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Let M_1 and M_2 be two Riemannian spaces¹ with Riemannian metrics d_1 and d_2 respectively whose sectional curvature is positive constant. We consider the product of the two Riemannian spaces $M_1 \times M_2$, then the Riemannian space $M_1 \times M_2$ has nonnegative sectional curvature with respect to the Riemannian metric $d_1 \times d_2$ but not strictly positive sectional curvature.

We can construct a Riemannian metric on $M_1 \times M_2$ which approaches the Riemannian metric $d_1 \times d_2$ as closely as we wish and which has strictly positive sectional curvature.

Now, our results can be stated as follows. We consider two manifolds $M_1(H_1 - E_1, q_1)$, $M_2(H_2 - E_2, q_2)$ such that each of them has only one chart where H_1, E_1 are the south hemisphere and the equator, respectively, of a k -dimensional sphere ($k \geq 2$) and E_2, H_2 are also the south hemisphere and the equator, respectively, of an n -dimensional sphere ($n \geq 2$), and q_1, q_2 are special mappings. We also consider on M_1 and M_2 particular Riemannian metrics d_1, d_2 , respectively, with positive constant sectional curvature. We obtain a special 1-parameter family of Riemannian metrics $F(t)$ on $M_1 \times M_2$ such that $F(0) = d_1 \times d_2$. We have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter t for $t = 0$ and for any plane of $(M_1 \times M_2)_P$, is strictly positive.

1. Let M_1 be a manifold which consists of one chart $(H_1 - E_1, q_1)$, where H_1, E_1 are the south hemisphere and the equator, respectively, of a k -dimensional sphere $S_1^k (k \geq 2)$ and the inverse mapping of q_1 is defined as follows

$$q_1^{-1} = \left\{ \begin{aligned} x^1 &= \frac{2u_1}{1 + u_1^2 + \dots + u_k^2}, \dots, x^k = \frac{2u_k}{1 + u_1^2 + \dots + u_k^2}, \\ x^{k+1} &= \frac{u_1^2 + \dots + u_k^2 - 1}{1 + u_1^2 + \dots + u_k^2} \end{aligned} \right\}.$$

q_1 maps the open set $H_1 - E_1$ onto the open ball $u_1^2 + \dots + u_k^2 < 1$.

On the manifold M_1 , we take the following Riemannian metric

¹ A Riemannian space is a Riemannian manifold covered with one chart ([5], p. 314).

$$(1.1) \quad d_1 = dS_1^2 = \left\{ d_{11} = \dots = d_{kk} = \frac{4}{(1 + u_1^2 + \dots + u_k^2)^2}, \right. \\ \left. d_{ij} = 0 \text{ if } i \neq j \right\},$$

whose sectional curvature is positive constant.

Let M_2 be another manifold which consists of one chart $(H_2 - E_2, q_2)$, where H_2, E_2 are the south hemisphere and the equator, respectively, of an n -dimensional sphere $S_2^n (n \geq 2)$ and the inverse mapping of q_2 is defined by

$$q_2^{-1} = \left\{ x^1 = \frac{2u_{k+1}}{1 + u_{k+1}^2 + \dots + u_{k+n}^2}, \dots, \right. \\ \left. x^n = \frac{2u_{k+n}}{1 + u_{k+1}^2 + \dots + u_{k+n}^2}, x^{n+1} = \frac{u_{k+1}^2 + \dots + u_{k+n}^2 - 1}{1 + u_{k+1}^2 + \dots + u_{k+n}^2} \right\}.$$

q_2 maps the open set $H_2 - E_2$ onto the open ball $u_{k+1}^2 + \dots + u_{k+n}^2 < 0$. On the manifold M_2 , we also take the following Riemannian metric

$$(1.2) \quad d_2 = dS_2^2 = \left\{ d_{k+1 k+1} = \dots = d_{k+n k+n} \right. \\ \left. = \frac{4}{(1 + u_{k+1}^2 + \dots + u_{k+n}^2)^2}, d_{ij} = 0 \text{ if } i \neq j \right\},$$

whose sectional curvature is positive constant.

Consider the product of the two manifolds $M_1 \times M_2$. Then $M_1 \times M_2$ is a manifold with one chart $\{(H_1 - E_1) \times (H_2 - E_2), q_1 \times q_2\}$.

We define a 1-parameter family of Riemannian metrics on the manifold $M_1 \times M_2$ defined by

$$(1.3) \quad dS^2(t) = \begin{cases} g_{11} = \dots = g_{kk} = \frac{4(1 + tf)}{(1 + u_1^2 + \dots + u_k^2)^2}, \\ g_{k+1 k+1} = \dots = g_{k+n k+n} \\ = \frac{4(1 + t\varphi)}{(1 + u_{k+1}^2 + \dots + u_{k+n}^2)^2}, g_{ij} = 0 \text{ if } i \neq j, \end{cases}$$

where $-b < t < b, \varphi = \varphi(u_1, \dots, u_k), f = f(u_{k+1}, \dots, u_{k+n})$.

The Riemannian metric $dS^2(0)$ coincides with the product Riemannian metric $dS_1^2 \times dS_2^2$ of the two manifolds M_1 and M_2 .

2. We shall calculate the components R_{hijk} of the Riemannian curvature tensor when the index $h = 1$, because the other cases are similar to these.

If $h = 1$, there exist the following distinguished cases in which R_{1ijk} do not vanish identically.

$$\begin{aligned}
 &R_{1j1j}, j = 2, \dots, k, R_{1k+j1k+j}, j = 1, \dots, n, \\
 &R_{1jjl}, j \neq l, j = 2, \dots, k, l = 2, \dots, k, \\
 &R_{1jjk+l}, j = 2, \dots, k, l = 1, \dots, n, \\
 &R_{1k+jk+jl}, j = 1, \dots, n, l = 2, \dots, k, \\
 &R_{1ijl}, i \neq j \neq l, i = 2, \dots, k+n, j = 2, \dots, k+n, l = 2, \dots, k+n.
 \end{aligned}$$

As it is known, R_{1ijk} is given by ([12], p. 18)

$$\begin{aligned}
 R_{1ijl} = &\frac{1}{2} \left(\frac{\partial^2 g_{1j}}{\partial u_i \partial u_l} + \frac{\partial^2 g_{il}}{\partial u_1 \partial u_j} - \frac{\partial^2 g_{ij}}{\partial u_1 \partial u_l} - \frac{\partial^2 g_{1l}}{\partial u_i \partial u_j} \right) \\
 &- g_{rs} \left(\begin{Bmatrix} r \\ ij \end{Bmatrix} \begin{Bmatrix} s \\ 1l \end{Bmatrix} - \begin{Bmatrix} r \\ il \end{Bmatrix} \begin{Bmatrix} s \\ 1j \end{Bmatrix} \right),
 \end{aligned}$$

where $\begin{Bmatrix} r \\ ij \end{Bmatrix}, \begin{Bmatrix} s \\ 1l \end{Bmatrix}, \begin{Bmatrix} r \\ il \end{Bmatrix}, \begin{Bmatrix} s \\ 1j \end{Bmatrix}$ are the Christoffel symbols of the second kind.

From the above formula by virtue of (1.3) we obtain

$$(2.1) \quad R_{1j1j} = -\frac{16(1+tf)}{A^4} + \frac{t^2}{1+t\varphi} \frac{B^2}{A^4} \sum_{i=1}^n \left(\frac{\partial f}{\partial u_{k+i}} \right)^2, j = 2, \dots, k,$$

$$\begin{aligned}
 (2.2) \quad R_{1k+j1k+j} = &\frac{2t}{(AB)^2} \left\{ A^2 \frac{\partial^2 \varphi}{\partial u_1^2} + 2A u_1 \frac{\partial \varphi}{\partial u_1} - 2A \sum_{i=2}^k u_i \frac{\partial \varphi}{\partial u_i} \right. \\
 &+ B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} \left. \right\} \\
 &- t^2 \left\{ \frac{\left(\frac{\partial f}{\partial u_{k+j}} \right)^2}{(1+tf)A^2} + \frac{\left(\frac{\partial \varphi}{\partial u_1} \right)^2}{(1+t\varphi)B^2} \right\}, j = 1, \dots, n,
 \end{aligned}$$

$$(2.3) \quad R_{1j1l} = 0, j \neq l, j = 2, \dots, k, l = 2, \dots, k,$$

$$(2.4) \quad R_{1jjk+l} = t^2 \frac{\frac{\partial f}{\partial u_{k+l}} \frac{\partial \varphi}{\partial u_1}}{(1+t\varphi)A^2}, j = 2, \dots, k, l = 1, \dots, n,$$

$$\begin{aligned}
 (2.5) \quad R_{1k+jk+jl} = &-\frac{2t}{B^2} \left\{ \frac{\partial^2 \varphi}{\partial u_1 \partial u_l} + \frac{2u_1}{A} \frac{\partial \varphi}{\partial u_l} + \frac{2u_l}{A} \frac{\partial \varphi}{\partial u_1} \right\} \\
 &+ t^2 \frac{\frac{\partial \varphi}{\partial u_1} \frac{\partial \varphi}{\partial u_l}}{(1+t\varphi)B^2}, j = 1, \dots, n, l = 2, \dots, k,
 \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad R_{1ijl} = &0, i \neq j \neq l, i = 2, \dots, k+n, \\
 &j = 2, \dots, k+n, l = 2, \dots, k+n,
 \end{aligned}$$

where

$$(2.7) \quad A = 1 + u_1^2 + \dots + u_k^2, \quad B = 1 + u_{k+1}^2 + \dots + u_{k+n}^2.$$

If the functions φ and f are chosen such that they satisfy the systems of partial differential equations

$$(2.8) \quad \frac{\partial^2 \varphi}{\partial u_i \partial u_j} + \frac{2u_i}{A} \frac{\partial \varphi}{\partial u_j} + \frac{2u_j}{A} \frac{\partial \varphi}{\partial u_i} = 0, \\ i \neq j, i = 1, \dots, k, j = 1, \dots, k,$$

$$(2.9) \quad \frac{\partial^2 f}{\partial u_h \partial u_l} + \frac{2u_h}{B} \frac{\partial f}{\partial u_l} + \frac{2u_l}{B} \frac{\partial f}{\partial u_h} = 0, \\ h \neq l, h = k + 1, \dots, k + n, l = k + 1, \dots, k + n,$$

respectively and if $m \in [1, \dots, k]$ and

$$i \in [k + 1, \dots, k + n], i \neq j \in [k + 1, \dots, k + n]$$

or if $m \in [k + 1, \dots, k + n]$ and $i \in [1, \dots, k], i \neq j \in [1, \dots, k]$, then we have

$$(2.10) \quad R_{immj} = t^2 \frac{\frac{\partial f}{\partial u_i} \frac{\partial f}{\partial u_j}}{(1 + tf)A^2}, \quad \text{or} \quad R_{immj} = t^2 \frac{\frac{\partial \varphi}{\partial u_i} \frac{\partial \varphi}{\partial u_j}}{(1 + t\varphi)B^2}.$$

We consider one partial differential equation of the system (2.8), for example,

$$\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{2u_1}{A} \frac{\partial \varphi}{\partial u_2} + \frac{2u_2}{A} \frac{\partial \varphi}{\partial u_1} = 0,$$

or

$$(2.11) \quad \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \log A}{\partial u_1} \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \frac{\partial \varphi}{\partial u_1} = 0.$$

From the first of (2.7), we conclude that

$$(2.12) \quad \frac{\partial^2 \log A}{\partial u_1 \partial u_2} = - \frac{\partial \log A}{\partial u_1} \frac{\partial \log A}{\partial u_2}.$$

Equation (2.11), by virtue of (2.12), takes the form

$$\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \log A}{\partial u_1} \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \frac{\partial \varphi}{\partial u_1} \\ + \frac{\partial^2 \log A}{\partial u_1 \partial u_2} \varphi + \frac{\partial \log A}{\partial u_1} \frac{\partial \log A}{\partial u_2} \varphi = 0,$$

or

$$\frac{\partial}{\partial u_1} \left\{ \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi \right\} + \frac{\partial \log A}{\partial u_1} \left\{ \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi \right\} = 0,$$

from which we obtain

$$(2.13) \quad \frac{\partial \varphi}{\partial u_2} + \frac{\partial \log A}{\partial u_2} \varphi - \frac{v}{A} = 0,$$

where v is an arbitrary function of u_2, \dots, u_k .

Equation (2.13) is a linear differential equation whose general solution is

$$(2.14) \quad \varphi = \frac{1}{A} \left(z + \int v du_2 \right),$$

where z is an arbitrary function of u_1, u_3, \dots, u_k .

Relation (2.14), by virtue of the first of (2.7), takes the form

$$(2.15) \quad \varphi = \alpha \frac{\mu(u_1, u_3, \dots, u_k) + \pi(u_2, \dots, u_k)}{1 + u_1^2 + \dots + u_k^2},$$

where $z = \alpha\mu$, $\int v du_2 = \alpha\pi$ and α is an arbitrary real constant.

In order for the function φ to satisfy the rest of partial differential equations of the system (2.8), as it is easily proved that it must have the form

$$(2.16) \quad \varphi = \alpha \frac{\varphi_1(u_1) + \dots + \varphi_k(u_k)}{1 + u_1^2 + \dots + u_k^2},$$

where $\varphi_1, \dots, \varphi_k$ are arbitrary functions of u_1, \dots, u_k , respectively.

Similarly, in order for the function f to satisfy the system of partial differential equations (2.9), it must have the form

$$(2.17) \quad f = \alpha \frac{f_{k+1}(u_{k+1}) + \dots + f_{k+n}(u_{k+n})}{1 + u_{k+1}^2 + \dots + u_{k+n}^2},$$

where f_{k+1}, \dots, f_{k+n} are arbitrary functions of u_{k+1}, \dots, u_{k+n} , respectively.

From (2.1), (2.2), (2.4) and (2.10), we obtain

$$(2.18) \quad R_{1j1j}(0) = -\frac{16}{A^4}, R'_{1j1j}(0) = -\frac{16f}{A^4}, j = 2, \dots, k,$$

$$(2.19) \quad \begin{aligned} R_{1k+j1k+j}(0) = 0, R'_{1k+j1k+j}(0) = \frac{2}{(AB)^2} \left\{ A^2 \frac{\partial^2 \varphi}{\partial u_1^2} + 2A u_1 \frac{\partial \varphi}{\partial u_1} \right. \\ \left. - 2A \sum_{i=2}^k u_i \frac{\partial \varphi}{\partial u_i} + B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} \right\}, \\ j = 1, \dots, n \end{aligned}$$

$$(2.20) \quad R_{1j j k+l}(0) = R'_{1j j k+l}(0) = 0, \quad j = 2, \dots, l = 1, \dots, n,$$

$$(2.21) \quad R_{1k+j k+j l}(0) = R'_{1k+j k+j l}(0) = 0, \quad j = 1, \dots, n, l = 1, \dots, n,$$

where $R'_{h i j l}$ denotes the derivative of $R_{h i j l}$ with respect to the parameter t .

From (1.1), (1.2) and (1.3), we obtain the following formulas

$$(2.22) \quad \begin{cases} g_{11}(0) = \dots = g_{kk}(0) = d_{11}, \\ g_{k+1 k+1}(0) = \dots = g_{k+n k+n}(0) = d_{k+n k+n}, \\ g'_{11}(0) = \dots = g'_{kk}(0) = f d_{11}, \\ g'_{k+1 k+1}(0) = \dots = g'_{k+n k+n}(0) = \varphi d_{k+n k+n}. \end{cases}$$

Relations (2.18) and (2.19) by means of (2.7) and (2.22) take the form

$$(2.23) \quad R_{1j 1j} = -d_{11}^2, R'_{1j 1j}(0) = -f d_{11}^2, \quad j = 2, \dots, k,$$

$$(2.24) \quad \begin{aligned} R_{1k+j 1k+j}(0) = 0, R'_{1k+j 1k+j}(0) &= \frac{d_{11} d_{k+1 k+1}}{8} \left\{ A^2 \frac{\partial^2 \varphi}{\partial u_1^2} + 2A u_1 \frac{\partial \varphi}{\partial u_1} \right. \\ &\left. - 2A \sum_{i=2}^k u_i \frac{\partial \varphi}{\partial u_i} + B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} \right\} \\ & \quad j = 1, \dots, k. \end{aligned}$$

3. Let P be any point of $M_1 \times M_2$. Then the $k + n$ vectors $\partial/\partial u_1, \dots, \partial/\partial u_k, \partial/\partial u_{k+1}, \dots, \partial/\partial u_{k+n}$ form an orthonormal basis of the tangent space $(M_1 \times M_2)_P$.

As it is known, the sectional curvature of the plane spanned by $\partial/\partial u_1, \partial/\partial u_j, j = 2, \dots, k$, is given by

$$K_{1j} = - \frac{R_{1j 1j}}{g_{11} g_{jj}}, \quad j = 2, \dots, k,$$

which implies

$$(3.1) \quad K'_{1j}(0) = - \frac{R'_{1j 1j}(0) g_{11}(0) g_{jj}(0) - R_{1j 1j}(0) \{g'_{11}(0) g_{jj}(0) + g_{11}(0) g'_{jj}(0)\}}{g_{11}^2(0) g_{jj}^2(0)}$$

Relation (3.1), by virtue of (2.22) and (2.23), takes the form

$$(3.2) \quad K'_{1j}(0) = - f.$$

Similarly, calculating $K'_{k+1 k+j}(0)$, we obtain

$$(3.3) \quad K'_{k+1 k+j}(0) = - \varphi.$$

Formulas (3.2) and (3.3), by means of (2.16) and (2.17), take the form

$$K'_{1j}(0) = -\alpha \frac{f_{k+1}(u_{k+1}) + \dots + f_{k+n}(u_{k+n})}{1 + u_{k+1}^2 + \dots + u_{k+n}^2},$$

$$K'_{k+1k+j}(0) = -\alpha \frac{\varphi_1(u_1) + \dots + \varphi_k(u_k)}{1 + u_1^2 + \dots + u_k^2},$$

respectively. In order for $K'_{1j}(0), K'_{k+1k+j}(0)$ to be positive, we must have $\alpha < 0, f_{k+j}(u_{k+j}) > 0, j = 1, \dots, n, \varphi_i(u_i) > 0, i = 1, \dots, k$, which means the real number α must be negative and the functions $f_{k+j}(u_{k+j})$ and $\varphi_i(u_i)$ must be positive when the corresponding variable takes values in the interval $(-1, 1)$.

The sectional curvature of the plane spanned by $\partial/\partial u_l, \partial/\partial u_{k+j}$ is given by

$$K_{lk+j} = -\frac{R_{lk+jlk+j}}{g_{ll}g_{k+jk+j}}, \quad l = 1, \dots, k, j = 1, \dots, n,$$

which, by virtue of (2.22) and either (2.24) or similar to (2.24), takes the form

$$(3.4) \quad K'_{lk+j}(0) = -\frac{1}{8} \left\{ A^2 \frac{\partial^2 \varphi}{\partial u_l^2} + 2A u_l \frac{\partial \varphi}{\partial u_l} - 2A \sum_{i \neq l}^k u_i \frac{\partial \varphi}{\partial u_i} + B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} \right\}.$$

In order for $K'_{lk+j}(0)$ to be positive and because the functions φ and f are independent, it must be

$$(3.5) \quad A^2 \frac{\partial^2 \varphi}{\partial u_l^2} + 2A u_l \frac{\partial \varphi}{\partial u_l} - 2A \sum_{i \neq l}^k u_i \frac{\partial \varphi}{\partial u_i} < 0, \quad l = 1, \dots, k,$$

$$(3.6) \quad B^2 \frac{\partial^2 f}{\partial u_{k+j}^2} + 2B u_{k+j} \frac{\partial f}{\partial u_{k+j}} - 2B \sum_{i \neq j}^n u_{k+i} \frac{\partial f}{\partial u_{k+i}} < 0, \quad j = 1, \dots, n.$$

Inequalities (3.5) and (3.6), by virtue of (2.16) and (2.17), become

$$\frac{\alpha}{A} \left\{ A^2 \frac{d^2 \varphi_l}{du_l^2} - 2A \sum_{i=1}^k u_i \frac{d\varphi_i}{du_i} - 2(2 - A) \sum_{i=1}^k \varphi_i \right\} < 0, \quad l = 1, \dots, k,$$

$$\frac{\alpha}{B} \left\{ B^2 \frac{d^2 f_{k+j}}{du_{k+j}^2} - 2B \sum_{i=1}^n u_{k+i} \frac{df_{k+i}}{du_{k+i}} - 2(2 - B) \sum_{i=1}^n f_{k+i} \right\} < 0, \quad j = 1, \dots, n,$$

which imply

$$(3.7) \quad \begin{cases} A^2 \frac{d^2 \varphi_l}{du_l^2} - 2A \sum_{i=1}^k u_i \frac{d\varphi_l}{du_i} - 2(2 - A) \sum_{i=1}^k \varphi_i > 0, & l = 1, \dots, k, \\ B^2 \frac{d^2 f_{k+j}}{du_{k+j}^2} - 2B \sum_{i=1}^n u_{k+i} \frac{df_{k+i}}{du_{k+i}} - 2(2 - B) \sum_{i=1}^n f_{k+i} > 0, & j = 1, \dots, n. \end{cases}$$

If the functions $f_{k+j} = f_{k+j}(u_{k+j})$, $\varphi_i = \varphi_i(u_i)$ are chosen to have the form

$$(3.8) \quad f_{k+j} = u_{k+j}^2 + \frac{1}{2n}, j = 1, \dots, n, \varphi_i = u_i^2 + \frac{1}{2k}, i = 1, \dots, k,$$

then the inequalities (3.7) take the form

$$2 - A > 0, \quad 2 - B > 0,$$

which, by virtue of (2.7), become

$$1 - u_1^2 - \dots - u_k^2 > 0, \quad 1 - u_{k+1}^2 - \dots - u_{k+n}^2 > 0,$$

which are valid on the open balls $u_1^2 + \dots + u_k^2 < 1, u_{k+1}^2 + \dots + u_{k+n}^2 < 1$, respectively.

Relations (2.16) and (2.17), by means of (3.8), take the form

$$(3.9) \quad f = \alpha \frac{u_{k+1}^2 + \dots + u_{k+n}^2 + 1/2}{u_{k+1}^2 + \dots + u_{k+n}^2 + 1}, \quad \varphi = \alpha \frac{u_1^2 + \dots + u_k^2 + 1/2}{u_1^2 + \dots + u_k^2 + 1}.$$

The second of (2.24) or similar to that and (3.4), by means of (3.9), become

$$\begin{aligned} R'_{lk+jlk+j}(0) &= \frac{2\alpha}{(1 + u_1^2 + \dots + u_k^2)^2(1 + u_{k+1}^2 + \dots + u_{k+n}^2)^2} \\ &\quad \times \left\{ \frac{1 - u_1^2 - \dots - u_k^2}{1 + u_1^2 + \dots + u_k^2} + \frac{1 - u_{k+1}^2 - \dots - u_{k+n}^2}{1 + u_{k+1}^2 + \dots + u_{k+n}^2} \right\}, \\ K'_{lk+j}(0) &= -\frac{\alpha}{8} \left\{ \frac{1 - u_1^2 - \dots - u_k^2}{1 + u_1^2 + \dots + u_k^2} + \frac{1 - u_{k+1}^2 - \dots - u_{k+n}^2}{1 + u_{k+1}^2 + \dots + u_{k+n}^2} \right\}, \\ &\quad l = 1, \dots, k, j = 1, \dots, n. \end{aligned}$$

Using the fact that $\alpha < 0$, then following inequalities are obtained from the above relations:

$$(3.10) \quad R'_{lk+jlk+j}(0) < 0, \quad K'_{lk+j}(0) > 0, \quad l = 1, \dots, k, j = 1, \dots, n,$$

which are valid on the open balls $u_1^2 + \dots + u_k^2 < 1, u_{k+1}^2 + \dots + u_{k+n}^2 < 1$.

Let $\xi(\xi^1, \dots, \xi^{k+n})$ and $z(z^1, \dots, z^{k+n})$ be any two vectors of the tangent space $(M_1 \times M_2)_P$. The sectional curvature of the plane spanned by ξ and z is given by ([11], p. 12)

$$K = \frac{R_{hijl}z^h z^j \xi^i \xi^l}{(g_{hi}g_{ij} - g_{hj}g_{il})z^h z^j \xi^i \xi^l},$$

or

$$(3.11) \quad K = \frac{A_1}{B_1},$$

where

$$(3.12) \quad A_1 = R_{hijl}z^h z^j \xi^i \xi^l, \quad B_1 = (g_{hi}g_{ij} - g_{hj}g_{il})z^h z^j \xi^i \xi^l.$$

From (3.11), the following is obtained:

$$(3.13) \quad K'(0) = \frac{A'_1(0)B_1(0) - A_1(0)B'_1(0)}{B_1^2(0)}.$$

From (3.12), by virtue of (2.3), (2.6), (2.20), (2.21), (2.22), (2.23), (2.24) and similar formulas to (2.23) and (2.24), we obtain

$$(3.14) \quad A_1(0) = -Cd_{11}^2 - Dd_{k+1k+1}^2,$$

$$A'_1(0) = -fCd_{11}^2 - \varphi Dd_{k+1k+1}^2 + T,$$

$$(3.15) \quad B_1(0) = -Cd_{11}^2 - Dd_{k+1k+1}^2 - Ed_{11}d_{k+1k+1},$$

$$(3.16) \quad B'_1(0) = -2fCd_{11}^2 - 2\varphi Dd_{k+1k+1}^2 - (f + \varphi)Ed_{11}d_{k+1k+1},$$

where

$$(3.17) \quad C = \sum_{i=1}^k \sum_{i < j=2}^k \alpha_{ij}^2, \quad D = \sum_{i=k+1}^{k+n} \sum_{i < j=k+2}^{k+n} \alpha_{ij}^2, \quad E = \sum_{i=1}^k \sum_{j=1}^n \alpha_{ik+j}^2,$$

$$(3.18) \quad T = \sum_{l=1}^k \sum_{j=1}^n R'_{lk+jlk+j}(0)\alpha_{lk+j}^2, \quad \alpha_{jm} = (z^i \xi^m - z^m \xi^i).$$

Relation (3.13), by means of (3.14), takes the form

$$(3.19) \quad K'(0) = \frac{TB_1(0) + CGd_{11}^2 + DJd_{k+1k+1}^2}{B_1^2(0)},$$

where

$$(3.20) \quad G = B'_1(0) - fB_1(0), \quad J = B'_1(0) - \varphi B_1(0).$$

Formulas (3.20), by virtue of (3.15), and (3.16), become

$$(3.21) \quad G = L - (2\varphi - f)Dd_{k+1k+1}^2, \quad J = N - (2f - \varphi)Cd_{11}^2,$$

where

$$(3.22) \quad \begin{aligned} L &= -\varphi Ed_{11}d_{k+1k+1} - fCd_{11}^2, \\ N &= -fEd_{11}d_{k+1k+1} - \varphi Dd_{k+1k+1}^2. \end{aligned}$$

Relation (3.19), by means of (3.21), takes the form

$$(3.23) \quad K'(0) = \frac{TB_1(0) + CLd_{11}^2 + DNd_{k+1, k+1}^2 - (f + \varphi)CDd_{11}^2d_{k+1, k+1}^2}{B_1^2(0)}.$$

From (3.15) and (3.22), by means of (3.17), and because the functions f and φ are negative, we conclude

$$(3.24) \quad B_1(0) < 0, \quad L \geq 0, \quad N \geq 0.$$

The first of (3.18), by virtue of the first inequality of (3.10), implies

$$(3.25) \quad T \leq 0.$$

Formula (3.23), by means of (3.17), (3.24), (3.25) and $f < 0$, $\varphi < 0$, implies

$$K'(0) > 0,$$

because it is not possible that simultaneously $C = D = T = 0$ for the two vectors ξ and z .

Hence, we have the following theorem.

THEOREM. *Let M_1 and M_2 be two special Riemannian spaces with constant positive sectional curvature defined in § 1. If we consider a special 1-parameter family of Riemannian metrics $F(t)$ on $M_1 \times M_2$ defined by (1.3), where the functions f, φ have the form (3.9), then the derivative of the sectional curvature with respect to the parameter t for $t = 0$ and for any plane of $(M_1 \times M_2)_P$ and $\forall P \in M_1 \times M_2$ is strictly positive.*

From the above, we conclude that, if the parameter t is positive and small enough, then the corresponding Riemannian metric $F(t)$ defined by (1.3) on $M_1 \times M_2$, where the functions f and φ have the form (3.9), has strictly positive sectional curvature.

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Martin Aigner, <i>On the tetrahedral graph</i>	219
Gregory Frank Bachelis, <i>Homomorphisms of annihilator Banach algebras</i>	229
Phillip Alan Griffith, <i>Transitive and fully transitive primary abelian groups</i>	249
Benjamin Rigler Halpern, <i>Fixed points for iterates</i>	255
James Edgar Keesling, <i>Mappings and dimension in general metric spaces</i>	277
Al (Allen Frederick) Kelley, Jr., <i>Invariance for linear systems of ordinary differential equations</i>	289
Hayri Korezlioglu, <i>Reproducing kernels in separable Hilbert spaces</i>	305
Gerson Louis Levin and Wolmer Vasconcelos, <i>Homological dimensions and Macaulay rings</i>	315
Leo Sario and Mitsuru Nakai, <i>Point norms in the construction of harmonic forms</i>	325
Barbara Osofsky, <i>Noncommutative rings whose cyclic modules have cyclic injective hulls</i>	331
Newton Tenney Peck, <i>Extreme points and dimension theory</i>	341
Jack Segal, <i>Quasi dimension type. II. Types in 1-dimensional spaces</i>	353
Michael Schilder, <i>Expected values of functionals with respect to the Ito distribution</i>	371
Grigorios Tsagas, <i>A Riemannian space with strictly positive sectional curvature</i>	381
John Alexander Williamson, <i>Random walks and Riesz kernels</i>	393