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**THE STRUCTURE SPACE OF A COMMUTATIVE LOCALLY
m-CONVEX ALGEBRA**

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If A is a commutative Banach algebra with identity, then the sets \mathcal{M} (all maximal ideals), \mathcal{M}_c (all closed maximal ideals), \mathcal{M}_1 (kernels of nonzero C -valued homomorphisms of A), and \mathcal{M}_0 (kernels of nonzero continuous C -valued homomorphisms of A) coincide. If A is a commutative complete locally m -convex algebra, one has only $\mathcal{M}_c = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}$, and the containments can be proper. Our goal is to investigate \mathcal{M} and its relationship to \mathcal{M}_0 ; specifically (1) to give a description of $\mathcal{M}(A)$ in terms of A and $\mathcal{M}_0(A)$ which is valid for at least the class of F -algebras, (2) to determine when $\mathcal{M}(A)$ is one of the standard compactifications (Wallman, Stone-Ćech) of $\mathcal{M}_0(A)$.

For many locally m -convex algebras, especially algebras of functions, one can determine \mathcal{M}_0 . However, descriptions of \mathcal{M} and its relationship to \mathcal{M}_0 seem to be limited to special cases; for example, Hewitt's description of $\mathcal{M}(C(X))$ [5] and Kakutani's description of \mathcal{M} for the algebra of analytic functions in the unit disc [6]. We show that a commutative complete locally m -convex algebra A generates a lattice \mathcal{L} on \mathcal{M}_0 , and that if we impose a rather natural restriction on A , then \mathcal{M} is the space of ultrafilters of \mathcal{L} . We give necessary and sufficient conditions on A in order that (1) \mathcal{M} is the Wallman compactification of $(\mathcal{M}_0, \text{hull-kernel})$, (2) \mathcal{M} is the Wallman compactification of $(\mathcal{M}_0, \text{Gelfand})$. In the second case, we show that $\mathcal{M} = \beta \mathcal{M}_0$ and obtain a correspondence between \mathcal{M}_1 and the A -realcompactification of \mathcal{M}_0 .

We then specialize to F -algebras and show (1) F -algebras always satisfy the condition imposed in the general situation, (2) \mathcal{M} is the Wallman compactification of $(\mathcal{M}_0, \text{hull-kernel})$, and (3) $\mathcal{M} = \beta \mathcal{M}_0$, whenever the algebra is regular.

1. The general case. A locally m -convex algebra (hereafter LMC algebra) is a locally convex Hausdorff topological algebra A whose topology is given by a family of pseudonorms (submultiplicative, convex, symmetric functionals). For the basic properties of these algebras the reader is referred to [1] or [9]. In this paper we shall restrict our attention to complete algebras with identity element 1. If λ is a complex number we shall write " λ " for " $\lambda \cdot 1$ ".

The *structure space* of A is the set \mathcal{M} of all maximal ideals of

A , endowed with the hull-kernel (hk —) topology. This space is always compact and satisfies the T_1 separation axiom. The *spectrum* of A is the set \mathcal{M}_0 of all closed maximal ideals of A .

DEFINITION 1.1. If $S \subseteq A$, $F \subseteq \mathcal{M}_0$, $G \subseteq \mathcal{M}$, $x \in A$, then

- (i) $H(S) = \{M \in \mathcal{M} : S \subseteq M\}$.
- (ii) $h(S) = \{M \in \mathcal{M}_0 : S \subseteq M\}$.
- (iii) $kF (= k(F)) = \bigcap \{M \in \mathcal{M}_0 : M \in F\} = \{x \in A : x \in M \text{ for each } M \in F\}$.
- (iv) $K(G) = \bigcap \{M \in \mathcal{M} : M \in G\} = \{x \in A : x \in M \text{ for each } M \in G\}$.
- (v) $H(x) = H(\{x\})$, $h(x) = h(\{x\})$.

The hull-kernel topology is defined in terms of the closure operator:

$$(1.1) \quad \text{Cl}_{\mathcal{M}}(F) = HK(F),$$

or

$$(1.2) \quad \text{Cl}_{\mathcal{M}}(F) = \bigcap \{H(x) : F \subseteq H(x)\}, \text{ for each } F \subseteq \mathcal{M}.$$

A simple computation yields

THEOREM 1.1. *The closure operator on \mathcal{M}_0 which defines the relative hull-kernel topology on \mathcal{M}_0 is given by*

$$(1.3) \quad \text{Cl}_{\mathcal{M}_0}(F) = hk(F)$$

or

$$(1.4) \quad \text{Cl}_{\mathcal{M}_0}(F) = \bigcap \{h(x) : F \subseteq h(x)\}, \text{ for each } F \subseteq \mathcal{M}_0.$$

The spectrum can also be endowed with a second natural topology. If $M \in \mathcal{M}_0$, then M is the kernel of a unique continuous homomorphism of A onto C [9, p. 11]. We identify M and the corresponding homomorphism, denoting the value of the homomorphism at an element x of A by $M(x)$, and endow \mathcal{M}_0 with the relative weak $-(w^* -)$ topology from A^* , the conjugate space of A . This topology is the weakest such that all of the functions $\hat{x} : \mathcal{M}_0 \rightarrow C$ defined by $\hat{x}(M) = M(x)$ for each $x \in A$ are continuous. We state without proof the basic properties of the mapping $x \rightarrow \hat{x}$ of A into $C(\mathcal{M}_0)$ (cf [9, Props. 7.3, 8.1, and 9.2] and [4, Ex. 7M]).

THEOREM 1.2. *The mapping $x \rightarrow \hat{x}$ is a homomorphism of A onto a subalgebra \hat{A} of $C(\mathcal{M}_0)$ which contains the constant functions and separates the points of \mathcal{M}_0 . The kernel of this homomorphism is the radical $\mathcal{R}(A)$ of A , and $\mathcal{R}(A) = \bigcap \{M : M \in \mathcal{M}_0\} = \bigcap \{M : M \in \mathcal{M}\} = \{x \in A : (1 + ax) \text{ is regular (invertible) for each } x \in A\}$ is a closed ideal in A . Hence, \mathcal{M}_0 is dense in \mathcal{M} .*

DEFINITION 1.2. A commutative LMC algebra A with identity is called *regular* provided that for each w^* -closed subset F on \mathcal{M}_0 and each point $M \in \mathcal{M}_0 - F$ there exists an element x of A such that $\hat{x}(M) = 1$ and $\hat{x} = 0$ on F (equivalently, $x \in kF - M$).

THEOREM 1.3. (Proposition II, p. 223 of Naimark [7]). *The hull-kernel topology on \mathcal{M}_0 is weaker than the w^* -topology. They agree if, and only if, A is regular.*

DEFINITION 1.3. A commutative LMC algebra A with identity is called w^* -normal (respectively, hk -normal) provided that for each pair F_1, F_2 of disjoint, w^* -closed (respectively, hk -closed) subsets of \mathcal{M}_0 there exists $x \in A$ such $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2 .

We note that if A is w^* -normal, then A is regular, the two topologies on \mathcal{M}_0 agree and \mathcal{M}_0 is a normal space. If A is hk -normal, we cannot conclude that \mathcal{M}_0 with the hk -topology is normal; since, in general, the elements of \hat{A} are not hk -continuous.

If $\{x_1, \dots, x_n\} \subseteq A$, we write $h(x_1, \dots, x_n)$ instead of $h(\{x_1, \dots, x_n\})$, and denote the ideal in A generated by this family by (x_1, \dots, x_n) . We note that $h(x_1, \dots, x_n) = h((x_1, \dots, x_n))$ and that $h(x_1, \dots, x_n) = \bigcap \{h(x_i) : i = 1, \dots, n\}$ and $H(x_1, \dots, x_n) = \bigcap \{H(x_i) : i = 1, \dots, n\}$.

THEOREM 1.4. *The first three statements about the finite family $\{x_1, \dots, x_n\} \subseteq A$ are equivalent. Each of these implies the fourth.*

- (i) $h(x_1, \dots, x_n) = \phi$ implies $(x_1, \dots, x_n) = A$.
- (hH) (ii) $h(x_1, \dots, x_n) = \phi$ implies $H(x_1, \dots, x_n) = \phi$.
- (iii) $H(x_1, \dots, x_n) = \text{Cl}_{\mathcal{M}} h(x_1, \dots, x_n)$.
- (iv) $\text{Cl}_{\mathcal{M}} h(x_1, \dots, x_n) = \bigcap \{\text{Cl}_{\mathcal{M}} h(x_i) : i = 1, \dots, n\}$.

Proof. (i) if, and only if, (ii): $H(x_1, \dots, x_n) = \phi$ if, and only if, (x_1, \dots, x_n) is not contained in any maximal ideal if, and only if, $(x_1, \dots, x_n) = A$.

(i) implies (iii):

$$\begin{aligned} \text{Cl}_{\mathcal{M}} h(x_1, \dots, x_n) &= Hk(h(x_1, \dots, x_n)) = HK(h(x_1, \dots, x_n)) \\ &\subseteq HK(H(x_1, \dots, x_n)) = H(x_1, \dots, x_n). \end{aligned}$$

Suppose $M \notin Hk(h(x_1, \dots, x_n))$. Then $kh(x_1, \dots, x_n) + M = A$ and there exist $z \in kh(x_1, \dots, x_n)$, $w \in M$ such that $z + w = 1$. Then $h(z, w) = \phi$ and $h(x_1, \dots, x_n, w) = \phi$ (since $h(x_1, \dots, x_n) \subseteq h(z)$). By (i) we have $(x_1, \dots, x_n, w) = A$ and $(w \in M)$ at least one $x_i \notin M$. Thus,

$$M \notin H(x_1, \dots, x_n).$$

(iii) implies (ii): obvious.

(iii) implies (iv): clear, since $H(x_1, \dots, x_n) = \bigcap_{i=1}^n H(x_i)$.

We consider throughout the remainder of this section only algebras which satisfy condition (hH) ((ii) of Theorem 1.4). We note the following formulation of (hH) . If $\{a_1, \dots, a_n\} \subseteq A$ we consider the equation $\sum_{i=1}^n a_i x_i = 1$ and ask for condition on A which insure solvability in A . (hH) is the assumption that the vacuousness of $h(a_1, \dots, a_n)$ is sufficient. Arens [2] gave sufficient conditions in terms of the solvability of certain related equations in Banach algebras. We shall show below that in F -algebras the vacuousness of $h(a_1, \dots, a_n)$ is sufficient for the solvability of the equation in A .

THEOREM 1.5. *Suppose F_1 and F_2 are disjoint subsets of \mathcal{M}_0 . The following statements are equivalent.*

- (i) $\text{Cl}_{\mathcal{M}} F_1 \cap \text{Cl}_{\mathcal{M}} F_2 = \phi$.
- (ii) *There exists $x \in A$ such that $\hat{x} = 0$ on F_1 , $\hat{x} = 1$ on F_2 .*
- (iii) $kF_1 + kF_2 = A$.

Proof. (i) if, and only if, (iii): $\text{Cl}_{\mathcal{M}} F_i = HkF_i, i = 1, 2$, and $HkF_1 \cap HkF_2 = H(kF_1 + kF_2)$. The equivalence follows (kF is always a closed ideal in A for $F \subseteq \mathcal{M}_0$).

(ii) if, and only if, (iii): If (iii) $kF_1 + kF_2 = A$ we choose $x \in kF_1, y \in kF_2$ such that $x + y = 1$. Then $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2 . The converse is immediate.

COROLLARY 1.5. *Disjoint w^* -closed (respectively, hk -closed) subsets of \mathcal{M}_0 have disjoint closures in \mathcal{M} if, and only if, A is w^* -normal (respectively, hk -normal).*

We now give our description of \mathcal{M} , assuming A satisfies (hH) . The result is stated in terms of a lattice compactification of \mathcal{M}_0 . The basic facts about these compactifications may be found in [12] and [13], and in the form used here in [3].

DEFINITION 1.4. A lattice \mathcal{L} (with respect to \cup and \cap) of subsets of \mathcal{M}_0 is called an α -lattice provided that for each $B \in \mathcal{L}$ and $M \in \mathcal{M}_0 - B$ there exists $D \in \mathcal{L}$ such that $M \in D, B \cap D = \phi$. \mathcal{L} is called a β -lattice provided that for each pair M_1, M_2 of distinct points of \mathcal{M}_0 there exists $B \in \mathcal{L}$ such that $M_1 \in B, M_2 \in \mathcal{M}_0 - B$. \mathcal{L} is said to be *normal* provided that for each pair B, D of disjoint members of \mathcal{L} there exists a pair B_1, D_1 of elements of \mathcal{L} such that $B \subseteq B_1, D \subseteq D_1, B \cap D_1 = \phi = B_1 \cap D$, and $B_1 \cup D_1$ belongs to every ultrafilter in \mathcal{L} (in the presence of (α) , this is equivalent to the statement that $B_1 \cup D_1 = \mathcal{M}_0$).

$w\mathcal{L} (= w(\mathcal{M}_0, \mathcal{L}))$ is the set of all ultrafilters in \mathcal{L} . For each

$E \in \mathcal{L}$ we define $C(E) = \{\mathcal{U} \in w\mathcal{L} : E \in \mathcal{U}\}$ and define a topology on $w\mathcal{L}$ by taking the family $\{C(E) : E \in \mathcal{L}\}$ as a base for the closed sets ($E \rightarrow C(E)$ is a lattice homomorphism of \mathcal{L} into the power set of $w\mathcal{L}$). The space $w\mathcal{L}$ is always compact and satisfies the T_1 separation axiom. The assumption that \mathcal{L} is an α -lattice is equivalent to assuming that for each $M \in \mathcal{M}_0$ the family $\mathcal{U}_M = \{E \in \mathcal{L} : M \in E\}$ is an ultrafilter in \mathcal{L} . If \mathcal{L} is an α -lattice then the function $\varphi: \mathcal{M}_0 \rightarrow w\mathcal{L}$ defined by $\varphi(M) = \mathcal{L}_M$ maps \mathcal{M}_0 onto a dense subspace of $w\mathcal{L}$. If \mathcal{L} is an α -lattice, then (β) is equivalent to the statement that φ is one-to-one. Normality of \mathcal{L} is equivalent to the statement that $w\mathcal{L}$ is Hausdorff. If we fix a topology \mathcal{T} on \mathcal{M}_0 then φ is continuous (assuming (α)) if, and only if, each element of \mathcal{L} is \mathcal{T} -closed, and φ is a homeomorphism if, and only if, such element of \mathcal{L} is \mathcal{T} -closed, \mathcal{L} is a β -lattice, and \mathcal{L} forms a base for the \mathcal{T} -closed subsets of \mathcal{M}_0 . (For proofs, see Theorems 2.5 and 2.7 of [3]). Finally, if $E \in \mathcal{L}$, then $(\varphi E)^-$ (the closure in $w\mathcal{L}$ will be denoted by “ $-$ ”) $= C(E)$, and for any subset F of \mathcal{M}_0 , $(\varphi F)^- = \bigcap \{C(A) : F \subseteq A\}$ (Theorem 2.6 of [3]).

LEMMA 1.6. *The family $\mathcal{L} = \{h(x_1, \dots, x_n) : \{x_1, \dots, x_n\} \subseteq A\}$ is an α -, β -lattice of hk -closed subsets of \mathcal{M}_0 which forms a base for the hk -closed sets. Thus, the mapping $\varphi(M \rightarrow \mathcal{U}_M)$ is a homeomorphism of (\mathcal{M}_0, hk) onto a dense subspace of $w\mathcal{L}$.*

Proof. The family \mathcal{L} is closed under finite intersections, since $h(x_1, \dots, x_n) = \bigcap_{i=1}^n h(x_1, \dots, x_n)$ for each finite family $\{x_1, \dots, x_n\}$ in A . Moreover, $h(x_1, \dots, x_n) \cup h(y_1, \dots, y_n) = \bigcap \{h(x_i y_i) : i = 1, \dots, n; j = 1, \dots, m\}$, the latter being an element of \mathcal{L} . Thus, \mathcal{L} is a lattice on \mathcal{M}_0 consisting of hk -closed sets which forms a base for the hk -closed sets of \mathcal{M}_0 .

If $M \in \mathcal{M}_0 - h(x_1, \dots, x_n)$, then $(x_1, \dots, x_n) + M = A$ and there exists $z \in M$ such that $\hat{z} = 1$ on $h(x_1, \dots, x_n)$. But this implies $M \in h(z)$ and $h(z) \cap h(x_1, \dots, x_n) = \phi$. Thus, \mathcal{L} is an α -lattice. That \mathcal{L} is an β -lattice is immediate.

We note that in general \mathcal{L} is not a normal lattice. For example, if A is the algebra of all functions on the open unit disc D to the complex plane which are analytic on D , then \mathcal{M}_0 and D are in a natural one-to-one correspondence. In this case, \mathcal{L} is the lattice of all discrete subsets of D plus the set D itself. It is clear that \mathcal{L} is not normal.

THEOREM 1.6. *\mathcal{M} is $w\mathcal{L}$ (i.e., there exists a homeomorphism σ of \mathcal{M} onto $w\mathcal{L}$ such that $\sigma(M) = \varphi(M)$ for each $M \in \mathcal{M}_0$).*

Proof. For each $M \in \mathcal{M}$ we let $\sigma(M)$ be the subfamily of \mathcal{L} consisting of all $E \in \mathcal{L}$ such that $M \in \text{Cl}_{\mathcal{M}} E$. It is clear that $\sigma(M)$ is a filter in \mathcal{L} . We use the criterion "A filter \mathcal{F} in \mathcal{L} is an ultrafilter if, and only if, for each $E \in \mathcal{L} - \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $E \cap F = \phi$ " ([12, p. 105]) to establish that $\sigma(M)$ is an ultrafilter. If $E \in \mathcal{L} - \sigma(M)$, then $E = h(x_1, \dots, x_n)$ for some family $\{x_1, \dots, x_n\}$ in A and $M \notin \text{Cl}_{\mathcal{M}} h(x_1, \dots, x_n) = H(x_1, \dots, x_n)$. We have $\{M\} = \bigcap \{H(y_1, \dots, y_m) : M \in H(y_1, \dots, y_m)\}$ and this family is a descending family of compact sets of \mathcal{M} whose intersection is contained in the open set $\mathcal{M} - H(x_1, \dots, x_n)$. Thus, there exists a family $\{y_1, \dots, y_m\}$ in A such that $M \in H(y_1, \dots, y_m) \subseteq \mathcal{M} - H(x_1, \dots, x_n)$. But then $h(y_1, \dots, y_m) \in \sigma(M)$ and is disjoint from $h(x_1, \dots, x_n)$.

If \mathcal{U} is an ultrafilter in \mathcal{L} , we let $\mathcal{U}^* = \{\text{Cl}_{\mathcal{M}} E : E \in \mathcal{U}\}$. Then, \mathcal{U}^* is a descending family of compact subsets of \mathcal{M} and has a nonempty intersection. It is easily verified that there is a unique element M of $\bigcap \mathcal{U}^*$ and that $\sigma(M) = \mathcal{U}$. It follows that the mapping σ is one-to-one, onto, and that for each $M \in \mathcal{M}$ $\sigma(M) = \mathcal{U}_M = \varphi(M)$. The equality $\sigma[H(x_1, \dots, x_n)] = C[h(x_1, \dots, x_n)]$ for each finite family $\{x_1, \dots, x_n\}$ in A yields the fact that σ is a homeomorphism.

We state without proof the following theorem on lattice compactifications (cf. [3, Th. 3.1]).

THEOREM 1.7. *If \mathcal{L}' is a second α -lattice on \mathcal{M}_0 , $\mathcal{L} \subseteq \mathcal{L}'$, and ψ is the mapping of \mathcal{M}_0 into $w\mathcal{L}'$, then the following statements are equivalent.*

- (i) *If $F_1, F_2 \in \mathcal{L}'$, then $F_1 \cap F_2 = \phi$ if, and only if, $(\varphi F_1)^- \cap (\varphi F_2)^- = \phi$.*
- (ii) *If $F_1, F_2 \in \mathcal{L}'$, then $\varphi(F_1 \cap F_2)^- = (\varphi F_1)^- \cap (\varphi F_2)^-$.*
- (iii) *$w\mathcal{L}' = w\mathcal{L}$ (i.e., there exists a homeomorphism τ of $w\mathcal{L}'$ onto $w\mathcal{L}$ such that $\tau\varphi(M) = \psi(M)$ for each $M \in \mathcal{M}_0$).*

We apply this theorem to our situation. We identify \mathcal{M} and $w\mathcal{L}$ here and let $\mathcal{C}(hk)$ and $\mathcal{C}(w^*)$ denote the lattices of all hk -closed subsets of \mathcal{M}_0 and all w^* -closed subsets of \mathcal{M}_0 , respectively. $W(\mathcal{M}_0, \mathcal{I})$ denotes the Wallman compactification of the topological space $(\mathcal{M}_0, \mathcal{I})$.

COROLLARY 1.7. *$\mathcal{M} = W(\mathcal{M}_0, hk)$ if, and only if, A is hk -normal. If A is regular, then (\mathcal{M}_0, w^*) is embedded homeomorphically in \mathcal{M} and $\mathcal{M} = W(\mathcal{M}_0, w^*)$ if, and only if, A is normal. In this case, \mathcal{M} is Hausdorff and $\mathcal{M} = \beta\mathcal{M}_0$.*

Proof. The first statement is clear in view of Theorem 1.7 and Corollary 1.5, where we let $\mathcal{L}' = \mathcal{C}(hk)$. The second statement

follows from the same two theorems, where $\mathcal{L}' = \mathcal{C}(w^*)$. Finally, if A is normal, then \mathcal{M}_0 is a normal space and $W(\mathcal{M}_0)$ (we suppress \mathcal{T} since the topologies agree) is Hausdorff [13, p. 119], hence $\mathcal{M} = W(\mathcal{M}_0) = \beta\mathcal{M}_0$ (cf. [7, Exercises 5P and 5R] or [3, Th. 3.2]).

EXAMPLE 1.1. We give an example to show first that in general a commutative LMC algebra can be completely regular, but not normal (the concepts are equivalent for F -algebras, see § 2), and secondly that \mathcal{M} may be $\beta\mathcal{M}_0$ while A is not normal. We let Ω be the first uncountable ordinal and w the first ordinal with countably many predecessors, Ω' is the set of all ordinals up to and including Ω , w' the set of all ordinals up to and including w , $T' = \Omega' \times w'$ with the product topology (each of Ω', w' being endowed with the order topology), and $T = T' - \{(\Omega, w)\}$. T is a locally compact Hausdorff space which is not normal and $\beta T = T'$ (cf. [4, pp. 123-124]). We let $A = C(T)$ with the compact-open topology. Then $(\mathcal{M}_0, w^*) = T$, $w^* = hk$ on T and $\mathcal{M} = \beta T = T'$. But A is not normal.

We next consider for a normal algebra A satisfying the condition (hH) the problem of identifying the subspace of \mathcal{M} which consists of the maximal ideals of A which are kernels of (possibly discontinuous) homomorphisms of A onto C . We denote this subspace by \mathcal{M}_1 .

Since A is normal, $\mathcal{M} = \beta\mathcal{M}_0$ and for each $x \in A$ the function \hat{x} on \mathcal{M}_0 is a continuous mapping of \mathcal{M}_0 into the one-point compactification $C^* = C \cup \{\infty\}$ of C . Thus \hat{x} has an extension x^* , a C^* -valued continuous function on $w\mathcal{L}(=\beta\mathcal{M}_0)$. Discussions of this extension and of the realcompactification of a space are found in Chapters 7 and 8 of [4]. The realcompactification of $\mathcal{M}_0, \nu\mathcal{M}_0$, is the subspace of $\beta\mathcal{M}_0$ consisting of all $\mathcal{V} \in \beta\mathcal{M}_0$ such that for each $z \in C(\mathcal{M}_0)$ $\mathcal{V} \in z^{*-1}(C)$, i.e. z^* does not take on the value ∞ at \mathcal{V} , where z^* is the extension of the mapping $z: \mathcal{M}_0 \rightarrow C^*$ to $\beta\mathcal{M}_0$.

DEFINITION 1.5. $\nu_A\mathcal{M}_0$ (the A -realcompactification of \mathcal{M}_0) = $\{\mathcal{U} \in w\mathcal{L}: x^*(\mathcal{U}) \in C \text{ for each } x \in A\}$.

THEOREM 1.8. If $\mathcal{U} \in \nu_A\mathcal{M}_0$ and $\mathcal{U} = \sigma(M)$, then $M = \{x \in A: x^*(\mathcal{U}) = 0\}$.

Proof. If $M \in \mathcal{M}$ and $\sigma(M) = \mathcal{U} \in \nu_A\mathcal{M}_0$, then the set $I = \{x \in A: x^*(\mathcal{U}) = 0\}$ is an ideal in A . Moreover, if $x \in M$, then $\mathcal{U} \in C[h(x)] = h(x)^-$ and since x^* is continuous on $w\mathcal{L}$ and agrees with \hat{x} on \mathcal{M}_0 , $\hat{x}^*(\mathcal{U}) = 0$. Therefore, $M \subseteq I$ and $I \neq A(1 \notin I)$. Hence, $M = I$.

THEOREM 1.9. The restriction of the mapping $\sigma: \mathcal{M} \rightarrow w\mathcal{L}$ to

\mathcal{H}_1 , is a homeomorphism of \mathcal{H}_1 onto $v_A \mathcal{H}_0$.

Proof. If $\mathcal{U} \in v_A \mathcal{H}_0$ and $\mathcal{U} = \sigma(M)$, then the mapping $x \rightarrow x^*(\mathcal{U})$ is a homomorphism of A onto C with kernel M and $M \in \mathcal{H}_1$.

If $M \in \mathcal{H}_1$ and $\mathcal{U} = \sigma(M)$, then for each $x \in A$ there exists $\lambda \in C$ ($\lambda = M(x)$) such that $x - \lambda \in M$. We fix $x \in A$ and the corresponding (unique) $\lambda \in C$. If $x - \lambda \in M$, then $M \in H(x - \lambda)$ and $\mathcal{U} \in C[h(x - \lambda)]$. This implies $(x - \lambda)^*(\mathcal{U}) = 0$. Since $\lambda^*(\mathcal{U}) = \lambda \in C$, we have $x^*(\mathcal{U}) = [(x - \lambda) + \lambda]^*(\mathcal{U}) = (x - \lambda)^*(\mathcal{U}) + \lambda^*(\mathcal{U}) = \lambda \in C$ and $\mathcal{U} \in v_A \mathcal{H}_0$.

We wish to acknowledge here our indebtedness to Donald L. Plank of the Case Western Reserve University who communicated to the author theorems analogous to 1.8 and 1.9 for a real algebra A of functions on a completely regular space X satisfying: $BC(X) \subseteq A \subseteq C(X)$, where $BC(X)$ is the algebra of all bounded real-valued functions on X to \mathbf{R} .

2. A special case. We consider in this section the special case: A is a commutative F -algebra with identity 1 — a complete LMC algebra whose topology is given by a countably family of pseudonorms. In this case we can assume that the family $\{p_n\}_{n=0}^\infty$ satisfies: $p_n(x) \leq p_{n+1}(x)$ for each $n \geq 0$ and each $x \in A$. The fact the F -algebras are inverse limits of Banach algebras is important for our purposes. We let $N_k = \{x \in A: p_k(x) = 0\}$, Π_k the natural map of A onto A/N_k and A_k the completion of A/N_k with respect to the norm defined by $\|\Pi_k x\| = p_k(x)$. Each A_n is a commutative Banach algebra with identity. For each $n \geq 0$ there is a norm-decreasing homomorphism Π_n^{n+1} of A_{n+1} onto a dense subalgebra of A_n which is defined on A/N_{n+1} by $\Pi_n^{n+1}(\Pi_{n+1}x) = \Pi_n x$ and extended to A_{n+1} . For $n \leq m$, $\Pi_n^m: A_m \rightarrow A_n$ is defined by the obvious composition. The resulting family of algebras and homomorphisms is an inverse limit system and A is isomorphic and pseudo-isometric to the inverse limit of this system. An important consequence of this is the following fact. If $\{\xi_n\}_{n=0}^\infty$ is a sequence where $\xi_n \in A_n$ and $\Pi_n^m \xi_m = \xi_n$ whenever $n \leq m$, then there exists $x \in A$ such that $\Pi_n x = \xi_n$ for each $n \geq 0$. For details of this construction and the basic facts about such systems, the reader is referred to [9].

We state without proof two theorems, the first is just Theorem 4.2 of [2] in our terminology, the second is immediate.

THEOREM 2.1. Suppose $\{a_1, \dots, a_m\}$ is a family of elements of A such that $(\Pi_n a_1, \dots, \Pi_n a_m) = A_n$ for each $n \geq 0$. Then $(a_1, \dots, a_m) = A$.

THEOREM 2.2. If $\{\xi_1, \dots, \xi_m\}$ is a finite family in A_n and $\hat{\xi}_1, \dots, \hat{\xi}_m$ have no common zeros on $\mathcal{H}(A_n)$ (the structure space of A_n), then $(\xi_1, \dots, \xi_m) = A_n$.

The spectrum \mathcal{M}_0 of A has the following structure. $\mathcal{M}_0 = \bigcup \{\mathcal{M}^k: k = 0, 1, 2, \dots\}$, where each \mathcal{M}^k is homeomorphic to $\mathcal{M}(A_k)$, the structure space of A_k . The homeomorphism σ_k of $\mathcal{M}(A_k)$ into \mathcal{M}_0 is defined by $[\sigma_k(M^k)](x) = M^k(\Pi_k x)$ for each $M^k \in \mathcal{M}(A_k)$ and $x \in A$.

THEOREM 2.3. *If A is a commutative F -algebra with identity, then A satisfies the condition (hH) .*

Proof. We fix a family $\{x_1, \dots, x_n\}$ in A satisfying $h(x_1, \dots, x_n) = \phi$ and show $H(x_1, \dots, x_n) = \phi$. We need only show that for each $k \geq 0$ the family $\{\Pi_k x_1, \dots, \Pi_k x_n\}$ generates the improper ideal A_k . We fix $k \geq 0$. For each i we have $(\Pi_k x_i)(M^k) = x_i(\sigma_k M^k)$. Therefore, the family $\{(\Pi_k x_1)^\wedge, \dots, (\Pi_k x_n)^\wedge\}$ has a common zero on $\mathcal{M}(A_k)$ if, and only if, the intersection of \mathcal{M}^k and $h(x_1, \dots, x_n)$ is nonempty. We have assumed that $h(x_1, \dots, x_n)$ is empty. Thus, Theorem 2.2 implies $(\Pi_k x_1, \dots, \Pi_k x_n) = A_k$ for each $k \geq 0$, and we obtain $(x_1, \dots, x_n) = A$.

Thus, F -algebras always satisfy the conditions of Theorem 1.4 and $\mathcal{M} = w\mathcal{L}$. We next extend Theorem 2.1 to pin down further the space \mathcal{M} . We note that Theorem 2.4 is immediate for Banach algebras (since $\mathcal{M}_0 = \mathcal{M}$) and false for commutative LMC algebras in general (cf. Example 1.1 above).

THEOREM 2.4. *If I_1 and I_2 are closed ideals in A and if $h(I_1) \cap h(I_2) = \phi$, then $I_1 + I_2 = A$.*

Proof. We shall construct two sequences in $\Pi_n A_n$, show that they yield elements of I_1 and I_2 whose sum is 1. We let $F_1 = h(I_1)$ and $F_2 = h(I_2)$. Since the tail of a sequence is the important thing in determining whether it corresponds to an element of A we assume that $F_1 \cap \mathcal{M}^0 \neq \phi$ and $F_2 \cap \mathcal{M}^0 \neq \phi$. If not we begin the construction with the first integer k so that both F_1 and F_2 meet \mathcal{M}^k and define the first k terms by the maps $\Pi_i^k, i = 0, \dots, k-1$.

Since $F_1 \cap F_2 = \phi$, $\sigma_n^{-1}(F_1 \cap \mathcal{M}^n) \cap \sigma_n^{-1}(F_2 \cap \mathcal{M}^n) = \phi$ in $\mathcal{M}(A_n)$ for each $n \geq 0$. We first note that for each $n \geq 0$ $\Pi_n(I_1)^-$ and $\Pi_n(I_2)^-$ are closed ideals in A_n and $\Pi_n(I_1)^- + \Pi_n(I_2)^- = A_n$. If not, then there exists $M^n \in (A_n)$ such that $\Pi_n(I_1)^-, \Pi_n(I_2)^- \subseteq M^n$. Then $I_1, I_2 \subseteq M = \sigma_n(M^n)$, and $M \in F_1 \cap F_2$, a contradiction. By Lemma 7.8 of [9] I_j is the inverse limit of the sequence $\{\Pi_n(I_j)^-\}$ with the restricted homomorphism, for $j = 1, 2$, and for each pair $n, m, n \leq m$, $\Pi_n^m[\Pi_m(I_j)^-]$ is dense in $\Pi_n(I_j)^-$, since the former contains $\Pi_n^m[\Pi_m(I_j)] = \Pi_n(I_j)$ which is dense in $\Pi_n(I_j)^-$.

We first choose a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers such that the series $\sum_{n=1}^\infty \varepsilon_n$ converges. Since $\Pi_0(I_1)^- + \Pi_0(I_2)^- = A_0$ we choose

$\xi_0^j \in \Pi_0(I_j)^-, j = 1, 2$ such that $\xi_0^1 + \xi_0^2 = 1$.

We next choose ζ_i^j in $\Pi_1(I_j)^-, j = 1, 2$, such that $\zeta_1^1 + \zeta_1^2 = 1$, then choose η_i^j in $\Pi_1(I_j)^-, j = 1, 2$, such that

$$\| \Pi_0^1 \gamma_1^j - \xi_0^j \| < \min \left(\varepsilon_1/4, \varepsilon_1/4 \max_{j=1,2} \| \xi_1^j \| \right).$$

This is possible because $\Pi_0^1[\Pi_1(I_j)^-]$ is dense in $\Pi_0(I_j)^-, j = 1, 2$. We let $\xi_1^j = \eta_1^j + \zeta_1^j(1 - \eta_1^j), i = 1, 2$. Then

$$\xi_1^j \in \Pi_1(I_j)^-, j = 1, 2; \xi_1^1 + \xi_1^2 = 1,$$

and $\| \Pi_0^1 \xi_1^j - \xi_0^j \| < \varepsilon_1$, for each j .

Proceeding inductively we choose for each $n \geq 1, j = 1, 2, \xi_n^j \in \Pi_n(I_j)^-$ such that $\| \Pi_{n-1}^n \xi_n^j - \xi_{n-1}^j \| < \varepsilon_n$, and $\xi_n^1 + \xi_n^2 = 1$. Then for $k = 0, 1, \dots, n-1$ we have

$$(3.1) \quad \| \Pi_{k \xi_n}^n - \Pi_{k \xi_{n-1}}^{n-1} \| < \varepsilon_n.$$

From this point on the construction is identical to that given in the proof of Theorem 4.2 of [2]. We sketch the important steps.

We first fix $n \geq 0$ and let $x_j(n)_k = \Pi_{k \xi_n}^k$ for each $k \geq n, j = 1, 2$. $\{x_j(n)_k\}_{k=n}^\infty$ is a sequence in $\Pi_n(I_j)^-$ and satisfies

- (i) $\Pi_n^{n+1}(x_j(n+1)_k) = x_j(n)_k$ for each $k \geq n+1, j = 1, 2$;
- (ii) $x_1(n)_k + x_2(n)_k = 1$,
- (iii) $\| x_j(n)_k - x_j(n)_{k+p} \| < \varepsilon_{k+1} + \dots + \varepsilon_{k+p}$.

Thus the sequences are Cauchy for each n, j and converge to elements $x_j(n)$ in $\Pi_n(I_j)^-$ for each $n \geq 0, j = 1, 2$. There exist $x_1 \in I_1, x_2 \in I_2$ such that $\Pi_n(x_j) = x_j(n)$ for each $n \geq 0, j = 1, 2$. Thus, $x_1 + x_2 = 1$.

COROLLARY 2.4.1. *If F_1 and F_2 are disjoint hk -closed subsets of \mathcal{M}_0 , then $\text{Cl}_{\mathcal{M}} F_1 \cap \text{Cl}_{\mathcal{M}} F_2 = \phi$.*

Proof. Letting $I_1 = kF_1$ and $I_2 = kF_2$ yields $I_1 + I_2 = A$. Apply Theorem 1.5.

COROLLARY 2.4.2. *If A is a commutative F -algebra with identity, then $\mathcal{M} = W(\mathcal{M}_0, hk)$. Moreover, if A is regular, then A is normal and $\mathcal{M} = W(\mathcal{M}_0) = \beta \mathcal{M}_0$.*

Proof. The first statement follows from Corollary 2.4.1, Theorem 1.5, and Corollary 1.7. The second follows from Corollaries 1.7 and 2.4.1.

We note that Rosenfeld [11] has indicated a proof of part of Corollary 2.4.2 (A regular implies A normal) using Silov's theorem. This theorem also yields a proof of Corollary 2.4.1, since $F_1 \cup F_2$ is

hk -closed in \mathcal{M}_0 and is $\mathcal{M}_0(B)$, where $B = A/(kF_1 \cap kF_2)$. However, since the application of this theorem yields an element a of A such that $\hat{a} \equiv 0$ on F_1 and $\hat{a} \equiv 1$ on F_2 , we can conclude only that $kF_1 + kF_2 = A$. Thus, it does not appear that the proof of Theorem 2.4 can be simplified by the use of this tool.

THEOREM 2.5. *Let I be a closed ideal in A and $B = A/I$. Then B is a commutative F -algebra with identity, $\mathcal{M}_0(B)$ is homeomorphic to $h(I)$ with respect to both the w^* - and hk -topologies, and $\mathcal{M}(B) = \text{Cl}_{\mathcal{M}(A)} h(I)$.*

Proof. The first conclusion follows from the open mapping theorem for F -spaces (cf. [8, Lemma 11.3]) and the fact that the natural map Π of A onto B is continuous and open. The range of $\Pi^*: \mathcal{M}_0(B) \rightarrow \mathcal{M}_0(A)$ is easily seen to be $h(I)$ and it is also immediate that Π^* is a w^* -homeomorphism. For convenience we let $F = h(I)$ for the remainder of the proof.

We show that for each $E \subseteq F$, $\Pi^{*-1}[hk(E)] = h'k'[\Pi^{*-1}(E)]$, where h' and k' are the h - and k -operators for B . $M' \in \Pi^{*-1}[hk(E)]$ if, and only if, $M \in hk(E)$ ($M = \Pi^*M'$) if, and only if, $M(x) = 0$ for each $x \in kE$. And $x \in kE$ if, and only if, $M_1(x) = 0$ for each $M_1 \in E$ if, and only if, $M'_1(\Pi x) = 0$ for each $M'_1 \in \Pi^{*-1}(E)$. So $x \in kE$ if, and only if, $\Pi x \in k'[\Pi^{*-1}(E)]$. Thus, from above, $M(x) = 0$ for each $x \in kE$ if, and only if, $M'(\Pi x) = 0$ for each $\Pi x \in k'[\Pi^{*-1}(E)]$, if, and only if, $M' \in h'k'[\Pi^{*-1}(E)]$. The equality is established and it is immediate that Π^* is a homeomorphism with respect to the hk -topologies in $\mathcal{M}_0(B)$ and F .

For each $x \in A$ we have $\Pi^*[h'(\Pi x)] = h(x) \cap F$. Thus, there is a lattice isomorphism of $\mathcal{L}' = \{h(\xi_1, \dots, \xi_n): \{\xi_1, \dots, \xi_n\} \subseteq B\}$ onto $\mathcal{L}_F = \{E \subseteq F: E = B \cap F \text{ for some } B \in \mathcal{L}\}$, and there is induced a homeomorphism of $w\mathcal{L}'$ onto $w\mathcal{L}_F$. Therefore, $\mathcal{M}(B)$ is homeomorphic to $w\mathcal{L}_F$. For each $M \in \text{Cl}_{\mathcal{M}(A)} F$ we define $\tau(M) = \{E \in \mathcal{L}_F: M \in \text{Cl}_{\mathcal{M}} E\}$. $M \rightarrow \tau(M)$ is a one-to-one mapping of $\text{Cl}_{\mathcal{M}(A)} F$ onto $w\mathcal{L}_F$. From the easily verified equation $H(x) \cap \text{Cl}_{\mathcal{M}(A)} F = C[h(x) \cap F]$ it follows that τ is a homeomorphism.

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