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Let M_1 and M_2 be two Riemannian spaces[1] with Riemannian metrics d_1 and d_2 respectively whose sectional curvature is positive constant. We consider the product of the two Riemannian spaces $M_1 \times M_2$, then the Riemannian space $M_1 \times M_2$ has nonnegative sectional curvature with respect to the Riemannian metric $d_1 \times d_2$ but not strictly positive sectional curvature.

We can construct a Riemannian metric on $M_1 \times M_2$ which approaches the Riemannian metric $d_1 \times d_2$ as closely as we wish and which has strictly positive sectional curvature.

Now, our results can be stated as follows. We consider two manifolds $M_1(H_1 - E_1, q_1)$, $M_2(H_2 - E_2, q_2)$ such that each of them has only one chart where H_1, E_1 are the south hemisphere and the equator, respectively, of a k-dimensional sphere ($k \geqq 2$) and E_2, H_2 are also the south hemisphere and the equator, respectively, of an n-dimensional sphere ($n \geqq 2$), and q_1, q_2 are special mappings. We also consider on M_1 and M_2 particular Riemannian metrics d_1, d_2, respectively, with positive constant sectional curvature. We obtain a special 1-parameter family of Riemannian metrics $F(t)$ on $M_1 \times M_2$ such that $F(0) = d_1 \times d_2$. We have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter t for $t = 0$ and for any plane of $(M_1 \times M_2)_P$, is strictly positive.

1. Let M_1 be a manifold which consists of one chart $(H_1 - E_1, q_1)$, where H_1, E_1 are the south hemisphere and the equator, respectively, of a k-dimensional sphere $S_1^k (k \geqq 2)$ and the inverse mapping of q_1 is defined as follows

$$q_1^{-1} = \left\{ x^1 = \frac{2u_1}{1 + u_1^2 + \cdots + u_k^2}, \cdots, x^k = \frac{2u_k}{1 + u_1^2 + \cdots + u_k^2}, \right.$$
$$\left. x^{k+1} = \frac{u_1^2 + \cdots + u_k^2 - 1}{1 + u_1^2 + \cdots + u_k^2} \right\} .$$

q_1 maps the open set $H_1 - E_1$ onto the open ball $u_1^2 + \cdots + u_k^2 < 1$.

On the manifold M_1, we take the following Riemannian metric

[1] A Riemannian space is a Riemannian manifold covered with one chart ([5], p. 314).

$$d_1 = dS_1^2 = \left\{ d_{11} = \cdots = d_{kk} = \frac{4}{(1 + u_1^2 + \cdots + u_k^2)^2} , \right. \tag{1.1}$$
$$\left. d_{ij} = 0 \text{ if } i \neq j \right\} ,$$

whose sectional curvature is positive constant.

Let M_2 be another manifold which consists of one chart $(H_2 - E_2, q_2)$, where H_2, E_2 are the south hemisphere and the equator, respectively, of an n-dimensional sphere $S_2^n (n \geqq 2)$ and the inverse mapping of q_2 is defined by

$$q_2^{-1} = \left\{ x^1 = \frac{2u_{k+1}}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}, \cdots, \right.$$
$$\left. x^n = \frac{2u_{k+n}}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}, x^{n+1} = \frac{u_{k+1}^2 + \cdots + u_{k+n}^2 - 1}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2} \right\} .$$

q_2 maps the open set $H_2 - E_2$ onto the open ball $u_{k+1}^2 + \cdots + u_{k+n}^2 < 0$.

On the manifold M_2, we also take the following Riemannian metric

$$d_2 = dS_2^2 = \left\{ d_{k+1\,k+1} = \cdots = d_{k+n\,k+n} \right. \tag{1.2}$$
$$\left. = \frac{4}{(1 + u_{k+1}^2 + \cdots + u_{k+n}^2)^2}, d_{ij} = 0 \text{ if } i \neq j \right\} ,$$

whose sectional curvature is positive constant.

Consider the product of the two manifolds $M_1 \times M_2$. Then $M_1 \times M_2$ is a manifold with one chart $\{(H_1 - E_1) \times (H_2 - E_2), q_1 \times q_2\}$.

We define a 1-parameter family of Riemannian metrics on the manifold $M_1 \times M_2$ defined by

$$dS^2(t) = \begin{cases} g_{11} = \cdots = g_{kk} = \dfrac{4(1 + tf)}{(1 + u_1^2 + \cdots + u_k^2)^2} , \\ g_{k+1\,k+1} = \cdots = g_{k+n\,k+n} \\ \qquad = \dfrac{4(1 + t\varphi)}{(1 + u_{k+1}^2 + \cdots + u_{k+n}^2)^2}, g_{ij} = 0 \text{ if } i \neq j , \end{cases} \tag{1.3}$$

where $-b < t < b, \varphi = \varphi(u_1, \cdots, u_k), f = f(u_{k+1}, \cdots, u_{k+n})$.

The Riemannian metric $dS^2(0)$ coincides with the product Riemannian metric $dS_1^2 \times dS_2^2$ of the two manifolds M_1 and M_2.

2. We shall calculate the components R_{hijk} of the Riemannian curvature tensor when the index $h = 1$, because the other cases are similar to these.

If $h = 1$, there exist the following distinguished cases in which R_{1ijk} do not vanish identically.

R_{1j1j}, $j = 2, \cdots, k$, $R_{1k+j1k+j}$, $j = 1, \cdots, n$,
R_{1jjl}, $j \neq l$, $j = 2, \cdots, k$, $l = 2, \cdots, k$,
R_{1jjk+l}, $j = 2, \cdots, k$, $l = 1, \cdots, n$,
$R_{1k+j\,k+jl}$, $j = 1, \cdots, n$, $l = 2, \cdots, k$,
R_{1ijl}, $i \neq j \neq l$, $i = 2, \cdots, k + n$, $j = 2, \cdots, k + n$, $l = 2, \cdots, k + n$.

As it is known, R_{1ijk} is given by ([12], p. 18)

$$R_{1ijl} = \frac{1}{2}\left(\frac{\partial^2 g_{1j}}{\partial u_i \partial u_l} + \frac{\partial^2 g_{il}}{\partial u_1 \partial u_j} - \frac{\partial^2 g_{ij}}{\partial u_1 \partial u_l} - \frac{\partial^2 g_{1l}}{\partial u_i \partial u_j}\right) - g_{rs}\left(\begin{Bmatrix} r \\ ij \end{Bmatrix}\begin{Bmatrix} s \\ 1l \end{Bmatrix} - \begin{Bmatrix} r \\ il \end{Bmatrix}\begin{Bmatrix} s \\ 1j \end{Bmatrix}\right),$$

where $\begin{Bmatrix} r \\ ij \end{Bmatrix}, \begin{Bmatrix} s \\ 1l \end{Bmatrix}, \begin{Bmatrix} r \\ il \end{Bmatrix}, \begin{Bmatrix} s \\ 1j \end{Bmatrix}$ are the Christoffel symbols of the second kind.

From the above formula by virtue of (1.3) we obtain

$$R_{1j1j} = -\frac{16(1 + tf)}{A^4} + \frac{t^2}{1 + t\varphi}\frac{B^2}{A^4}\sum_{i=1}^{n}\left(\frac{\partial f}{\partial u_{k+i}}\right)^2, j = 2, \cdots, k, \tag{2.1}$$

$$\begin{aligned} R_{1\,k+j1k+j} = {} & \frac{2t}{(AB)^2}\left\{A^2\frac{\partial^2 \varphi}{\partial u_1^2} + 2Au_1\frac{\partial \varphi}{\partial u_1} - 2A\sum_{i=2}^{k} u_i\frac{\partial \varphi}{\partial u_i}\right. \\ & \left. + B^2\frac{\partial^2 f}{\partial u_{k+j}^2} + 2Bu_{k+j}\frac{\partial f}{\partial u_{k+j}} - 2B\sum_{i\neq j}^{n} u_{k+i}\frac{\partial f}{\partial u_{k+i}}\right\} \\ & - t^2\left\{\frac{\left(\dfrac{\partial f}{\partial u_{k+j}}\right)^2}{(1 + tf)A^2} + \frac{\left(\dfrac{\partial \varphi}{\partial u_1}\right)^2}{(1 + t\varphi)B^2}\right\}, j = 1, \cdots, n, \end{aligned} \tag{2.2}$$

$$R_{1jjl} = 0, j \neq l, j = 2, \cdots, k, l = 2, \cdots, k, \tag{2.3}$$

$$R_{1jj\,k+l} = t^2\frac{\dfrac{\partial f}{\partial u_{k+l}}\dfrac{\partial \varphi}{\partial u_1}}{(1 + t\varphi)A^2}, j = 2, \cdots, k, l = 1, \cdots, n, \tag{2.4}$$

$$\begin{aligned} R_{1k+j\,k+jl} = {} & -\frac{2t}{B^2}\left\{\frac{\partial^2 \varphi}{\partial u_1 \partial u_l} + \frac{2u_1}{A}\frac{\partial \varphi}{\partial u_l} + \frac{2u_l}{A}\frac{\partial \varphi}{\partial u_1}\right\} \\ & + t^2\frac{\dfrac{\partial \varphi}{\partial u_1}\dfrac{\partial \varphi}{\partial u_l}}{(1 + t\varphi)B^2}, j = 1, \cdots, n, l = 2, \cdots, k, \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_{1ijl} = 0, \ & i \neq j \neq l, i = 2, \cdots, k + n, \\ & j = 2, \cdots, k + n, l = 2, \cdots, k + n, \end{aligned} \tag{2.6}$$

where

$$(2.7) \qquad A = 1 + u_1^2 + \cdots + u_k^2 , \qquad B = 1 + u_{k+1}^2 + \cdots + u_{k+n}^2 .$$

If the functions φ and f are chosen such that they satisfy the systems of partial differential equations

$$(2.8) \qquad \frac{\partial^2 \varphi}{\partial u_i \partial u_j} + \frac{2u_i}{A}\frac{\partial \varphi}{\partial u_j} + \frac{2u_j}{A}\frac{\partial \varphi}{\partial u_i} = 0 , \qquad i \neq j, i = 1, \cdots, k, j = 1, \cdots, k ,$$

$$(2.9) \qquad \frac{\partial^2 f}{\partial u_h \partial u_l} + \frac{2u_h}{B}\frac{\partial f}{\partial u_l} + \frac{2u_l}{B}\frac{\partial f}{\partial u_h} = 0 , \qquad h \neq l, h = k+1, \cdots, k+n, l = k+1, \cdots, k+n ,$$

respectively and if $m \in [1, \cdots, k]$ and

$$i \in [k+1, \cdots, k+n], i \neq j \in [k+1, \cdots, k+n]$$

or if $m \in [k+1, \cdots, k+n]$ and $i \in [1, \cdots, k], i \neq j \in [1, \cdots, k]$, then we have

$$(2.10) \qquad R_{immj} = t^2 \frac{\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j}}{(1+tf)A^2} , \quad \text{or} \quad R_{immj} = t^2 \frac{\frac{\partial \varphi}{\partial u_i}\frac{\partial \varphi}{\partial u_j}}{(1+t\varphi)B^2} .$$

We consider one partial differential equation of the system (2.8), for example,

$$\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{2u_1}{A}\frac{\partial \varphi}{\partial u_2} + \frac{2u_2}{A}\frac{\partial \varphi}{\partial u_1} = 0 ,$$

or

$$(2.11) \qquad \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \log A}{\partial u_1}\frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2}\frac{\partial \varphi}{\partial u_1} = 0 .$$

From the first of (2.7), we conclude that

$$(2.12) \qquad \frac{\partial^2 \log A}{\partial u_1 \partial u_2} = - \frac{\partial \log A}{\partial u_1}\frac{\partial \log A}{\partial u_2} .$$

Equation (2.11), by virtue of (2.12), takes the form

$$\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \log A}{\partial u_1}\frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2}\frac{\partial \varphi}{\partial u_1} + \frac{\partial^2 \log A}{\partial u_1 \partial u_2}\varphi + \frac{\partial \log A}{\partial u_1}\frac{\partial \log A}{\partial u_2}\varphi = 0 ,$$

or

$$\frac{\partial}{\partial u_1}\left\{\frac{\partial \varphi}{\partial u_2}+\frac{\partial \log A}{\partial u_2}\varphi\right\}+\frac{\partial \log A}{\partial u_1}\left\{\frac{\partial \varphi}{\partial u_2}+\frac{\partial \log A}{\partial u_2}\varphi\right\}=0\,,$$

from which we obtain

$$\frac{\partial \varphi}{\partial u_2}+\frac{\partial \log A}{\partial u_2}\varphi-\frac{v}{A}=0\,, \tag{2.13}$$

where v is an arbitrary function of $u_2, \cdots, u_k$.

Equation (2.13) is a linear differential equation whose general solution is

$$\varphi=\frac{1}{A}\left(z+\int v du_2\right)\,, \tag{2.14}$$

where z is an arbitrary function of $u_1, u_3, \cdots, u_k$.

Relation (2.14), by virtue of the first of (2.7), takes the form

$$\varphi=\alpha\frac{\mu(u_1, u_3, \cdots, u_k)+\pi(u_2, \cdots, u_k)}{1+u_1^2+\cdots+u_k^2}\,, \tag{2.15}$$

where $z=\alpha\mu$, $\int v du_2=\alpha\pi$ and α is an arbitrary real constant.

In order for the function φ to satisfy the rest of partial differential equations of the system (2.8), as it is easily proved that it must have the form

$$\varphi=\alpha\frac{\varphi_1(u_1)+\cdots+\varphi_k(u_k)}{1+u_1^2+\cdots+u_k^2}\,, \tag{2.16}$$

where $\varphi_1, \cdots, \varphi_k$ are arbitrary functions of $u_1, \cdots, u_k$, respectively.

Similarly, in order for the function f to satisfy the system of partial differential equations (2.9), it must have the form

$$f=\alpha\frac{f_{k+1}(u_{k+1})+\cdots+f_{k+n}(u_{k+n})}{1+u_{k+1}^2+\cdots+u_{k+n}^2}\,, \tag{2.17}$$

where $f_{k+1}, \cdots, f_{k+n}$ are arbitrary functions of $u_{k+1}, \cdots, u_{k+n}$, respectively.

From (2.1), (2.2), (2.4) and (2.10), we obtain

$$R_{1j1j}(0)=-\frac{16}{A^4},\ R'_{1j1j}(0)=-\frac{16f}{A^4},\ j=2, \cdots, k\,, \tag{2.18}$$

$$\begin{aligned}R_{1k+j\,1k+j}(0)=0,\ R'_{1k+j\,1k+j}(0)&=\frac{2}{(AB)^2}\left\{A^2\frac{\partial^2\varphi}{\partial u_1^2}+2Au_1\frac{\partial\varphi}{\partial u_1}\right.\\ &\left.-2A\sum_{i=2}^{k}u_i\frac{\partial\varphi}{\partial u_i}+B^2\frac{\partial^2 f}{\partial u_{k+j}^2}+2Bu_{k+j}\frac{\partial f}{\partial u_{k+j}}-2B\sum_{i\neq j}^{n}u_{k+i}\frac{\partial f}{\partial u_{k+i}}\right\}\,,\\ &j=1, \cdots, n\end{aligned} \tag{2.19}$$

$$R_{1jjk+l}(0) = R'_{1jjk+l}(0) = 0 \ , \quad j = 2, \cdots, l = 1, \cdots, n \ , \tag{2.20}$$

$$R_{1k+j\,k+jl}(0) = R'_{1k+j\,k+jl}(0) = 0 \ , \quad j = 1, \cdots, n, l = 1, \cdots, n \ , \tag{2.21}$$

where R'_{hijl} denotes the derivative of R_{hijl} with respect to the parameter t.

From (1.1), (1.2) and (1.3), we obtain the following formulas

$$\begin{cases} g_{11}(0) = \cdots = g_{kk}(0) = d_{11}, \\ \qquad g_{k+1\,k+1}(0) = \cdots = g_{k+n\,k+n}(0) = d_{k+n\,k+n} \ , \\ g'_{11}(0) = \cdots = g'_{kk}(0) = fd_{11}, \\ \qquad g'_{k+1\,k+1}(0) = \cdots = g'_{k+n\,k+n}(0) = \varphi d_{k+n\,k+n} \ . \end{cases} \tag{2.22}$$

Relations (2.18) and (2.19) by means of (2.7) and (2.22) take the form

$$R_{1j1j} = -d_{11}^2, R'_{1j1j}(0) = -fd_{11}^2 \ , \qquad j = 2, \cdots, k \ , \tag{2.23}$$

$$\begin{gathered} R_{1k+j\,1k+j}(0) = 0, R'_{1k+j\,1k+j}(0) = \frac{d_{11}d_{k+1\,k+1}}{8}\left\{A^2\frac{\partial^2\varphi}{\partial u_1^2} + 2Au_1\frac{\partial\varphi}{\partial u_1}\right. \\ \left. - 2A\sum_{i=2}^{k} u_i\frac{\partial\varphi}{\partial u_i} + B^2\frac{\partial^2 f}{\partial u_{k+j}^2} + 2Bu_{k+j}\frac{\partial f}{\partial u_{k+j}} - 2B\sum_{i\neq j}^{n} u_{k+j}\frac{\partial f}{\partial u_{k+i}}\right\} \\ j = 1, \cdots, k \ . \end{gathered} \tag{2.24}$$

3. Let P be any point of $M_1 \times M_2$. Then the $k + n$ vectors $\partial/\partial u_1, \cdots, \partial/\partial u_k, \partial/\partial u_{k+1}, \cdots, \partial/\partial u_{k+n}$ form an orthonormal basis of the tangent space $(M_1 \times M_2)_P$.

As it is known, the sectional curvature of the plane spanned by $\partial/\partial u_1, \partial/\partial u_j, j = 2, \cdots, k$, is given by

$$K_{1j} = -\frac{R_{1j1j}}{g_{11}g_{jj}} \ , \qquad j = 2, \cdots, k \ ,$$

which implies

$$K'_{1j}(0) = -\frac{R'_{1j1j}(0)g_{11}(0)g_{jj}(0) - R_{1j1j}(0)\{g'_{11}(0)g_{jj}(0) + g_{11}(0)g'_{jj}(0)\}}{g_{11}^2(0)g_{jj}^2(0)} \tag{3.1}$$

Relation (3.1), by virtue of (2.22) and (2.23), takes the form

$$K'_{1j}(0) = -f \ . \tag{3.2}$$

Similarly, calculating $K'_{k+1\,k+j}(0)$, we obtain

$$K'_{k+1\,k+j}(0) = -\varphi \ . \tag{3.3}$$

Formulas (3.2) and (3.3), by means of (2.16) and (2.17), take the form

$$K'_{1j}(0) = -\alpha\frac{f_{k+1}(u_{k+1}) + \cdots + f_{k+n}(u_{k+n})}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2} ,$$

$$K'_{k+1\,k+j}(0) = -\alpha\frac{\varphi_1(u_1) + \cdots + \varphi_k(u_k)}{1 + u_1^2 + \cdots + u_k^2} ,$$

respectively. In order for $K'_{ij}(0)$, $K'_{k+1\,k+j}(0)$ to be positive, we must have $\alpha < 0$, $f_{k+j}(u_{k+j}) > 0$, $j = 1, \cdots, n$, $\varphi_i(u_i) > 0$, $i = 1, \cdots, k$, which means the real number α must be negative and the functions $f_{k+j}(u_{k+j})$ and $\varphi_i(u_i)$ must be positive when the corresponding variable takes values in the interval$(-1, 1)$.

The sectional curvature of the plane spanned by $\partial/\partial u_l$, $\partial/\partial u_{k+j}$ is given by

$$K_{l k+j} = -\frac{R_{l k+j\, l k+j}}{g_{ll} g_{k+j\, k+j}} , \qquad l = 1, \cdots, k, j = 1, \cdots, n ,$$

which, by virtue of (2.22) and either (2.24) or similar to (2.24), takes the form

$$\begin{aligned} K'_{l k+j}(0) = -\frac{1}{8}\Big\{ & A^2\frac{\partial^2\varphi}{\partial u_l^2} + 2Au_l\frac{\partial\varphi}{\partial u_l} - 2A\sum_{i\neq l}^{k} u_i\frac{\partial\varphi}{\partial u_i} \\ & + B^2\frac{\partial f^2}{\partial u_{k+j}^2} + 2Bu_{k+j}\frac{\partial f}{\partial u_{k+j}} - 2B\sum_{i\neq j}^{n} u_{k+i}\frac{\partial f}{\partial u_{k+i}}\Big\} . \end{aligned} \tag{3.4}$$

In order for $K'_{l k+j}(0)$ to be positive and because the functions φ and f are independent, it must be

$$A^2\frac{\partial^2\varphi}{\partial u_l^2} + 2Au_l\frac{\partial\varphi}{\partial u_l} - 2A\sum_{i\neq l}^{k} u_i\frac{\partial\varphi}{\partial u_i} < 0 , \qquad l = 1, \cdots, k , \tag{3.5}$$

$$B^2\frac{\partial^2 f}{\partial u_{k+j}^2} + 2Bu_{k+j}\frac{\partial f}{\partial u_{k+j}} - 2B\sum_{i\neq j}^{n} u_{k+i}\frac{\partial f}{\partial u_{k+i}} < 0 , \qquad j = 1, \cdots, n . \tag{3.6}$$

Inequalities (3.5) and (3.6), by virtue of (2.16) and (2.17), become

$$\frac{\alpha}{A}\Big\{A^2\frac{d^2\varphi_l}{du_l^2} - 2A\sum_{i=1}^{k} u_i\frac{d\varphi_i}{du_i} - 2(2 - A)\sum_{i=1}^{k}\varphi_i\Big\} < 0 , \qquad l = 1, \cdots, k ,$$

$$\frac{\alpha}{B}\Big\{B^2\frac{d^2 f_{k+j}}{du_{k+j}^2} - 2B\sum_{i=1}^{n} u_{k+i}\frac{df_{k+i}}{du_{k+i}} - 2(2 - B)\sum_{i=1}^{n} f_{k+i}\Big\} < 0 , \qquad j = 1, \cdots, n ,$$

which imply

$$(3.7)\quad \begin{cases} A^2\dfrac{d^2\varphi_l}{du_l^2} - 2A\sum\limits_{i=1}^{k} u_i\dfrac{d\varphi_i}{du_i} - 2(2-A)\sum\limits_{i=1}^{k}\varphi_i > 0\ , \qquad l = 1, \cdots, k\ , \\ B^2\dfrac{d^2 f_{k+j}}{du_{k+j}^2} - 2B\sum\limits_{i=1}^{n} u_{k+i}\dfrac{df_{k+i}}{du_{k+i}} - 2(2-B)\sum\limits_{i=1}^{n} f_{k+i} > 0\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \cdots, n\ . \end{cases}$$

If the functions $f_{k+j} = f_{k+j}(u_{k+j})$, $\varphi_i = \varphi_i(u_i)$ are chosen to have the form

$$(3.8)\qquad f_{k+j} = u_{k+j}^2 + \frac{1}{2n}, j = 1, \cdots, n, \varphi_i = u_i^2 + \frac{1}{2k}, i = 1, \cdots, k\ ,$$

then the inequalities (3.7) take the form

$$2 - A > 0\ , \qquad 2 - B > 0\ ,$$

which, by virtue of (2.7), become

$$1 - u_1^2 - \cdots - u_k^2 > 0\ , \qquad 1 - u_{k+1}^2 - \cdots - u_{k+n}^2 > 0\ ,$$

which are valid on the open balls $u_1^2 + \cdots + u_k^2 < 1, u_{k+1}^2 + \cdots + u_{k+n}^2 < 1$, respectively.

Relations (2.16) and (2.17), by means of (3.8), take the form

$$(3.9)\quad f = \alpha\frac{u_{k+1}^2 + \cdots + u_{k+n}^2 + 1/2}{u_{k+1}^2 + \cdots + u_{k+n}^2 + 1}\ , \qquad \varphi = \alpha\frac{u_1^2 + \cdots + u_k^2 + 1/2}{u_1^2 + \cdots + u_k^2 + 1}\ .$$

The second of (2.24) or similar to that and (3.4), by means of (3.9), become

$$\begin{aligned} R'_{lk+j\,lk+j}(0) &= \frac{2\alpha}{(1 + u_1^2 + \cdots + u_k^2)^2(1 + u_{k+1}^2 + \cdots + u_{k+n}^2)^2} \\ &\quad\times\left\{\frac{1 - u_1^2 - \cdots - u_k^2}{1 + u_1^2 + \cdots + u_k^2} + \frac{1 - u_{k+1}^2 - \cdots - u_{k+n}^2}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}\right\}\ , \\ K'_{lk+j}(0) &= -\frac{\alpha}{8}\left\{\frac{1 - u_1^2 - \cdots - u_k^2}{1 + u_1^2 + \cdots + u_k^2} + \frac{1 - u_{k+1}^2 - \cdots - u_{k+n}^2}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}\right\}\ , \\ &\qquad\qquad\qquad\qquad l = 1, \cdots, k, j = 1, \cdots, n\ . \end{aligned}$$

Using the fact that $\alpha < 0$, then following inequalities are obtained from the above relations:

$$(3.10)\quad R'_{lk+j\,lk+j}(0) < 0\ , \quad K'_{lk+j}(0) > 0\ , \quad l = 1, \cdots, k, j = 1, \cdots, n\ ,$$

which are valid on the open balls $u_1^2 + \cdots + u_k^2 < 1, u_{k+1}^2 + \cdots + u_{k+n}^2 < 1$.

Let $\xi(\xi^1, \cdots, \xi^{k+n})$ and $z(z^1, \cdots, z^{k+n})$ be any two vectors of the tangent space $(M_1 \times M_2)_P$. The sectional curvature of the plane spanned by ξ and z is given by ([11], p. 12)

$$K = \frac{R_{hijl} z^h z^j \xi^i \xi^l}{(g_{hl} g_{ij} - g_{hj} g_{il}) z^h z^j \xi^i \xi^l} ,$$

or

$$K = \frac{A_1}{B_1} , \tag{3.11}$$

where

$$A_1 = R_{hijl} z^h z^j \xi^i \xi^l , \qquad B_1 = (g_{hl} g_{ij} - g_{hj} g_{il}) z^h z^j \xi^i \xi^l . \tag{3.12}$$

From (3.11), the following is obtained:

$$K'(0) = \frac{A_1'(0) B_1(0) - A_1(0) B_1'(0)}{B_1^2(0)} . \tag{3.13}$$

From (3.12), by virtue of (2.3), (2.6), (2.20), (2.21), (2.22), (2.23), (2.24) and similar formulas to (2.23) and (2.24), we obtain

$$\begin{aligned} A_1(0) &= - C d_{11}^2 - D d_{k+1\,k+1}^2 , \\ A_1'(0) &= - f C d_{11}^2 - \varphi D d_{k+1\,k+1}^2 + T , \end{aligned} \tag{3.14}$$

$$B_1(0) = - C d_{11}^2 - D d_{k+1\,k+1}^2 - E d_{11} d_{k+1\,k+1} , \tag{3.15}$$

$$B_1'(0) = - 2 f C d_{11}^2 - 2 \varphi D d_{k+1\,k+1}^2 - (f + \varphi) E d_{11} d_{k+1\,k+1} , \tag{3.16}$$

where

$$C = \sum_{i=1}^{k} \sum_{i<j=2}^{k} \alpha_{ij}^2 , \quad D = \sum_{i=k+1}^{k+n} \sum_{i<j=k+2}^{k+n} \alpha_{ij}^2 , \quad E = \sum_{i=1}^{k} \sum_{j=1}^{n} \alpha_{ik+j}^2 , \tag{3.17}$$

$$T = \sum_{l=1}^{k} \sum_{j=1}^{n} R'_{lk+j\,lk+j}(0) \alpha_{lk+j}^2 , \; \alpha_{jm} = (z^i \xi^m - z^m \xi^i) . \tag{3.18}$$

Relation (3.13), by means of (3.14), takes the form

$$K'(0) = \frac{T B_1(0) + C G d_{11}^2 + D J d_{k+1\,k+1}^2}{B_1^2(0)} , \tag{3.19}$$

where

$$G = B_1'(0) - f B_1(0) , \qquad J = B_1'(0) - \varphi B_1(0) . \tag{3.20}$$

Formulas (3.20), by virtue of (3.15), and (3.16), become

$$G = L - (2\varphi - f) D d_{k+1\,k+1}^2 , \qquad J = N - (2f - \varphi) C d_{11}^2 , \tag{3.21}$$

where

$$\begin{aligned} L &= - \varphi E d_{11} d_{k+1\,k+1} - f C d_{11}^2 , \\ N &= - f E d_{11} d_{k+1\,k+1} - \varphi D d_{k+1\,k+1}^2 . \end{aligned} \tag{3.22}$$

Relation (3.19), by means of (3.21), takes the form

$$(3.23)\quad K'(0) = \frac{TB_1(0) + CLd_{11}^2 + DNd_{k+1\,k+1}^2 - (f + \varphi)CDd_{11}^2 d_{k+1\,k+1}^2}{B_1^2(0)} .$$

From (3.15) and (3.22), by means of (3.17), and because the functions f and φ are negative, we conclude

$$(3.24)\qquad B_1(0) < 0 , \quad L \geqslant 0 , \quad N \geqslant 0 .$$

The first of (3.18), by virtue of the first inequality of (3.10), implies

$$(3.25)\qquad T \leqq 0 .$$

Formula (3.23), by means of (3.17), (3.24), (3.25) and $f < 0$, $\varphi < 0$, implies

$$K'(0) > 0 ,$$

because it is not possible that simultaneously $C = D = T = 0$ for the two vectors ξ and z.

Hence, we have the following theorem.

THEOREM. *Let M_1 and M_2 be two special Riemannian spaces with constant positive sectional curvature defined in* § 1. *If we consider a special* 1*-parameter family of Riemannian metrics $F(t)$ on $M_1 \times M_2$ defined by* (1.3), *where the functions f, φ have the form* (3.9), *then the derivative of the sectional curvature with respect to the parameter t for $t = 0$ and for any plane of $(M_1 \times M_2)_P$ and $\forall P \in M_1 \times M_2$ is strictly positive.*

From the above, we conclude that, if the parameter t is positive and small enough, then the corresponding Riemannian metric $F(t)$ defined by (1.3) on $M_1 \times M_2$, where the functions f and φ have the form (3.9), has strictly positive sectional curvature.

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