B_t \mid 0 \leqq t \leqq 1\}$ be the unique arc in $C(\psi)$ from $\{y\}$ to ψ with $B_0 = \{y\}$ and $B_1 = \psi$ (see [14, Th. 8.4]). Let $f'\colon C(\psi) \to C(X)$ be defined by $f'(Y) = f[Y]$ for each $Y \in C(\psi)$. Then f' is a mapping. Denote $f'(B_t)$ by B'_t and $f'[B]$

by B'. Then B' is a subcontinuum of $C(X)$ which contains X and $\{x\}$.

Now since $B'_t \cap W \neq \emptyset$ by property (2) of M we have B'_t contains W or W contains B'_t for $0 \leqq t \leqq 1$. However, since M is not the continuous image of ψ neither is W so that B'_t properly contains W or W properly contains B'_t. Therefore

$$B'_t \in C(X, W) \cup C(W) , \qquad \text{for } 0 \leqq t \leqq 1$$

and so

$$B' \subset C(X, W) \cup C(W) .$$

It is possible to define a real-valued continuous function, μ, of $C(X)$ onto the unit interval I (see [23]) with the properties:

(1) If $Y_1 \subset Y_2$, $Y_1 \neq Y_2$ then $\mu(Y_1) < \mu(Y_2)$.

(2) $\mu(X) = 1$, and for any $x \in X$, $\mu(\{x\}) = 0$.

Therefore since B' contains $\{x\}$ and X we have that the continuum $\mu[B']$ contains 0 and 1 and so $\mu[B'] = I$. Thus there is t_0, $0 < t_0 < 1$, such that $\mu(B'_{t_0}) = \mu(W)$. Now in $C(X, W) \cup C(W)$ the only element with μ-value equal to $\mu(W)$ is W itself since all other elements either are properly contained or properly contain W and (1) applies. So finally we have $B'_{t_0} = W$ which is a contradiction.

COROLLARY 1.3. *The quasi dimension types of the pseudo-arc ψ and Whyburn's continuum M are not comparable.*

Question. Are there snake-like continua whose quasi dimension types are not comparable?

Question. What is the cardinality of the set of quasi dimension types of snake-like continua?

LEMMA 1.3. *Suppose X is a compactum, K is a component of X and V is a set open in K. Then there is a set V' open in X such that $V' \cap K = V$ and $\mathrm{Cl}(V') \cap K = \mathrm{Cl}(V)$.*

LEMMA 1.4. *Suppose X is a compactum with a snake-like continuum K as a component and $C = (c_1, \cdots, c_n)$ is a chain covering K where each c_i is an open set in K. Then there is a chain $D = (d_1, \cdots, d_n)$ covering K where each d_i is an open set in X and $d_i \cap K = c_i$, $i = 1, \cdots, n$.*

Proof. By the previous lemma we have that there is a set d'_i open in X such that $d'_i \cap K = c_i$ and $\mathrm{Cl}(d'_i) \cap K = \mathrm{Cl}(c_i)$ for $i = 1, \cdots, n$. Let

$$d_i = d_i' - \bigcup \{\mathrm{Cl}\,(d_j') \mid j \in \varphi(i)\}$$

where $\varphi(i) = \{j \mid j \text{ is an integer, } 1 \leqq j \leqq n, |j - i| > 1\}$. Then $D = (d_1, \cdots, d_n)$ is a chain covering K and $d_i \cap K = c_i$, $i = 1, \cdots, n$.

NOTATION. If $C = \{C_\alpha \mid \alpha \in A\}$ and $D = \{D_\beta \mid \beta \in B\}$ are two collections of sets, $C \cap D = \{C_\alpha \cap D_\beta \mid \alpha \in A, \beta \in B, C_\alpha \cap D_\beta \neq \varnothing\}$. If, in addition, E is a set $E \cap C = \{E \cap C_\alpha \mid \alpha \in A, E \cap C_\alpha \neq \varnothing\}$. Finally, $C^* = \bigcup \{C_\alpha \mid \alpha \in A\}$.

THEOREM 1.4. *Suppose X is a compactum each nondegenerate component of which is a snake-like continuum. Then X is embeddable in R^2.*

Proof. First we show that X has arbitrarily small coverings whose nerves are embeddable in R^1. Let $U = (u_1, \cdots, u_m)$ be an arbitrary finite irreducible covering of X and let $\{K_\alpha \mid \alpha \in A\}$ be the collection of components of X. Denote $\{u_k \mid u_k \in U, u_k \cap K_\alpha \neq \varnothing\}$ by U_α. There exists a a chain $C_\alpha = (c_1^\alpha, \cdots, c_{n(\alpha)}^\alpha)$ covering K_α each of whose links is open in K_α and such that $C_\alpha > U_\alpha$ (i.e., C_α refines U_α). By the previous lemma there is a chain $D_\alpha = (d_1^\alpha, \cdots, d_{n(\alpha)}^\alpha)$ covering K_α each of whose links is open in X and such that $D_\alpha \cap K_\alpha = C_\alpha$.

Now letting $v_j^\alpha = \bigcap \{u_k \mid u_k \in U_\alpha, c_j^\alpha \subset u_k\}$ we define E_α to be $\{d_j^\alpha \cap v_j^\alpha \mid j = 1, \cdots, n(\alpha)\}$. Then E_α is a chain covering K_α whose links are open in X and such that $E_\alpha > U_\alpha$. Finally we define

$$E_\alpha' = \{d_j^\alpha \cap v_j^\alpha - \bigcup \{K_\beta \mid \beta \in A, K_\beta \not\subset E_\alpha^*\} \mid d_j^\alpha \cap v_j^\alpha \in E_\alpha\}\,.$$

Then E_α' is a chain covering K_α whose links are open in X and such that $E_\alpha' > U_\alpha$. We can partition X as follows. There is a collection $P = (P_1, \cdots, P_n)$ of disjoint open and closed sets whose union is X. Then $\{P \cap E_\alpha' \mid \alpha \in A\}$ is a covering of X and since the latter is compact we need only elements from a finite number of the chains $\{P \cap E_\alpha' \mid \alpha \in A\}$ to cover X, say $Q = (Q_1, \cdots, Q_s)$ where $Q_j = P_j \cap E_\alpha'$ for some j, $1 \leqq j \leqq n$ and $\alpha \in A$. Then Q is a covering of X, $Q > U$ and each nondegenerate component of the nerve of Q is an arc. Hence the nerve of Q is embeddable in R^1 and since there is a Q-mapping of X into the nerve of Q we have that X is the inverse limit of an inverse sequence each coordinate space of which is contained in R^1 (see [9, Lemma 1.6]). Thus by Theorem 1 of [13] X is embeddable in R^2.

THEOREM 1.5. ([8]) *If X is a planar compactum each nondegenerate component of which is a snake-like continuum (pseudo-arc), then X is embeddable in a snake-like continuum (pseudo-arc).*

THEOREM 1.6. *If X is a compactum, then the following are equivalent*:

(a) *every nondegenerate component of X is a snake-like continuum.*

(b) $dX \leqq dY$ *for some snake-like continuum Y (i.e., X is embeddable in Y).*

(c) $qX \leqq qI$.

Proof. First, (a) implies (b) by Theorems 1.4 and 1.5. Second, (b) implies (c) since $qX \leqq qY \leqq qI$. Third, (c) implies (a) since, for any nondegenerate component K of X, $qK \leqq qI$ so that K is a snake-like continuum by Lemma 1.2.

THEOREM 1.7. *If X is a compactum, then the following are equivalent*:

(a) *every nondegenerate component of X is a pseudo-arc.*

(b) $dX \leqq d\psi$.

(c) $qX \leqq q\psi$.

Proof. First, (a) implies (b) by Theorems 1.4 and 1.5. Second, it is obvious that (b) implies (c). Third, (c) implies (a) since, for any nondegenerate component K of X, $qK \leqq q\psi$ so that K is a pseudo-arc by Lemma 1.1 and 1.2.

THEOREM 1.8. *In the class of metric spaces the Cantor discontinuum C immediately precedes the pseudo-arc ψ (i.e., $qC \ll q\psi$).*

Proof. Suppose not, i.e., $qC < qX < q\psi$ for some metric space X. Then X is compact and not totally disconnected (otherwise $qX \leqq qC$). Therefore X contains a nondegenerate continuum and so $qX = q\psi$ by Theorem 1.1. But this is a contradiction.

2. Quasi dimension type of linear graphs.

DEFINITION 2.1. By a *linear graph* we mean a finite connected 1-dimensional geometric complex. A *tree* is an acyclic linear graph. An *end point* of a linear graph is a vertex which is contained in only one edge (1-simplex). A *branch point* is a vertex which is contained in more than two edges. A tree which is homeomorphic to the cone over $n(\geqq 2)$ points is called an n-od. A *dendrite* is a locally connected continuum which contains no simple closed curve. In a tree the unique arc with end points p, q is denoted by $[pq]$; $[pq] = [pq] - p$, etc.

DEFINITION 2.2. For a positive integer n, we say the curve X

is of order $\leqq n$ *at the point* p (in symbols: $\operatorname{ord}_p X \leqq n$) if, for every $\varepsilon > 0$, there exists an open set U of X such that $p \in U$, $\delta(U) < \varepsilon$ and cardinality of the boundary of $U \leqq n$. Furthermore, if $\operatorname{ord}_p X \leqq n$ and it is false that $\operatorname{ord}_p X \leqq m$ for any $m < n$, then we say $\operatorname{ord}_p X = n$. We also define for a linear graph X the following

$$\sigma(X) = 2 + \sum (\operatorname{ord}_p X - 2)$$

and

$$\beta(X) = \sum \operatorname{ord}_p X$$

the summations being extended over all branch points of X.

LEMMA 2.1. *If* X *is a linear graph, then* $\beta(X) - \sigma(X) = 2(m-1)$ *where* m *is the number of branch points of* X.

LEMMA 2.2. ([14, Th. 5.5]) *If* X *is a linear graph, then* $\dim C(X) = \sigma(X)$. *Moreover,* $C(X)$ *is a polyhedron.*

THEOREM 2.1. *If* X *and* Y *are linear graphs and, for every* $\varepsilon > 0$, *there is an* ε*-map* $f_\varepsilon\colon X \to Y$, *then for* ε *sufficiently small*

$$\sigma(X) \leqq \sigma(f_\varepsilon[X]) .$$

Proof. For any $\varepsilon > 0$ we have $f_\varepsilon[X]$ is subcontinuum of Y and hence is also a linear graph. The map f'_ε induced by f_ε (see Theorem 1.3) is an ε-mapping of $C(X)$ into $C(f_\varepsilon[X])$. Therefore, for ε sufficiently small we have by Alexandroff's Theorem on the approximation to compact spaces by polyhedra (see [12, p. 72]) that

$$\sigma(X) = \dim C(X) \leqq \dim C(f_\varepsilon[X]) = \sigma(f_\varepsilon[X]) .$$

COROLLARY 2.1. *If* X *and* Y *are linear graphs and* $qX \leqq qY$, *then* $\sigma X \leqq \sigma Y$.

COROLLARY 2.2. *If* X *and* Y *are linear graphs and* $qX \leqq qY$, *then* $\beta(X) \leqq \beta(Y)$.

NOTATION. For any metric space X and $\varepsilon > 0$ we define

$$V(x, \varepsilon) = \{y \in X \mid \rho(x, y) \leqq \varepsilon\} .$$

THEOREM 2.2. *If* X *and* Y *are linear graphs, then the following statements are equivalent.*

(a) $qX = qY$.

(b) $dX = dY$.

(c) *X and Y are homeomorphic.*

Proof. The implications (c) $\rightarrow$ (b) $\rightarrow$ (a) are trivial. Now assume $qX = qY$. Denote the branch points of X by $p_1, \cdots, p_m$ and the branch points of Y by $q_1, \cdots, q_n$. We consider X and Y to have the decompositions determined by taking the branch points and end points as the vertices. Let $\nu = 1/2 \min \{\rho(v_i, v_j) \mid v_i, v_j$ are distinct vertices of $Y\}$. First we show that $m = n$. There is a Δ_i, $0 < \Delta_i < \nu/2$, such that, for any positive number $\Delta \leqq \Delta_i$,

$$\operatorname{ord}_{q_i} V(q_i, \Delta) = \operatorname{ord}_{q_i} Y , \qquad \text{for } i = 1, \cdots, n .$$

This in turn implies that $\sigma(V(q_i, \Delta)) = \operatorname{ord}_{q_i} Y$. Let $V_i = V(q_i, \Delta_i)$ and $\sigma_i = \sigma(V_i)$. By hypothesis for every $\varepsilon > 0$ there is an ε-map $g_\varepsilon \colon Y \rightarrow X$. So

$$2 < \operatorname{ord}_{q_i} V_i = \sigma_i \leqq \sigma(g_{\eta_1}[V_i])$$

where $\eta_1 = \min \{\varepsilon_1, \cdots, \varepsilon_n\}$ and ε_i is sufficiently small so that $\sigma(V_i) \leqq \sigma(g_{\varepsilon_i}[V_i])$ for $i = 1, \cdots, n$. Now since V_i is a σ_i-od we have

$$V_i = \bigcup_{r=1}^{\sigma_i} [q_i y_i^r]$$

where the $(q_i y_i^r]$ are disjoint arcs.

We now define Y' to be the subdivision of Y induced by taking as the vertices of Y' those of Y and all the points y_i^r, $i = 1, \cdots, n$; $r = 1, \cdots, \sigma_i$. (Note that Y and Y' are homeomorphic and have the same branch points and end points.) Let the end points Y' be denoted by e_k^r where the subscript and superscript correspond to those of the y_k^r which is adjacent to e_k^r, (i.e., $[e_k^r y_k^r]$ is an edge of Y'). Then note that $Y' - \bigcup_{i=1}^n V_i$ is the union of disjoint arcs of two kinds:

(1) $[e_k^r y_k^r)$ for every end point e_k^r of Y', and

(2) $(y_k^r y_m^s)$ where $[y_k^r y_m^s]$ is an edge of Y' and y_k^r and y_m^s are end points of V_k and V_m, respectively.

Let

$$\eta = \min \{\eta_1, \lambda_1, \cdots, \lambda_n\}$$

where

$$\begin{aligned} \lambda_i = \min [& \min \{\rho(V_i, [e_k^r y_k^r]) \mid [e_k^r y_k^r] \cap V_t = \varnothing \text{ for any } t \neq k\} , \\ & \min \{\rho(V_i, [y_k^r y_m^s]) \mid [y_k^r y_m^s] \text{ is an edge of } Y' \text{ and} \\ & \quad [y_k^r y_m^s] \cap V_t = \varnothing \text{ for any } t \neq k \text{ or } m\}] . \end{aligned}$$

Now for any η-map, g_η, we have that the sets $g_\eta[V_i], \cdots, g_\eta[V_n]$ are

disjoint (since $\rho(V_i, V_j) \geqq \nu > \eta$). Since $\sigma(g_\eta[V_i]) > 2$ we have that $g_\eta[V_i]$ must contain at least one branch point of X for $i = 1, \cdots, n$. Therefore $n \leqq m$ and by a symmetric argument $m \leqq n$ and so $m = n$. Moreover, $g_\eta[V_i]$ contains exactly one branch point of X, say $p_{j(i)}$. Likewise $f_\gamma[W_i]$ contains exactly one branch point of Y' say $q_{k(i)}$ (where $W_i = V(p_i, \delta_i)$, with δ_i chosen like Δ_i, γ chosen like η and X' determined like Y'). So $\operatorname{ord}_{p_{j(i)}} X \geqq \sigma(g_\eta[V_i]) \geqq \sigma_i$ and $\operatorname{ord}_{q_{k(i)}} Y' \geqq \sigma(f_\gamma[W_i]) \geqq \sigma[W_i]$. Therefore,

$$\sigma(X') = 2 + \sum_{i=1}^{m} (\operatorname{ord}_{p_{j(i)}} X' - 2) \geqq 2 + \sum_{i=1}^{m} (\sigma_i - 2)$$
$$= 2 + \sum_{i=1}^{m} (\operatorname{ord}_{q_i} Y' - 2) = \sigma(Y')$$

and since $\operatorname{ord}_{p_{j(i)}} X' \geqq \sigma_i$ for each i we have $\operatorname{ord}_{p_{j(i)}} X' = \operatorname{ord}_{q_i} Y'$ for $i = 1, \cdots, m$. Likewise $\operatorname{ord}_{q_{k(i)}} Y' = \operatorname{ord}_{p_i} X'$ for $i = 1, \cdots, m$.

We can now write $g_\eta[V_i]$ as follows: $g_\eta[V_i] = \bigcup_{r=1}^{\sigma_i} [p_{j(i)} x^r_{j(i)}]$ and the $(p_{j(i)} x^r_{j(i)}]$ are disjoint arcs. Thus relying on the way in which η was chosen we have $X' - \bigcup_{i=1}^{n} g_\eta[V_i]$ is the union of disjoint arcs of two kinds:

(3) $[d^r_{j(k)} x^r_{j(k)})$ for every end point $d^r_{j(k)}$ of X' which is in $X' - \bigcup_{i=1}^{n} g_\eta[V_i]$ (the subscript and superscript of d being determined by the fact that $[d^r_{j(k)} x^r_{j(k)})$ does not contain and end point of any $g_\eta[V_i]$, for $i = 1, \cdots, n$, and

(4) $(x^r_{j(k)} x^s_{j(m)})$ for end points $x^r_{j(k)}$ and $x^s_{j(m)}$ of $g_\eta[V_k]$ and $g_\eta[V_m]$, respectively, and where $g_\eta(y^r_i) \in (p_{j(i)} x^r_{j(i)}]$, $i = 1, \cdots, n$.

We now define a homeomorphism h of Y' onto X' as follows:

(5) $(h \mid V_i)$: $V_i \to g_\eta[V_i]$ is a homeomorphism such that $h(q_i) = p_{j(i)}$ and

$$h(y^r_i) = \begin{cases} z^r_{j(i)} \ (= \text{midpoint of } [p_{j(i)} x^r_{j(i)}]); \text{ if } d^r_{j(i)} \in g_\eta[V_i] \\ x^r_{j(i)}, \text{ otherwise} \end{cases}$$

where $g_\eta(y^r_i) \in (p_{j(i)} x^r_{j(i)}]$, $i = 1, \cdots, n$, and $r = 1, \cdots, \sigma_i$.

(6) $(h \mid [y^r_k y^s_m])$: $[y^r_k y^s_m] \to [x^r_{j(k)} x^s_{j(m)}]$ is a homeomorphism such that $h(y^r_k) = x^r_{j(k)}$ and $h(y^s_m) = x^s_{j(m)}$, for any edge $[y^r_k y^s_m]$ in Y' whose end points are also end points of V_k and V_m, respectively, and $k \neq m$.

(7) $(h \mid [e^r_i y^r_i])$: $[e^r_i y^r_i] \to [d^r_{j(i)} h(y^r_i)]$ is a homeomorphism such that $h(e^r_i) = d^r_{j(i)}$; $h(y^r_i)$ is as in (5).

This determines a homeomorphism h of Y' onto X' and so Y and X are homeomorphic. Therefore we have (a) $\to$ (c).

The first of following examples shows that $qX \leqq qY$ does not imply $dX \leqq dY$ even in the case that X and Y are trees. The second shows that Theorem 2.2 is not true for dendrites.

EXAMPLE 2.1. Let X be a 4-od and Y the space obtained by taking the union of two vertical segments with end points $(1/4, 0)$ and $(3/4, 0)$ and the closed unit interval $[0, 1]$. Then $qX \leqq qY$ since an arc of arbitrarily small diameter with the branch point of X as one end point can be projected onto an arc of the same kind on another branch of X thereby obtaining a space homeomorphic to Y. But obviously neither X nor Y is embeddable in the other.

EXAMPLE 2.2. Let X be the union of the sets (1), (2) and (3) and let Y be the union of the sets (1) and (2) where

(1) the closed unit interval $[0, 1]$,

(2) the vertical segments with end points $(1/n, 0)$ and $(1/n, 1/n)$, $n = 2, 3, \cdots$,

(3) the segment given by $y = (1/2) - x$, $3/8 \leqq x \leqq 1/2$.

Then $qX \leqq qY$ (see the previous example) and $qY \leqq qX$ since Y is a closed subset of X. So $qX = qY$ but $dX \nleqq dY$.

3. Quasi dimension type of Peano curves.

DEFINITION 3.1. A *curve* is a 1-dimensional continuum and a *Peano curve* (*continuum*) is a locally connected curve (continuum). A curve X is *essentially nonplanar*, if there is an $\varepsilon > 0$ such that X cannot be ε-mapped onto any planar curve. If a space is the unit sphere of R^{n+1}, then it is called a *n-sphere* and is denoted by S^n. A space homeomorphic to $S^1(S^2)$ is called a *simple closed curve* (*spherical surface*).

DEFINITION 3.2. Kuratowski [15] described four Peano curves K_1 (= 1-skeleton of a tetrahedron with midpoints of a pair of nonadjacent edges joined by a segment), K_2 (= complete graph on 5 vertices), K_3 and K_4 which are indicated in Figures 1, 2, 3, and 4, respectively. A topological image of anyone of the four curves K_1, K_2, K_3, K_4 is called a *skew curve*. A topological image of K_1 or K_2 (K_3 or K_4) is called a *primitive* (*secondary*) *skew curve*.

Kuratowski showed that every nonplanar linear graph contains a primitive skew curve. Claytor [7] managed to use the secondary skew curves to obtain the following elegant result: *A Peano continuum which is not homeomorphic with a subset of S^2 necessarily contains a topological image of a skew curve.* As a special case he also proved: *Every cyclic Peano continuum which is not homeomorphic with a subset of S^2 contains a topological image of a primitive skew curve.*

DEFINITION 3.3. The Sierpinski (or universal plane) curve, $\mathfrak{S}$, and the Menger (or universal) curve, $\mathfrak{M}$, are different 1-dimensional locally

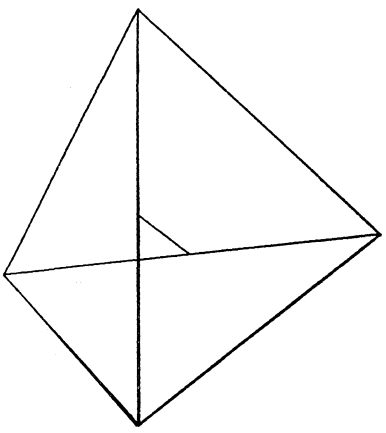

FIG. 1

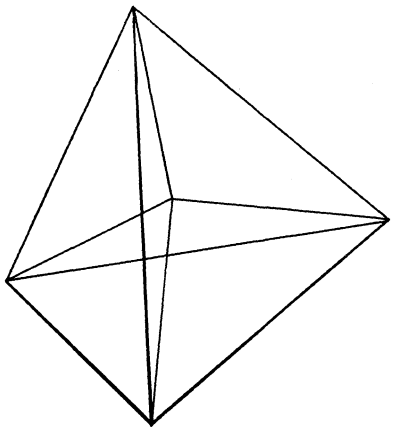

FIG. 2

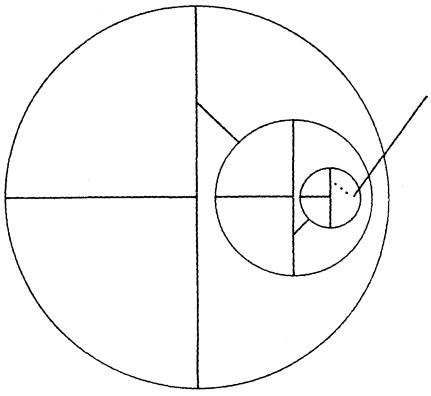

FIG. 3

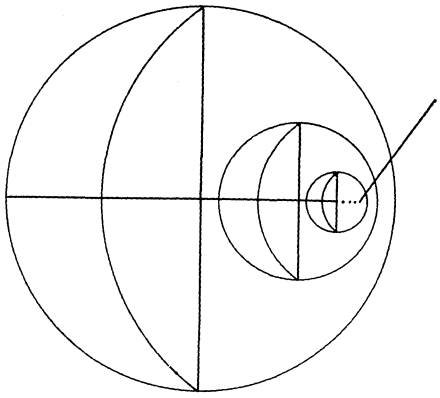

FIG. 4

connected analogues of the Cantor discontinuum. The Sierpinski curve can be obtained by considering a square and by successively deleting first the open middle-ninth of that square, second the middle-ninths of each of the eight squares remaining, third the middle-ninths of each of the 64 squares remaining, etc. The Menger curve may be defined as the set of all points of the unit cube which project in each of the three directions (of the edges) into Sierpinski curves constructed as above on the various faces of the cube. The Menger (Sierpinski) curve contains homeomorphic images of all 1-dimensional (planar) continua. Actually a more general statement is true, i.e., any 1-dimensional separable metric space can be embedded in $\mathfrak{M}$ and any 1-dimensional planar set can be embedded in $\mathfrak{S}$ (see [20, p. 106]). So for any subset X of R^3 (R^2) we have that $qX \leqq q\mathfrak{M}(q\mathfrak{S})$ implies $dX \leqq d\mathfrak{M}(d\mathfrak{S})$.

Mazurkiewicz [18] has shown the following.

THEOREM 3.1. *An essentially nonplanar Peano curve contains a*

primitive skew curve.

COROLLARY. *If X is a Peano curve which does not contain a primitive skew curve, then $qX \leqq q\mathfrak{S}$.*

THEOREM 3.2. [11] *If X is a compact polyhedron of dimension $n \neq 2$ such that $qX \leqq qR^{2n}$, then $dX \leqq dR^{2n}$.*

Combining these two results we have the next theorem.

THEOREM 3.3. *If X is a Peano curve, then $qX \leqq qR^2$ if and only if X does not contain a primitive skew curve.*

Proof. Suppose X contains a primitive skew curve, K, and suppose $qX \leqq qR^2$, then $qK \leqq qX \leqq qR^2$. Therefore, by the previous theorem $dK \leqq dR^2$, but this is impossible. The other half of the theorem is just a restatement of Theorem 3.1.

Claytor's fundamental result [7] in this area is stated next.

THEOREM 3.4. *A Peano continuum X is embeddable in S^2 if and only if X does not contain a skew curve.*

COROLLARY. *A Peano continuum X which is not a spherical surface is planar if and only if X does not contain a skew curve.*

The next two theorems show how quasi dimension type is related to the above results in a particular setting.

THEOREM 3.5. *If X is a Peano curve which does not contain a secondary skew curve, then the following are equivalent.*

(a) $dX \leqq dR^2$.
(b) $qX \leqq q\mathfrak{S}$.
(c) $qX \leqq qR^2$.
(d) *X does not contain a primitive skew curve.*

Proof. First, (a) implies (b) since if X is planar it is embeddable in $\mathfrak{S}$. Second, (b) implies (c) since $q\mathfrak{S} \leqq qR^2$. Third, (c) implies (d) is just half of Theorem 3.3. Fourth, (d) implies (a) follows from Theorem 3.4.

THEOREM 3.6. *If X is a Peano continuum which does not contain a secondary skew curve, then the following are equivalent.*

(a) $dX \leqq dR^2$.
(b) $qX \leqq qR^2$.

(c) *X is not a spherical surface and does not contain a primitive skew curve.*

Proof. First, (a) implies (b) is trivial. Second, (b) implies (c) follows from Theorem 3.3 and the Borsuk-Ulam Theorem, i.e., for any mapping of $f: S^n \to R^n$, there is a pair of antipodal points of S^n with the same image under f (so that $qS^2 \not\leqq qR^2$). Finally (c) implies (a) as a result of Theorem 3.4.

REMARK 3.1. Note that K_3 and K_4 are such that $qK_3 \leqq qR^2$ and $qK_4 \leqq qR^2$. To see that the former is true choose any $\varepsilon > 0$. Then take a circle S in the defining sequence of K_3 of diameter $< \varepsilon/3$. Then shrinking S and its interior (in R^2) to a point clearly yields a space embeddable in R^2. The same technique works for showing that $qK_4 \leqq qR^2$. This shows that necessity of requiring that X contain no secondary skew curve in Theorem 3.5 and 3.6. Since the dyadic solenoid D is such that $qD \leqq qS^1 < qR^2$ but $dD \not\leqq dR^2$, the need for requiring X to be locally connected in the previous two theorems is evinced.

Mazurkiewicz [18] also pointed out that the following two theorems are consequences of Theorem 3.1.

THEOREM 3.7. *If X is a curve and $qK_1 \leqq qK$, then $dK_1 \leqq dX$.*

THEOREM 3.8. *If X is a curve and $qK_2 \leqq qX$, then $dK_1 \leqq dX$ or $dK_2 \leqq dX$.*

As a result of Theorems 3.7 and 2.1 we have the following.

THEOREM 3.9. *The primitive skew curves, K_1 and K_2, are not comparable with respect to quasi dimension type.*

THEOREM 3.10. *The secondary skew curves, K_3 and K_4 are not comparable with respect to quasi dimension type.*

Proof. Suppose $qK_3 \leqq qK_4$. Let A_1 be a subset of K_3 which is a circumscribed figure T (i.e., $A_1 = S \cup B$ where S is a simple closed curve, B is a 3-od with center x and end points a_1, a_2, and a_3; and $S \cap B = a_1 \cup a_2 \cup a_3$). Let $A = A_1 \cup [yb] \cup [zc]$ where $[yb]$ and $[zc]$ are arcs, $[yb] \cap A_1 = y \in (xa_1)$ (an open arc contained in B), and $[zc] \cap A_1 = z$ (an element of the open arc of S from a_1 to a_2 which does not contain a_3). Then, for ε sufficiently small, the ε-image A contains a curve homeomorphic to A. However, K_4 contains no such curve so we have a contradiction.

Suppose $qK_4 \leqq qK_3$. Let A denote a circumscribed figure X (i.e., $A = S \cup B$ where S is a simple closed curve, B is a 4-od with center x and end points a_1, a_2, a_3 and a_4; and $S \cap B = a_1 \cup a_2 \cup a_3 \cup a_4$). Then, for ε sufficiently small, the ε-mage of A contains either a curve which is homeomorphic to A, or a circumscribed figure H. However, K_3 contains no such curves so we have a contradiction.

COROLLARY. *No two of the skew curves are comparable with respect to quasi dimension type.*

4. Quasi dimension type of curves. Recall the following notation: I = closed unit interval, D = dyadic solenoid, ψ = pseudo-arc, $\mathfrak{S}$ = Sierpinski curve and $\mathfrak{M}$ = Menger curve. Then from the results of § 1 and § 2 we have the following.

THEOREM 4.1. *If X is a snake-like continuum, then*
(1) $q\psi \leqq qX \leqq qI$;
(2) *X contains an arc if and only if $qI \leqq qX$;*
(3) *X is a pseudo-arc if and only if $qX \leqq q\psi$.*

DEFINITION 4.1. A *circular chain* is a chain subjected to the added condition that its first and last links intersect. A continuum X is called *circle-like* if for each $\varepsilon > 0$, X can be covered with a circular chain of mesh $< \varepsilon$.

THEOREM 4.2. *If X is a circle-like continuum, then*
(1) $q\psi \leqq qX \leqq qS^1$;
(2) *X is a simple closed curve if and only if $qS^1 \leqq qX$;*
(3) *X is a pseudo-arc if and only if $qX \leqq q\psi$.*
(4) *$qX \leqq qD$ implies that $qX \leqq qI$ or X is not planar.*

Proof. The first inequality of (1) follows from the fact that any circle-like continuum contains a snake continuum and Theorem 1.2′. The second inequality of (1) follows from the fact that a continuum has ε-mappings into the nerves of its coverings.

Only one implication of (2) is nontrivial. Assume $qS^1 \leqq qX$. This implies that X contains a simple closed curve S and since $qX \leqq qS^1$ there are ε-mappings, f_ε, of X into S^1. Then, for ε sufficiently small, f_ε must take S onto all of S^1. Therefore $S = X$. Condition (4) follows from the fact that D is not the continuous image of any planar continuum (see [10]).

DEFINITION 4.2. A *tree chain* is a finite collection of open sets whose nerve is a tree. A continuum X is called *tree-like* if for each

$\varepsilon > 0$, X can be covered with a tree chain of mesh $> \varepsilon$.

DEFINITION 4.3. For any positive integer n, *a curve X is said to be of order less than or equal to n* (in symbols: $\operatorname{ord} X \leqq n$), if $\operatorname{ord}_p X \leqq n$ for each $p \in X$ (see Definition 2.2). Furthermore, if $\operatorname{ord} X \leqq n$ and it is false that $\operatorname{ord} X \leqq m$ for any $m < n$ (i.e., $\operatorname{ord}_p X = n$ for some $p \in X$), then $\operatorname{ord} X = n$.

DEFINITION 4.4. For any positive integer n, *a continuum X is said to be of τ-order less than or equal to n* (in symbols: $\tau\text{-ord } X \leqq n$), if X has coverings of arbitrarily small whose nerves are trees of order $\leqq n$. (The usual convention prevails in defining $\tau\text{-ord } X = n$.) If we replace "τ" by "γ" and "trees" by "linear graphs", then we obtain the definition of $\gamma\text{-ord } X \leqq n$.

Menger [19, p. 318] has shown that there is a dendrite $\mathfrak{U}$ (in the plane) which is universal for dendrites (i.e., any dendrite can be embedded in $\mathfrak{U}$). Moreover, he showed that for any positive integer $n \geqq 2$ there is a dendrite $\mathfrak{U}_n$ of order n which is universal for all dendrites of order n.

LEMMA 4.1. *If X is a tree, then $\tau\text{-ord } X \leqq 3$.*

Proof. Let $b_1, \cdots, b_n$ denote the branch points of X, $k_i = \operatorname{ord}_{b_i} X$ and $\nu = 1/2 \min\{\rho(v_i, v_j) \mid v_i, v_j \text{ are distinct vertices of } X\}$. There is a Δ_i, $0 < \Delta_i < \nu/2$, such that, for any positive number $\Delta \leqq \Delta_i$, $\operatorname{ord}_{b_i} V(b_i, \Delta) = k_i$, for $i = 1, \cdots, n$. Let $r = (1/2)\min\{\Delta_i \mid i = 1, \cdots, n\}$. We will replace each k_i-od in X by a $(k_i\text{-2})$-comb (a *m-comb* is a tree homeomorphic to the union of the closed interval $[0, m+1]$ and m vertical segments from $(i, 0)$ to $(i, 1)$ for $i = 1, \cdots, m$). This will yield a new tree Y of order $\leqq 3$ which is the ε-image of X for each $\varepsilon > 0$.

For a fixed branch point b_i consider the r-neighborhood $S(b_i, r)$ and its boundary points $y_1, \cdots, y_{k_i}$. Let A_1 be an arc which intersects X only in its end points y_1 and y_2. Let $z_3, \cdots, z_{k_i}$ be $(k_i - 2)$ distinct points of $A_1 - (y_1 \cup y_2)$. Now, for $3 \leqq j \leqq k_i$, let A_{j-1} be an arc which intersects $X \cup A_1 \cup \cdots \cup A_{j-2}$ only in its end points z_j and y_j. Finally let $B_i = \bigcup \{A_j \mid j = 1, \cdots, k_i - 1\}$ (so that B_i is a $(k_i - 2)$-comb) and let $Y = [X - \bigcup_{i=1}^{n} S(b_i, r)] \cup [\bigcup_{i=1}^{n} B_i]$. Using the techniques of Theorem 2.2 and Example 2.1 it is not difficult to see that X can be ε-mapped onto Y for all $\varepsilon > 0$.

Now let V be a covering of X with Lebesque number η. There exists a subdivision of Y fine enough so that, for the associated

covering U by stars of vertices, there exists an ε-mapping f of X onto Y with the property that the mesh $f^{-1}(U) < \eta$. Then $f^{-1}(U)$ refines V and its nerve is homeomorphic to Y. Hence X has coverings of arbitrarily small mesh whose nerves are trees of order $\leqq 3$, i.e., τ-ord $X < 3$.

THEOREM 4.3. *A continuum X is tree-like if and only if* τ-ord $X \leqq 3$.

Proof. Suppose X is tree-like (the other implication follows immediately from the definition of τ-order). Then X has coverings $\{V_i\}$ whose nerves $N(V_i) = N_i$ are trees and such that mesh $V_i \leqq 1/i$. Moreover, for any $\varepsilon > 0$, there is an ε-mapping f_ε of X into some N_i. Now by the previous lemma each N_i has coverings $\{W_i^j\}$ whose nerves are trees of order $\leqq 3$ and such that mesh $W_i^j \leqq 1/j$.

In view of the compactness of X, for any $\varepsilon > 0$, there is a positive integer i and a $\eta > 0$ such that: if $A \subset N_i$ and $\delta(A) < \eta$, then $\delta(f_\varepsilon^{-1}(A)) < \varepsilon$. So for j large enough so that mesh $W_i^j < \eta$ we have that mesh $f_\varepsilon^{-1}(W_i^j) < \varepsilon$. Moreover, $N(f_\varepsilon^{-1}(W_i^j))$ is a tree of order $\leqq 3$ and therefore τ-ord $X \leqq 3$.

LEMMA 4.2. *If X and Y are continua, Y is tree-like and $qX \leqq qY$, then X is tree-like.*

THEOREM 4.4. *A continuum X is tree-like if and only if* $qX \leqq q\mathfrak{U}_3$.

Proof. Suppose X is tree-like. Then by the previous theorem Y has coverings of arbitrarily small mesh whose nerves are trees of τ-order $\leqq 3$. So these nerves are embeddable in $\mathfrak{U}_3$. Therefore, for any $\varepsilon > 0$, X can be ε-mapped into $\mathfrak{U}_3$, i.e., $qX \leqq q\mathfrak{U}_3$. Conversely, suppose $qX \leqq q\mathfrak{U}_3$. Now $\mathfrak{U}_3$ is a dendrite and dendrites (being planar and not containing a subcontinuum which separates the plane) are tree-like (see [5, p. 656]). Thus we have by Lemma 4.2 that X is tree-like.

COROLLARY. *For every positive integer i, $q\mathfrak{U}_3 = q\mathfrak{U}_{3+i} = q\mathfrak{U}$.*

Proof. Since $\mathfrak{U}_i \subset \mathfrak{U}_{i+1} \subset \mathfrak{U}$, we have that $q\mathfrak{U}_i \leqq q\mathfrak{U}_{i+1} \leqq q\mathfrak{U}$ for each positive integer i. Since $\mathfrak{U}_i$ and $\mathfrak{U}$ are dendrites they are also tree-like. Thus by the previous theorem $q\mathfrak{U}_i$ and $q\mathfrak{U} \leqq q\mathfrak{U}_3$ and so the desired result follows.

Since there are tree-like continua which fail to contain a snake-like subcontinuum we have as an open question the following.

Question. Is $q\psi \leqq qX$ for any tree-like continuum X?

The appropriate modifications of Lemma 4.1, and Theorems 4.3 and 4.4 yield the following three results.

LEMMA 4.3. *If X is a linear graph, then* γ-ord $X \leqq 3$.

THEOREM 4.5. *A continuum X is 1-dimensional if and only if* γ-ord $X \leqq 3$.

THEOREM 4.6. *A continuum X is 1-dimensional if and only if* $qX \leqq q\mathfrak{M}$.

From the results of § 3 we obtain the next two theorems.

THEOREM 4.7. *If X is a Peano curve which does not contain a primitive skew curve, then*

(1) $qI \leqq qX \leqq q\mathfrak{S}$;

(2) *X is an arc if and only if* $qX \leqq qI$.

COROLLARY. *If X is a continuum with coverings of arbitrarily small mesh whose nerves are planar linear graphs (i.e., do not contain a primitive skew curve), then* $qX \leqq q\mathfrak{S}$ $(< qR^2)$. (For example, tree-like continua have such coverings.)

THEOREM 4.8. *If X is a Peano curve which contains a primitive skew curve, then*

(1) $qX \nleqq qR^2$;

(2) $\left\{\begin{matrix} qK_1 \\ qK_2 \end{matrix}\right\} \leqq qX \leqq q\mathfrak{M}$;

(3) *X is homeomorphic to K_1 if and only if* $qX = qK_1$;

(4) *X is homeomorphic to K_2 if and only if* $qX = qK_2$.

Proof. We consider only one implication from each of (3) and (4) (in that order). Assuming that $qX = qK_1$ by Theorem 3.7 we have that X contains K, a topological image of K_1. So the ε-mappings of X into K_1 must take K onto all of K_1 for ε sufficiently small. So we must have that $X = K$.

Assuming that $qX = qK_2$ we have by Theorem 3.8 that X contains K, a topological image of K_1 or K_2. The former case is impossible since K_1 and K_2 are not comparable with respect to quasi dimension type. So ε-mappings of X into K_2 must take K onto all of K_2 for ε sufficiently small. So we must have that $X = K$.

Added in proof. I intended to construct a snake-like continuum with 2^c different quasi dimension types among its subsets. However, this and the two questions in § 1 remain unsettled. The situation as regards circle-like continua is clearer. H. Cook [Fund. Math. 60 (1967)

241–249] has constructed a 1-dimensional continuum M_1 containing $\boldsymbol{c}$ continua which are circle-like but not snake-like and no one which is the continuous image of the other. It follows that no two of these circle-like continua are comparable with respect to quasi dimension type. Furthermore, since there are at most $\boldsymbol{c}$ topologically different circle-like continua, we may conclude that the cardinality of the set of quasi dimension types of circle-like continua is $\boldsymbol{c}$.

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