A RIEMANNIAN SPACE WITH STRICTLY POSITIVE
SECTIONAL CURVATURE

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Let $M_1$ and $M_2$ be two Riemannian spaces with Riemannian metrics $d_1$ and $d_2$ respectively whose sectional curvature is positive constant. We consider the product of the two Riemannian spaces $M_1 \times M_2$, then the Riemannian space $M_1 \times M_2$ has nonnegative sectional curvature with respect to the Riemannian metric $d_1 \times d_2$ but not strictly positive sectional curvature.

We can construct a Riemannian metric on $M_1 \times M_2$ which approaches the Riemannian metric $d_1 \times d_2$ as closely as we wish and which has strictly positive sectional curvature.

Now, our results can be stated as follows. We consider two manifolds $M_i(H_i - E_i, q_i), M_2(H_2 - E_2, q_2)$ such that each of them has only one chart where $H_i, E_i$ are the south hemisphere and the equator, respectively, of a $k$-dimensional sphere $(k \geq 2)$ and $E_2, H_2$ are also the south hemisphere and the equator, respectively, of an $n$-dimensional sphere $(n \geq 2)$, and $q_1, q_2$ are special mappings. We also consider on $M_1$ and $M_2$ particular Riemannian metrics $d_1, d_2$, respectively, with positive constant sectional curvature. We obtain a special 1-parameter family of Riemannian metrics $F(t)$ on $M_1 \times M_2$ such that $F(0) = d_1 \times d_2$. We have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter $t$ for $t = 0$ and for any plane of $(M_1 \times M_2)_P$, is strictly positive.

1. Let $M_i$ be a manifold which consists of one chart $(H_i - E_i, q_i)$, where $H_i, E_i$ are the south hemisphere and the equator, respectively, of a $k$-dimensional sphere $S_k^i(k \geq 2)$ and the inverse mapping of $q_i$ is defined as follows

$$q_i^{-1} = \left\{ x^1 = \frac{2u_i}{1 + u_i^1 + \cdots + u_i^k}, \ldots, x^k = \frac{2u_k}{1 + u_1^1 + \cdots + u_k^k}, \right\} \quad x^{k+1} = \frac{u_1^2 + \cdots + u_k^2 - 1}{1 + u_1^1 + \cdots + u_k^k}).$$

$q_i$ maps the open set $H_i - E_i$ onto the open ball $u_1^2 + \cdots + u_k^2 < 1$.

On the manifold $M_i$, we take the following Riemannian metric

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1 A Riemannian space is a Riemannian manifold covered with one chart ([5], p. 314).
where sectional curvature is positive constant.

Let $M_2$ be another manifold which consists of one chart $(H_2 - E_2, q_2)$, where $H_2, E_2$ are the south hemisphere and the equator, respectively, of an $n$-dimensional sphere $S_n^*(n \geq 2)$ and the inverse mapping of $q_2$ is defined by

\[
q_2^{-1} = \left\{ x^i = \frac{2u_{k+1}}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}, \cdots, \\
x^n = \frac{2u_{k+n}}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2}, x^{n+1} = \frac{u_{k+1}^2 + \cdots + u_{k+n}^2 - 1}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2} \right\},
\]

$q_2$ maps the open set $H_2 - E_2$ onto the open ball $u_{k+1}^2 + \cdots + u_{k+n}^2 < 0$.

On the manifold $M_2$, we also take the following Riemannian metric

\[
d_2 = dS_2^* = \left\{ d_{k+1,k+1} = \cdots = d_{k+n,k+n} = \frac{4}{(1 + u_{k+1}^2 + \cdots + u_{k+n}^2)^2}, d_{ij} = 0 \text{ if } i \neq j \right\},
\]

whose sectional curvature is positive constant.

Consider the product of the two manifolds $M_1 \times M_2$. Then $M_1 \times M_2$ is a manifold with one chart $\{(H_1 - E_1) \times (H_2 - E_2), q_1 \times q_2\}$.

We define a 1-parameter family of Riemannian metrics on the manifold $M_1 \times M_2$ defined by

\[
dS^*(t) = \left\{ g_{11} = \cdots = g_{kk} = \frac{4(1 + tf)}{(1 + u_1^2 + \cdots + u_k^2)^2}, \cdots, g_{k+1,k+1} = \cdots = g_{k+n,k+n} = \frac{4(1 + t\varphi)}{(1 + u_{k+1}^2 + \cdots + u_{k+n}^2)^2}, g_{ij} = 0 \text{ if } i \neq j \right\},
\]

where $-b < t < b$, $\varphi = \varphi(u_1, \cdots, u_k)$, $f = f(u_{k+1}, \cdots, u_{k+n})$.

The Riemannian metric $dS^*(0)$ coincides with the product Riemannian metric $dS_1^* \times dS_2^*$ of the two manifolds $M_1$ and $M_2$.

2. We shall calculate the components $R_{hijk}$ of the Riemannian curvature tensor when the index $h = 1$, because the other cases are similar to these.

If $h = 1$, there exist the following distinguished cases in which $R_{1ijk}$ do not vanish identically.
\[ R_{ij}, j = 2, \ldots, k, R_{ik+jk+}, j = 1, \ldots, n, \]
\[ R_{ijl}, j \neq l, j = 2, \ldots, k, l = 2, \ldots, k, \]
\[ R_{ijkl+}, j = 2, \ldots, k, l = 1, \ldots, n, \]
\[ R_{ikt+k+l}, j = 1, \ldots, n, l = 2, \ldots, k, \]
\[ R_{ijkl}, i \neq j \neq l, i = 2, \ldots, k + n, j = 2, \ldots, k + n, l = 2, \ldots, k + n. \]

As it is known, \( R_{ij} \) is given by \((12), p. 18)\)
\[
R_{ij} = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial u_i} + \frac{\partial g_{kl}}{\partial u_k} - \frac{\partial g_{ik}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_j} \right) - g_{ij} \left( \frac{\partial r_i}{\partial u_i} \frac{s_j}{\partial u_j} - \frac{\partial r_i}{\partial u_i} \frac{s_j}{\partial u_j} \right),
\]
where \( \{r_i\}, \{s_j\}, \{r_i\}, \{s_j\} \) are the Christoffel symbols of the second kind.

From the above formula by virtue of \( (1.3) \) we obtain
\[
(2.1) \quad R_{ij} = - \frac{16(1 + tf)}{A^2} + \frac{t^2}{1 + t\varphi} \frac{B^2}{A} \sum_{i=1}^{n} \left( \frac{\partial f_i}{\partial u_{k+i}} \right)^2, j = 2, \ldots, k,
\]
\[
(2.2) \quad R_{ik+jk+j} = \frac{2t}{(AB)^2} \left\{ A \frac{\partial \varphi}{\partial u_i} + 2Au_i \frac{\partial \varphi}{\partial u_i} - 2A \sum_{i=1}^{k} u_i \frac{\partial \varphi}{\partial u_i} + B^2 \frac{\partial f_i}{\partial u_{k+j}} + 2Bu_{k+j} \frac{\partial f_i}{\partial u_{k+j}} - 2B \sum_{i=1}^{n} u_{k+i} \frac{\partial f_i}{\partial u_{k+i}} \right\}, j = 1, \ldots, n,
\]
\[
(2.3) \quad R_{ijl} = 0, j \neq l, j = 2, \ldots, k, l = 2, \ldots, k,
\]
\[
(2.4) \quad R_{ijl}, l = 2, \ldots, k, l = 1, \ldots, n,
\]
\[
(2.5) \quad R_{ik+jk+j} = - \frac{2t}{B^2} \left\{ \frac{\partial \varphi}{\partial u_{k+i}} \frac{\partial u_i}{\partial u_i} + 2u_i \frac{\partial \varphi}{\partial u_i} + 2u_i \frac{\partial \varphi}{\partial u_i} \right\} + t^2 \frac{\partial \varphi}{\partial u_i} \frac{\partial \varphi}{\partial u_i} \frac{\partial f_i}{\partial u_{k+i}} , j = 1, \ldots, n, l = 2, \ldots, k,
\]
\[
(2.6) \quad R_{i+j}, i \neq j \neq l, i = 2, \ldots, k + n,
\]
\[
\quad j = 2, \ldots, k + n, l = 2, \ldots, k + n,
\]
where
If the functions $\phi$ and $f$ are chosen such that they satisfy the systems of partial differential equations

\[
\frac{\partial^2 \phi}{\partial u_i \partial u_j} + \frac{2u_i}{A} \frac{\partial \phi}{\partial u_i} + \frac{2u_j}{A} \frac{\partial \phi}{\partial u_j} = 0 ,
\]

\[i \neq j, i = 1, \ldots, k, j = 1, \ldots, k,
\]

\[
\frac{\partial^2 f}{\partial u_h \partial u_l} + \frac{2u_h}{B} \frac{\partial f}{\partial u_h} + \frac{2u_l}{B} \frac{\partial f}{\partial u_l} = 0 ,
\]

\[h \neq l, h = k + 1, \ldots, k + n, l = k + 1, \ldots, k + n,
\]

respectively and if $m \in [1, \ldots, k]$ and

\[i \in [k + 1, \ldots, k + n], i \neq j \in [k + 1, \ldots, k + n]
\]

or if $m \in [k + 1, \ldots, k + n]$ and $i \in [1, \ldots, k], i \neq j \in [1, \ldots, k]$, then we have

\[
R_{i,i,m,j} = t^2 \frac{\frac{\partial f}{\partial u_i} \frac{\partial f}{\partial u_j}}{(1 + t^2)A^2}, \quad \text{or} \quad R_{i,i,m,j} = t^2 \frac{\frac{\partial \phi}{\partial u_i} \frac{\partial \phi}{\partial u_j}}{(1 + t^2)B^2}.
\]

We consider one partial differential equation of the system (2.8), for example,

\[
\frac{\partial^2 \phi}{\partial u_i \partial u_z} + \frac{2u_i}{A} \frac{\partial \phi}{\partial u_i} + \frac{2u_z}{A} \frac{\partial \phi}{\partial u_z} = 0 ,
\]

or

\[
\frac{\partial^2 \phi}{\partial u_i \partial u_z} + \frac{\partial \log A}{\partial u_i} \frac{\partial \phi}{\partial u_z} + \frac{\partial \log A}{\partial u_z} \frac{\partial \phi}{\partial u_i} = 0 .
\]

From the first of (2.7), we conclude that

\[
\frac{\partial^2 \log A}{\partial u_i \partial u_z} = - \frac{\partial \log A}{\partial u_i} \frac{\partial \log A}{\partial u_z}.
\]

Equation (2.11), by virtue of (2.12), takes the form

\[
\frac{\partial^2 \phi}{\partial u_i \partial u_z} + \frac{\partial \log A}{\partial u_i} \frac{\partial \phi}{\partial u_z} + \frac{\partial \log A}{\partial u_z} \frac{\partial \phi}{\partial u_i} + \frac{\partial^3 \log A}{\partial u_i \partial u_z} \phi + \frac{\partial \log A}{\partial u_i} \frac{\partial \log A}{\partial u_z} \phi = 0 ,
\]

or
\[
\frac{\partial}{\partial u_1} \left\{ \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi \right\} + \frac{\partial \log A}{\partial u_1} \left\{ \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi \right\} = 0 ,
\]
from which we obtain
\[
(2.13) \quad \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi - \frac{v}{A} = 0 ,
\]
where \(v\) is an arbitrary function of \(u_2, \ldots, u_k\).

Equation (2.13) is a linear differential equation whose general solution is
\[
(2.14) \quad \varphi = \frac{1}{A} \left( z + \int v \, du_2 \right) ,
\]
where \(z\) is an arbitrary function of \(u_1, u_2, \ldots, u_k\).

Relation (2.14), by virtue of the first of (2.7), takes the form
\[
(2.15) \quad \varphi = \alpha \mu (u_1, u_2, \ldots, u_k) + \pi (u_2, \ldots, u_k) ,
\]
where \(z = \alpha \mu, \int v \, du_2 = \alpha \pi\) and \(\alpha\) is an arbitrary real constant.

In order for the function \(\varphi\) to satisfy the rest of partial differential equations of the system (2.8), as it is easily proved that it must have the form
\[
(2.16) \quad \varphi = \alpha \frac{\varphi_1 (u_1) + \cdots + \varphi_k (u_k)}{1 + u_1^2 + \cdots + u_k^2} ,
\]
where \(\varphi_1, \ldots, \varphi_k\) are arbitrary functions of \(u_1, \ldots, u_k\), respectively.

Similarly, in order for the function \(f\) to satisfy the system of partial differential equations (2.9), it must have the form
\[
(2.17) \quad f = \alpha \frac{f_{k+1} (u_{k+1}) + \cdots + f_{k+n} (u_{k+n})}{1 + u_{k+1}^2 + \cdots + u_{k+n}^2} ,
\]
where \(f_{k+1}, \ldots, f_{k+n}\) are arbitrary functions of \(u_{k+1}, \ldots, u_{k+n}\), respectively.

From (2.1), (2.2), (2.4) and (2.10), we obtain
\[
(2.18) \quad R_{ij} (0) = - \frac{16}{A^4} , R'_{ij} (0) = - \frac{16}{A^4} f , j = 2, \ldots, k ,
\]
\[
R_{1k+j+1} (0) = 0 , R'_1 k+j+1 (0) = \frac{2}{(AB)^2} \left\{ A^2 \frac{\partial^2 \varphi}{\partial u_1^2} + 2Au_1 \frac{\partial \varphi}{\partial u_1} \right\} ,
\]
\[
(2.19) \quad -2A \sum_{i=1}^{k} u_i \frac{\partial \varphi}{\partial u_i} + B \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i=1}^{k} u_{k+i} \frac{\partial f}{\partial u_{k+i}} \right\} ,
\]
\[j = 1, \ldots, n \]
(2.20) \[ R_{ij,l+1}(0) = R'_{ij,l+1}(0) = 0, \quad j = 2, \ldots, l = 1, \ldots, n, \]
(2.21) \[ R_{k+l,j,k+l}(0) = R'_{k+l,j,k+l}(0) = 0, \quad j = 1, \ldots, n, \]
where \( R'_{k+l,j} \) denotes the derivative of \( R_{k+l,j} \) with respect to the parameter \( t \).

From (1.1), (1.2) and (1.3), we obtain the following formulas

\[
\begin{cases}
g_{ii}(0) = \cdots = g_{kk}(0) = d_{ii}, \\
g_{k+1,1}(0) = \cdots = g_{k+n,k+n}(0) = d_{k+n,k+n}, \\
g'_{ii}(0) = \cdots = g'_{kk}(0) = Fd_{ii}, \\
g'_{k+1,1}(0) = \cdots = g'_{k+n,k+n}(0) = \phi d_{k+n,k+n}.
\end{cases}
\]

Relations (2.18) and (2.19) by means of (2.7) and (2.22) take the form

\[
R_{ij,j} = -d_{ij}, \quad R'_{ij,j}(0) = -fd_{ij}, \quad j = 2, \ldots, k.
\]

(2.24) \[
-2A \sum_{i=1}^{k} u_{i} \frac{\partial \phi}{\partial u_{i}} + B^{2} \frac{\partial^{2} f}{\partial u_{k+j}^{2}} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i=1}^{n} u_{k+j} \frac{\partial f}{\partial u_{k+n+i}}
\]

(2.23)

3. Let \( P \) be any point of \( M_{1} \times M_{2} \). Then the \( k+n \) vectors \( \partial/\partial u_{i}, \ldots, \partial/\partial u_{k}, \partial/\partial u_{k+1}, \ldots, \partial/\partial u_{k+n} \) form an orthonormal basis of the tangent space \( (M_{1} \times M_{2})_{P} \).

As it is known, the sectional curvature of the plane spanned by \( \partial/\partial u_{i}, \partial/\partial u_{j}, j = 2, \ldots, k \), is given by

\[ K_{ij} = -\frac{R_{ij,j}}{g_{ij}}, \quad j = 2, \ldots, k, \]

which implies

(3.1) \[ K'_{ij}(0) = -\frac{R'_{ij,j}(0)g_{ij}(0) - R_{ij,j}(0)(g'_{ij}(0)g_{ij}(0) + g_{ii}(0)g'_{jj}(0))}{g_{ii}(0)g_{jj}(0)} \]

Relation (3.1), by virtue of (2.22) and (2.23), takes the form

(3.2) \[ K'_{ij}(0) = -f. \]

Similarly, calculating \( K'_{k+1,j}(0) \), we obtain

(3.3) \[ K'_{k+1,j}(0) = -\phi. \]

Formulas (3.2) and (3.3), by means of (2.16) and (2.17), take the form
\[ K'_{ij}(0) = -\alpha f_{k+j}(u_{k+j}) + \cdots + f_{k+n}(u_{k+n}) + \frac{\psi_i(u_i)}{1 + u_i^2 + \cdots + u_k^2}, \]

\[ K'_{k+1,k+j}(0) = -\alpha \phi_i(u_i) + \cdots + \phi_k(u_k) + \frac{\psi_i(u_i)}{1 + u_i^2 + \cdots + u_k^2}, \]

respectively. In order for \( K'_{ij}(0) \), \( K'_{k+1,k+j}(0) \) to be positive, we must have \( \alpha < 0, f_{k+j}(u_{k+j}) > 0, j = 1, \ldots, n, \phi_i(u_i) > 0, i = 1, \ldots, k \), which means the real number \( \alpha \) must be negative and the functions \( f_{k+j}(u_{k+j}) \) and \( \phi_i(u_i) \) must be positive when the corresponding variable takes values in the interval \((-1, 1)\).

The sectional curvature of the plane spanned by \( \partial/\partial u_l, \partial/\partial u_{k+j} \) is given by

\[ K_{lk+j} = -\frac{R_{lk+j,k+l}}{g_{lk+j,k+l}}, \quad l = 1, \ldots, k, j = 1, \ldots, n, \]

which, by virtue of (2.22) and either (2.24) or similar to (2.24), takes the form

\[ K'_{lk+j}(0) = -\frac{1}{8} \left\{ A^2 \frac{\partial^2 \phi}{\partial u_l^2} + 2A u_l \frac{\partial \phi}{\partial u_l} - 2A \sum_{i \neq i}^k u_i \frac{\partial \phi}{\partial u_i} + B^2 - \frac{\partial^2 f^2}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} \right\}. \]

(3.4)

In order for \( K'_{lk+j}(0) \) to be positive and because the functions \( \phi \) and \( f \) are independent, it must be

\[ A^2 \frac{\partial^2 \phi}{\partial u_l^2} + 2A u_l \frac{\partial \phi}{\partial u_l} - 2A \sum_{i \neq i}^k u_i \frac{\partial \phi}{\partial u_i} < 0, \quad l = 1, \ldots, k, \]

\[ B^2 - \frac{\partial^2 f^2}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} < 0, \quad j = 1, \ldots, n. \]

(3.5) (3.6)

Inequalities (3.5) and (3.6), by virtue of (2.16) and (2.17), become

\[ \frac{\alpha}{A} \left\{ A^2 \frac{\partial^2 \phi}{\partial u_l^2} - 2A \sum_{i=1}^k u_i \frac{\partial \phi}{\partial u_i} - 2(2 - A) \sum_{i=1}^k \phi_i \right\} < 0, \quad l = 1, \ldots, k, \]

\[ \frac{\alpha}{B} \left\{ B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} - 2B \sum_{i=1}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} - 2(2 - B) \sum_{i=1}^n f_{k+i} \right\} < 0, \quad j = 1, \ldots, n, \]

which imply
If the functions $f_{k+j} = f_{k+j}(u_{k+j})$, $\varphi_i = \varphi_i(u_i)$ are chosen to have the form

$$f_{k+j} = u_{k+j}^i + \frac{1}{2n}, \quad j = 1, \ldots, n,$$

then the inequalities (3.7) take the form

$$2 - A > 0, \quad 2 - B > 0,$$

which, by virtue of (2.7), become

$$1 - u_i^1 - \cdots - u_i^k > 0, \quad 1 - u_{k+1}^1 - \cdots - u_{k+n}^n > 0,$$

which are valid on the open balls $u_i^1 + \cdots + u_i^k < 1$, $u_{k+1}^1 + \cdots + u_{k+n}^n < 1$, respectively.

Relations (2.16) and (2.17), by means of (3.8), take the form

$$f = \alpha \frac{u_{k+1}^1 + \cdots + u_{k+n}^n + 1/2}{u_{k+1}^1 + \cdots + u_{k+n}^n + 1}, \quad \varphi = \alpha \frac{u_i^1 + \cdots + u_i^k + 1/2}{u_i^1 + \cdots + u_i^k + 1}.$$

The second of (2.24) or similar to that and (3.4), by means of (3.9), become

$$R'_{l,k+j} = \frac{2\alpha}{(1 + u_i^1 + \cdots + u_i^k)^2 (1 + u_{k+1}^1 + \cdots + u_{k+n}^n)^2} \times \left\{ \frac{1 - u_i^1 - \cdots - u_i^k}{1 + u_i^1 + \cdots + u_i^k} + \frac{1 - u_{k+1}^1 - \cdots - u_{k+n}^n}{1 + u_{k+1}^1 + \cdots + u_{k+n}^n} \right\},$$

$$K'_{l,k+j} = -\frac{\alpha}{8} \left\{ \frac{1 - u_i^1 - \cdots - u_i^k}{1 + u_i^1 + \cdots + u_i^k} + \frac{1 - u_{k+1}^1 - \cdots - u_{k+n}^n}{1 + u_{k+1}^1 + \cdots + u_{k+n}^n} \right\},$$

$$l = 1, \ldots, k, \quad j = 1, \ldots, n.$$

Using the fact that $\alpha < 0$, then following inequalities are obtained from the above relations:

$$R'_{l,k+j} < 0, \quad K'_{l,k+j} > 0, \quad l = 1, \ldots, k, \quad j = 1, \ldots, n,$$

which are valid on the open balls $u_i^1 + \cdots + u_i^k < 1$, $u_{k+1}^1 + \cdots + u_{k+n}^n < 1$.

Let $\xi(\xi^1, \ldots, \xi^{k+n})$ and $z(z^1, \ldots, z^{k+n})$ be any two vectors of the tangent space $(M_1 \times M_2)_p$. The sectional curvature of the plane spanned by $\xi$ and $z$ is given by ([11], p. 12)
\[ K = \frac{R_{hijl} z^h z^i z^j z^l}{(g_{hi} g_{lj} - g_{hl} g_{ij}) z^h z^i z^j z^l}, \]

or

\[(3.11) \quad K = \frac{A_1}{B_1}, \]

where

\[(3.12) \quad A_1 = R_{hijl} z^h z^i z^j z^l, \quad B_1 = (g_{hi} g_{lj} - g_{hl} g_{ij}) z^h z^i z^j z^l. \]

From (3.11), the following is obtained:

\[(3.13) \quad K'(0) = \frac{A'_1(0) B_i(0) - A_i(0) B'_1(0)}{B_i'(0)}. \]

From (3.12), by virtue of (2.3), (2.6), (2.20), (2.21), (2.22), (2.23), (2.24) and similar formulas to (2.23) and (2.24), we obtain

\[(3.14) \quad A_i(0) = - C d_{ij}^m - D d_{i+1,k+1}^j, \]

\[(3.15) \quad B_i(0) = - C d_{ij}^m - D d_{i+1,k+1}^j - E d_{i+1,k+1}. \]

\[(3.16) \quad B'_i(0) = - 2 f C d_{ij}^m - 2 \varphi D d_{i+1,k+1}^j - (f + \varphi) E d_{i+1,k+1}. \]

where

\[(3.17) \quad C = \sum_{i=1}^{k} \alpha_i^2, \quad D = \sum_{i=1}^{k} \sum_{j=1}^{n} \alpha_{ij}^2, \quad E = \sum_{i=1}^{k} \sum_{j=1}^{n} \alpha_{ik+j}, \]

\[(3.18) \quad T = \sum_{i=1}^{k} \sum_{j=1}^{n} R_{ik+j l k+j}(0) \alpha_{ik+j}^2, \quad \alpha_{jm} = (z^i z^j = z^m z^l). \]

Relation (3.13), by means of (3.14), takes the form

\[(3.19) \quad K'(0) = \frac{TB_i(0) + CG d_{ij}^m + DJ d_{i+1,k+1}^j}{B_i'(0)}, \]

where

\[(3.20) \quad G = B'_i(0) - f B_i(0), \quad J = B'_i(0) - \varphi B_i(0). \]

Formulas (3.20), by virtue of (3.15), and (3.16), become

\[(3.21) \quad G = L - (2 \varphi - f) D d_{i+1,k+1}^j, \quad J = N - (2 f - \varphi) C d_{ij}^m, \]

where

\[(3.22) \quad L = - \varphi E d_{i+1,k+1}^j - f C d_{ij}^m, \]

\[N = - f E d_{i+1,k+1}^j - \varphi D d_{i+1,k+1}^j. \]
Relation (3.19), by means of (3.21), takes the form

\[(3.23) \quad K'(0) = \frac{TB(0) + CLd_i + DNd_i + (f + \varphi)CDd_i + d_i}{B_i(0)} \]

From (3.15) and (3.22), by means of (3.17), and because the functions \( f \) and \( \varphi \) are negative, we conclude

\[(3.24) \quad B_i(0) < 0, \quad L \geq 0, \quad N \geq 0.\]

The first of (3.18), by virtue of the first inequality of (3.10), implies

\[(3.25) \quad T \leq 0.\]

Formula (3.23), by means of (3.17), (3.24), (3.25) and \( f < 0, \varphi < 0 \), implies

\[K'(0) > 0,\]

because it is not possible that simultaneously \( C = D = T = 0 \) for the two vectors \( \xi \) and \( z \).

Hence, we have the following theorem.

**Theorem.** Let \( M_1 \) and \( M_2 \) be two special Riemannian spaces with constant positive sectional curvature defined in § 1. If we consider a special 1-parameter family of Riemannian metrics \( F(t) \) on \( M_1 \times M_2 \) defined by (1.3), where the functions \( f, \varphi \) have the form (3.9), then the derivative of the sectional curvature with respect to the parameter \( t \) for \( t = 0 \) and for any plane of \( (M_1 \times M_2)_P \) and \( \forall P \in M_1 \times M_2 \) is strictly positive.

From the above, we conclude that, if the parameter \( t \) is positive and small enough, then the corresponding Riemannian metric \( F(t) \) defined by (1.3) on \( M_1 \times M_2 \), where the functions \( f \) and \( \varphi \) have the form (3.9), has strictly positive sectional curvature.

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**References**


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