Pacific Journal of Mathematics

THE STRUCTURE SPACE OF A COMMUTATIVE LOCALLY *m*-CONVEX ALGEBRA

ROBERT MORGAN BROOKS

Vol. 25, No. 3 November 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 3, 1968

THE STRUCTURE SPACE OF A COMMUTATIVE LOCALLY M-CONVEX ALGEBRA

R. M. BROOKS

If A is a commutative Banach algebra with identity, then the sets $\mathscr{M}$ (all maximal ideals), $\mathscr{M}_c$ (all closed maximal ideals), $\mathscr{M}_1$ (kernels of nonzero C-valued homomorphisms of A), and $\mathscr{M}_0$ (kernels of nonzero continuous C-valued hommorphisms of A) coincide. If A is a commutative complete locally m-convex algebra, one has only $\mathscr{M}_c = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}$, and the containments can be proper. Our goal is to investigate $\mathscr{M}$ and its relationship to $\mathscr{M}_0$; specifically (1) to give a description of $\mathscr{M}(A)$ in terms of A and $\mathscr{M}_0(A)$ which is valid for at least the class of F-algebras, (2) to determine when $\mathscr{M}(A)$ is one of the standard compactifications (Wallman, Stone-Čech) of $\mathscr{M}_0(A)$.

For many locally m-convex algebras, especially algebras of functions, one can determine $\mathscr{M}_0$. However, descriptions of $\mathscr{M}$ and its relationship to $\mathscr{M}_0$ seem to be limited to special cases; for example, Hewitt's description of $\mathscr{M}(C(X))$ [5] and Kakutani's description of $\mathscr{M}$ for the algebra of analytic functions in the unit disc [6]. We show that a commutative complete locally m-convex algebra A generates a lattice $\mathscr{L}$ on $\mathscr{M}_0$, and that if we impose a rather natural restriction on A, then $\mathscr{M}$ is the space of ultrafilters of $\mathscr{L}$. We give necessary and sufficient conditions on A in order that (1) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, hull-kernel), (2) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, Gelfand). In the second case, we show that $\mathscr{M} = \beta\mathscr{M}_0$ and obtain a correspondence between $\mathscr{M}_1$ and the A-realcompactification of $\mathscr{M}_0$.

We then specialize to F-algebras and show (1) F-algebras always satisfy the condition imposed in the general situation, (2) $\mathscr{M}$ is the Wallman compactification of ($\mathscr{M}_0$, hull-kernel), and (3) $\mathscr{M} = \beta\mathscr{M}_0$, whenever the algebra is regular.

1. The general case. A locally m-convex algebra (hereafter LMC algebra) is a locally convex Hausdorff topological algebra A whose topology is given by a family of pseudonorms (submultiplicative, convex, symmetric functionals). For the basic properties of these algebras the reader is referred to [1] or [9]. In this paper we shall restrict our attention to complete algebras with identity element 1. If λ is a complex number we shall write "λ" for "$\lambda \cdot 1$".

The *structure space* of A is the set $\mathscr{M}$ of all maximal ideals of

A, endowed with the hull-kernel ($hk-$) topology. This space is always compact and satisfies the T_1 separation axiom. The *spectrum* of A is the set $\mathscr{M}_0$ of all closed maximal ideals of A.

DEFINITION 1.1. If $S \subseteqq A, F \subseteqq \mathscr{M}_0, G \subseteqq \mathscr{M}, x \in A$, then

(i) $H(S) = \{M \in \mathscr{M} : S \subseteqq M\}$.

(ii) $h(S) = \{M \in \mathscr{M}_0 : S \subseteqq M\}$.

(iii) $kF(=k(F)) = \bigcap \{M \in \mathscr{M}_0 : M \in F\} = \{x \in A : x \in M$ for each $M \in F\}$.

(iv) $K(G) = \bigcap \{M \in \mathscr{M} : M \in G\} = \{x \in A : x \in M$ for each $M \in F\}$.

(v) $H(x) = H(\{x\}), h(x) = h(\{x\})$.

The hull-kernel topology is defined in terms of the closure operator:

$$\mathrm{Cl}_{\mathscr{M}}(F) = HK(F)\ , \tag{1.1}$$

or

$$\mathrm{Cl}_{\mathscr{M}}(F) = \bigcap \{H(x) : F \subseteqq H(x)\}, \text{ for each } F \subseteqq \mathscr{M}. \tag{1.2}$$

A simple computation yields

THEOREM 1.1. *The closure operator on $\mathscr{M}_0$ which defines the relative hull-kernel topology on $\mathscr{M}_0$ is given by*

$$\mathrm{Cl}_{\mathscr{M}_0}(F) = hk(F) \tag{1.3}$$

or

$$\mathrm{Cl}_{\mathscr{M}_0}(F) = \bigcap \{h(x) : F \subseteqq h(x)\}, \text{ for each } F \subseteqq \mathscr{M}_0\,. \tag{1.4}$$

The spectrum can also be endowed with a second natural topology. If $M \in \mathscr{M}_0$, then M is the kernel of a unique continuous homomorphism of A onto C [9, p. 11]. We identify M and the corresponding homomorphism, denoting the value of the homomorphism at an element x of A by $M(x)$, and endow $\mathscr{M}_0$ with the relative weak $-(w^*-)$ topology from A^*, the conjugate space of A. This topology is the weakest such that all of the functions $\hat{x} : \mathscr{M}_0 \to C$ defined by $\hat{x}(M) = M(x)$ for each $x \in A$ are continuous. We state without proof the basic properties of the mapping $x \to \hat{x}$ of A into $C(\mathscr{M}_0)$ (cf [9, Props. 7.3, 8.1, and 9.2] and [4, Ex. 7M]).

THEOREM 1.2. *The mapping $x \to \hat{x}$ is a homomorphism of A onto a subalgebra $\hat{A}$ of $C(\mathscr{M}_0)$ which contains the constant functions and separates the points of $\mathscr{M}_0$. The kernel of this homomorphism is the radical $\mathscr{R}(A)$ of A, and $\mathscr{R}(A) = \bigcap \{M : M \in \mathscr{M}_0\} = \bigcap \{M : M \in \mathscr{M}\} = \{x \in A : (1 + ax)$ is regular (invertible) for each $x \in A\}$ is a closed ideal in A. Hence, $\mathscr{M}_0$ is dense in $\mathscr{M}$.*

DEFINITION 1.2. A commutative LMC algebra A with identity is called *regular* provided that for each w^*-closed subset F on $\mathscr{M}_0$ and each point $M \in \mathscr{M}_0 - F$ there exists an element x of A such that $\hat{x}(M) = 1$ and $\hat{x} = 0$ on F (equivalently, $x \in kF - M$).

THEOREM 1.3. (Proposition II, p. 223 of Naimark [7]). *The hull-kernel topology on $\mathscr{M}_0$ is weaker than the w^*-topology. They agree if, and only if, A is regular.*

DEFINITION 1.3. A commutative LMC algebra A with identity is called *w^*-normal* (respectively, *hk-normal*) provided that for each pair F_1, F_2 of disjoint, w^*-closed (respectively, hk-closed) subsets of $\mathscr{M}_0$ there exists $x \in A$ such $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2.

We note that if A is w^*-normal, then A is regular, the two topologies on $\mathscr{M}_0$ agree and $\mathscr{M}_0$ is a normal space. If A is hk-normal, we cannot conclude that $\mathscr{M}_0$ with the hk-topology is normal; since, in general, the elements of $\hat{A}$ are not hk-continuous.

If $\{x_1, \cdots, x_n\} \subseteqq A$, we write $h(x_1, \cdots, x_n)$ instead of $h(\{x_1, \cdots, x_n\})$, and denote the ideal in A generated by this family by $(x_1, \cdots, x_n)$. We note that $h(x_1, \cdots, x_n) = h((x_1, \cdots, x_n))$ and that $h(x_1, \cdots, x_n) = \bigcap \{h(x_i): i = 1, \cdots, n\}$ and $H(x_1, \cdots, x_n) = \bigcap \{H(x_i): i = 1, \cdots, n\}$.

THEOREM 1.4. *The first three statements about the finite family $\{x_1, \cdots, x_n\} \subseteqq A$ are equivalent. Each of these implies the fourth.*

(i) $h(x_1, \cdots, x_n) = \phi$ *implies* $(x_1, \cdots, x_n) = A$.

(hH)(ii) $h(x_1, \cdots, x_n) = \phi$ *implies* $H(x_1, \cdots, x_n) = \phi$.

(iii) $H(x_1, \cdots, x_n) = \mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n)$.

(iv) $\mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n) = \bigcap \{\mathrm{Cl}_{\mathscr{M}}\, h(x_i): i = 1, \cdots, n\}$.

Proof. (i) if, and only if, (ii): $H(x_1, \cdots, x_n) = \phi$ if, and only if, $(x_1, \cdots, x_n)$ is not contained in any maximal ideal if, and only if, $(x_1, \cdots, x_n) = A$.

(i) implies (iii):

$$\begin{aligned}\mathrm{Cl}_{\mathscr{M}}\, h(x_1, \cdots, x_n) &= Hk(h(x_1, \cdots, x_n)) = HK(h(x_1, \cdots, x_n)) \\ &\subseteqq HK(H(x_1, \cdots, x_n)) = H(x_1, \cdots, x_n)\,.\end{aligned}$$

Suppose $M \notin Hk(h(x_1, \cdots, x_n))$. Then $kh(x_1, \cdots, x_n) + M = A$ and there exist $z \in kh(x_1, \cdots, x_n)$, $w \in M$ such that $z + w = 1$. Then $h(z, w) = \phi$ and $h(x_1, \cdots, x_n, w) = \phi$ (since $h(x_1, \cdots, x_n) \subseteqq h(z)$). By (i) we have $(x_1, \cdots, x_n, w) = A$ and $(w \in M)$ at least one $x_i \notin M$. Thus,

$$M \notin H(x_1, \cdots, x_n)\,.$$

(iii) implies (ii): obvious.

(iii) implies (iv): clear, since $H(x_1, \cdots, x_n) = \bigcap_{i=1}^n H(x_i)$.

We consider throughout the remainder of this section only algebras which satisfy condition (hH) ((ii) of Theorem 1.4). We note the following formulation of (hH). If $\{a_1, \cdots, a_n\} \subseteqq A$ we consider the equation $\sum_{i=1}^n a_i x_i = 1$ and ask for condition on A which insure solvability in A. (hH) is the assumption that the vacuousness of $h(a_1, \cdots, a_n)$ is sufficient. Arens [2] gave sufficient conditions in terms of the solvability of certain related equations in Banach algebras. We shall show below that in F-algebras the vacuousness of $h(a_1, \cdots, a_n)$ is sufficient for the solvability of the equation in A.

THEOREM 1.5. *Suppose F_1 and F_2 are disjoint subsets of $\mathscr{M}_0$. The following statements are equivalent.*

(i) $\mathrm{Cl}_{\mathscr{M}} F_1 \cap \mathrm{Cl}_{\mathscr{M}} F_2 = \phi$.

(ii) *There exists $x \in A$ such that $\hat{x} = 0$ on F_1, $\hat{x} = 1$ on F_2.*

(iii) $kF_1 + kF_2 = A$.

Proof. (i) if, and only if, (iii): $\mathrm{Cl}_{\mathscr{M}} F_i = HkF_i$, $i = 1, 2$, and $HkF_1 \cap HkF_2 = H(kF_1 + kF_2)$. The equivalence follows (kF is always a closed ideal in A for $F \subseteqq \mathscr{M}_0$).

(ii) if, and only if, (iii): If (iii) $kF_1 + kF_2 = A$ we choose $x \in kF_1$, $y \in kF_2$ such that $x + y = 1$. Then $\hat{x} = 0$ on F_1 and $\hat{x} = 1$ on F_2. The converse is immediate.

COROLLARY 1.5. *Disjoint w^*-closed (respectively, hk-closed) subsets of $\mathscr{M}_0$ have disjoint closures in $\mathscr{M}$ if, and only if, A is w^*-normal (respectively, hk-normal).*

We now give our description of $\mathscr{M}$, assuming A satisfies (hH). The result is stated in terms of a lattice compactification of $\mathscr{M}_0$. The basic facts about these compactifications may be found in [12] and [13], and in the form used here in [3].

DEFINITION 1.4. A lattice $\mathscr{L}$ (with respect to $\cup$ and $\cap$) of subsets of $\mathscr{M}_0$ is called an *α-lattice* provided that for each $B \in \mathscr{L}$ and $M \in \mathscr{M}_0 - B$ there exists $D \in \mathscr{L}$ such that $M \in D$, $B \cap D = \phi$. $\mathscr{L}$ is called a *β-lattice* provided that for each pair M_1, M_2 of distinct points of $\mathscr{M}_0$ there exists $B \in \mathscr{L}$ such that $M_1 \in B$, $M_2 \in \mathscr{M}_0 - B$. $\mathscr{L}$ is said to be *normal* provided that for each pair B, D of disjoint members of $\mathscr{L}$ there exists a pair B_1, D_1 of elements of $\mathscr{L}$ such that $B \subseteqq B_1$, $D \subseteqq D_1$, $B \cap D_1 = \phi = B_1 \cap D$, and $B_1 \cup D_1$ belongs to every ultrafilter in $\mathscr{L}$ (in the presence of (α), this is equivalent to the statement that $B_1 \cup D_1 = \mathscr{M}_0$).

$w\mathscr{L} (= w(\mathscr{M}_0, \mathscr{L}))$ is the set of all ultrafilters in $\mathscr{L}$. For each

$E \in \mathscr{L}$ we define $C(E) = \{\mathscr{U} \in w\mathscr{L}: E \in \mathscr{U}\}$ and define a topology on $w\mathscr{L}$ by taking the family $\{C(E): E \in \mathscr{L}\}$ as a base for the closed sets ($E \to C(E)$ is a lattice homomorphism of $\mathscr{L}$ into the power set of $w\mathscr{L}$). The space $w\mathscr{L}$ is always compact and satisfies the T_1 separation axiom. The assumption that $\mathscr{L}$ is an α-lattice is equivalent to assuming that for each $M \in \mathscr{M}_0$ the family $\mathscr{U}_M = \{E \in \mathscr{L}: M \in E\}$ is an ultrafilter in $\mathscr{L}$. If $\mathscr{L}$ is an α-lattice then the function $\varphi: \mathscr{M}_0 \to w\mathscr{L}$ defined by $\varphi(M) = \mathscr{L}_M$ maps $\mathscr{M}_0$ onto a dense subspace of $w\mathscr{L}$. If $\mathscr{L}$ is an α-lattice, then (β) is equivalent to the statement that φ is one-to-one. Normality of $\mathscr{L}$ is equivalent to the statement that $w\mathscr{L}$ is Hausdorff. If we fix a topology $\mathscr{T}$ on $\mathscr{M}_0$ then φ is continuous (assuming (α)) if, and only if, each element of $\mathscr{L}$ is $\mathscr{T}$-closed, and φ is a homeomorphism if, and only if, such element of $\mathscr{L}$ is $\mathscr{T}$-closed, $\mathscr{L}$ is a β-lattice, and $\mathscr{L}$ forms a base for the $\mathscr{T}$-closed subsets of $\mathscr{M}_0$. (For proofs, see Theorems 2.5 and 2.7 of [3]). Finally, if $E \in \mathscr{L}$, then $(\varphi E)^-$ (the closure in $w\mathscr{L}$ will be denoted by "$-$") $= C(E)$, and for any subset F of $\mathscr{M}_0$, $(\varphi F)^- = \bigcap \{C(A): F \subseteqq A\}$ (Theorem 2.6 of [3]).

LEMMA 1.6. *The family* $\mathscr{L} = \{h(x_1, \cdots, x_u): \{x_1, \cdots, x_u\} \subseteqq A\}$ *is an* $\alpha-$, β*-lattice of hk-closed subsets of* $\mathscr{M}_0$ *which forms a base for the hk-closed sets. Thus, the mapping* $\varphi(M \to \mathscr{U}_M)$ *is a homeomorphism of* $(\mathscr{M}_0, hk)$ *onto a dense subspace of* $w\mathscr{L}$.

Proof. The family $\mathscr{L}$ is closed under finite intersections, since $h(x_1, \cdots, x_n) = \bigcap_{i=1}^n h(x_1, \cdots, x_n)$ for each finite family $\{x_1, \cdots, x_n\}$ in A. Moreover, $h(x_1, \cdots, x_n) \cup h(y_1, \cdots, y_n) = \bigcap \{h(x_i y_i): i = 1, \cdots, n;$ $j = 1, \cdots, m\}$, the latter being an element of $\mathscr{L}$. Thus, $\mathscr{L}$ is a lattice on $\mathscr{M}_0$ consisting of hk-closed sets which forms a base for the hk-closed sets of $\mathscr{M}_0$.

If $M \in \mathscr{M}_0 - h(x_1, \cdots, x_n)$, then $(x_1, \cdots, x_n) + M = A$ and there exists $z \in M$ such that $\hat{z} = 1$ on $h(x_1, \cdots, x_n)$. But this implies $M \in h(z)$ and $h(z) \cap h(x_1, \cdots, x_n) = \phi$. Thus, $\mathscr{L}$ is an α-lattice. That $\mathscr{L}$ is an β-lattice is immediate.

We note that in general $\mathscr{L}$ is not a normal lattice. For example, if A is the algebra of all functions on the open unit disc D to the complex plane which are analytic on D, then $\mathscr{M}_0$ and D are in a natural one-to-one correspondence. In this case, $\mathscr{L}$ is the lattice of all discrete subsets of D plus the set D itself. It is clear that $\mathscr{L}$ is not normal.

THEOREM 1.6. $\mathscr{M}$ *is* $w\mathscr{L}$ *(i.e., there exists a homeomorphism* σ *of* $\mathscr{M}$ *onto* $w\mathscr{L}$ *such that* $\sigma(M) = \varphi(M)$ *for each* $M \in \mathscr{M}_0$*).*

Proof. For each $M \in \mathscr{M}$ we let $\sigma(M)$ be the subfamily of $\mathscr{L}$ consisting of all $E \in \mathscr{L}$ such that $M \in \mathrm{Cl}_{\mathscr{M}} E$. It is clear that $\sigma(M)$ is a filter in $\mathscr{L}$. We use the criterion "A filter $\mathscr{F}$ in $\mathscr{L}$ is an ultrafilter if, and only if, for each $E \in \mathscr{L} - \mathscr{F}$ there exists $F \in \mathscr{F}$ such that $E \cap F = \phi$" ([12, p. 105]) to establish that $\sigma(M)$ is an ultrafilter. If $E \in \mathscr{L} - \sigma(M)$, then $E = h(x_1, \cdots, x_n)$ for some family $\{x_1, \cdots, x_n\}$ in A and $M \notin \mathrm{Cl}_{\mathscr{M}} h(x_1, \cdots, x_n) = H(x_1, \cdots, x_n)$. We have $\{M\} = \bigcap \{H(y_1, \cdots, y_n): M \in H(y_1, \cdots, y_m)\}$ and this family is a descending family of compact sets of $\mathscr{M}$ whose intersection is contained in the open set $\mathscr{M} - H(x_1, \cdots, x_n)$. Thus, there exists a family $\{y_1, \cdots, y_m\}$ in A such that $M \in H(y_1, \cdots, y_m) \subseteqq \mathscr{M} - H(x_1, \cdots, x_n)$. But then $h(y_1, \cdots, y_m) \in \sigma(M)$ and is disjoint from $h(x_1, \cdots, x_n)$.

If $\mathscr{U}$ is an ultrafilter in $\mathscr{L}$, we let $\mathscr{U}^* = \{\mathrm{Cl}_{\mathscr{M}} E: E \in \mathscr{U}\}$. Then, $\mathscr{U}^*$ is a descending family of compact subsets of $\mathscr{M}$ and has a nonempty intersection. It is easily verified that there is a unique element M of in $\cap \mathscr{U}^*$ and that $\sigma(M) = \mathscr{U}$. It follows that the mapping σ is one-to-one, onto, and that for each $M \in \mathscr{M}_0$ $\sigma(M) = \mathscr{U}_M = \varphi(M)$. The equality $\sigma[H(x_1, \cdots, x_n)] = C[h(x_1, \cdots, x_n)]$ for each finite family $\{x_1, \cdots, x_n\}$ in A yields the fact that σ is a homeomorphism.

We state without proof the following theorem on lattice compactifications (cf. [3, Th. 3.1]).

THEOREM 1.7. *If $\mathscr{L}'$ is a second α-lattice on $\mathscr{M}_0$, $\mathscr{L} \subseteqq \mathscr{L}'$, and ψ is the mapping of $\mathscr{M}_0$ into $w\mathscr{L}'$, then the following statements are equivalent.*

(i) *If $F_1, F_2 \in \mathscr{L}'$, then $F_1 \cap F_2 = \phi$ if, and only if, $(\varphi F_1)^- \cap (\varphi F_2)^- = \phi$.*

(ii) *If $F_1, F_2 \in \mathscr{L}'$, then $\varphi(F_1 \cap F_2)^- = (\varphi F_1)^- \cap (\varphi F_2)^-$.*

(iii) *$w\mathscr{L}' = w\mathscr{L}$ (i.e., there exists a homeomorphism τ of $w\mathscr{L}'$ onto $w\mathscr{L}$ such that $\tau\varphi(M) = \psi(M)$ for each $M \in \mathscr{M}_0$).*

We apply this theorem to our situation. We identity $\mathscr{M}$ and w$\mathscr{L}$ here and let $\mathscr{C}(hk)$ and $\mathscr{C}(w^*)$ denote the lattices of all hk-closed subsets of $\mathscr{M}_0$ and all w^*-closed subsets of $\mathscr{M}_0$, respectively. $W(\mathscr{M}_0, \mathscr{T})$ denotes the Wallman compactification of the topological space $(\mathscr{M}_0, \mathscr{T})$.

COROLLARY 1.7. *$\mathscr{M} = W(\mathscr{M}_0, hk)$ if, and only if, A is hk-normal. If A is regular, then $(\mathscr{M}_0, w^*)$ is embedded homeomorphically in $\mathscr{M}$ and $\mathscr{M} = W(\mathscr{M}_0, w^*)$ if, and only if, A is normal. In this case, $\mathscr{M}$ is Hausdorff and $\mathscr{M} = \beta\mathscr{M}_0$.*

Proof. The first statement is clear in view of Theorem 1.7 and Corollary 1.5, where we let $\mathscr{L}' = \mathscr{C}(hk)$. The second statement follows from the same two theorems, where $\mathscr{L}' = \mathscr{C}(w^*)$. Finally, if A is normal, then $\mathscr{M}_0$ is a normal space and $W(\mathscr{M}_0)$ (we suppress $\mathscr{T}$ since the topologies agree) is Hausdorff [13, p. 119], hence $\mathscr{M} = W(\mathscr{M}_0) = \beta\mathscr{M}_0$ (cf. [7, Exercises 5P and 5R] or [3, Th. 3.2]).

EXAMPLE 1.1. We give an example to show first that in general a commutative LMC algebra can be completely regular, but not normal (the concepts are equivalent for F-algebras, see § 2), and secondly that $\mathscr{M}$ may be $\beta\mathscr{M}_0$ while A is not normal. We let Ω be the first uncountable ordinal and w the first ordinal with countably many prodecessors, Ω' is the set of all ordinals up to and including Ω, w' the set of all ordinals up to and including w, $T' = \Omega' x w'$ with the product topology (each of Ω', w' being endowed with the order topology), and $T = T' - \{(\Omega, w)\}$. T is a locally compact Hausdorff space which is not normal and $\beta T = T'$ (cf. [4, pp. 123–124]). We let $A = C(T)$ with the compact-open topology. Then $(\mathscr{M}_0, w^*) = T$, $w^* = hk$ on T and $\mathscr{M} = \beta T = T'$. But A is not normal.

We next consider for a normal algebra A satisfying the condition (hH) the problem of identifying the subspace of $\mathscr{M}$ which consists of the maximal ideals of A which are kernels of (possibly discontinuous) homomorphisms of A onto C. We denote this subspace by $\mathscr{M}_1$.

Since A is normal, $\mathscr{M} = \beta\mathscr{M}_0$ and for each $x \in A$ the function $\hat{x}$ on $\mathscr{M}_0$ is a continuous mapping of $\mathscr{M}_0$ into the one-point compactification $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$ of $\boldsymbol{C}$. Thus $\hat{x}$ has an extension x^*, a $\boldsymbol{C}^*$-valued continuous function on $w\mathscr{L}(=\beta\mathscr{M}_0)$. Discussions of this extension and of the realcompactification of a space are found in Chapters 7 and 8 of [4]. The realcompactification of $\mathscr{M}_0$, $\upsilon\mathscr{M}_0$, is the subspace of $\beta\mathscr{M}_0$ consisting of all $\mathscr{V} \in \beta\mathscr{M}_0$ such that for each $z \in C(\mathscr{M}_0)$ $\mathscr{V} \in z^{*-1}(\boldsymbol{C})$, i.e. z^* does not take on the value ∞ at $\mathscr{V}$; where z^* is the extension of the mapping $z: \mathscr{M}_0 \to \boldsymbol{C}^*$ to $\beta\mathscr{M}_0$.

DEFINITION 1.5. $\upsilon_A\mathscr{M}_0$ *(the A-realcompactification of* $\mathscr{M}_0) = \{\mathscr{U} \in w\mathscr{L}: x^*(\mathscr{U}) \in \boldsymbol{C}$ *for each* $x \in A\}$.

THEOREM 1.8. *If* $\mathscr{U} \in \upsilon_A\mathscr{M}_0$ *and* $\mathscr{U} = \sigma(M)$, *then* $M = \{x \in A: x^*(\mathscr{U}) = 0\}$.

Proo. If $M \in \mathscr{M}$ and $\sigma(M) = \mathscr{U} \in \upsilon_A\mathscr{M}_0$, then the set $I = \{x \in A: x^*(\mathscr{U}) = 0\}$ is an ideal in A. Moreover, if $x \in M$, then $\mathscr{U} \in C[h(x)] = h(x)^-$ and since x^* is continuous on $w\mathscr{L}$ and agrees with $\hat{x}$ on $\mathscr{M}_0$, $\hat{x}^*(\mathscr{U}) = 0$. Therefore, $M \subseteqq I$ and $I \neq A (1 \notin I)$. Hence, $M = I$.

THEOREM 1.9. *The restriction of the mapping* $\sigma: \mathscr{M} \to w\mathscr{L}$ *to*

$\mathscr{M}_1$, *is a homeomorphism of* $\mathscr{M}_1$ *onto* $\upsilon_A \mathscr{M}_0$.

Proof. If $\mathscr{U} \in \upsilon_A \mathscr{M}_0$ and $\mathscr{U} = \sigma(M)$, then the mapping $x \rightarrow x^*(\mathscr{U})$ is a homomorphism of A onto C with kernel M and $M \in \mathscr{M}_1$.

If $M \in \mathscr{M}_1$ and $\mathscr{U} = \sigma(M)$, then for each $x \in A$ there exists $\lambda \in C(\lambda = M(x))$ such that $x - \lambda \in M$. We fix $x \in A$ and the corresponding (unique) $\lambda \in C$. If $x - \lambda \in M$, then $M \in H(x - \lambda)$ and $\mathscr{U} \in C[h(x - \lambda)]$. This implies $(x - \lambda)*(\mathscr{U}) = 0$. Since $\lambda*(\mathscr{U}) = \lambda \in C$, we have $x*(\mathscr{U}) = [(x - \lambda) + \lambda]*(\mathscr{U}) = (x - \lambda)*(\mathscr{U}) + \lambda*(\mathscr{U}) = \lambda \in C$ and $\mathscr{U} \in \upsilon_A \mathscr{M}_0$.

We wish to acknowledge here our indebtedness to Donald L. Plank of the Case Western Reserve University who communciated to the author theorems analogous to 1.8 and 1.9 for a real algebra A of functions on a completely regular space X satisfying: $BC(X) \subseteqq A \subseteqq C(X)$, where $BC(X)$ is the algebra of all bounded real-valued functions on X to $\boldsymbol{R}$.

2. A special case. We consider in this section the special case: A is a commutative F-algebra with identity 1 – a complete LMC algebra whose topology is given by a countably family of pseudonorms. In this case we can assume that the family $\{p_n\}_{n=0}^{\infty}$ satisfies: $p_n(x) \leqq p_{n+1}(x)$ for each $n \geqq 0$ and each $x \in A$. The fact the F-algebras are inverse limits of Banach algebras is important for our purposes. We let $N_k = \{x \in A: p_k(x) = 0\}$, Π_k the natural map of A onto A/N_k and A_k the completion of A/N_k with respect to the norm defined by $\|\Pi_k x\| = p_k(x)$. Each A_n is a commutative Banach algebra with identity. For each $n \geqq 0$ there is a norm-decreasing homomorphism Π_n^{n+1} of A_{n+1} onto a dense subalgebra of A_n which is defined on A/N_{n+1} by $\Pi_n^{n+1}(\Pi_{n+1}x) = \Pi_n x$ and extended to A_{n+1}. For $n \leqq m$, $\Pi_n^m: A_m \rightarrow A_n$ is defined by the obvious composition. The resulting family of algebras and homomorphisms is an inverse limit system and A is isomorphic and pseudo-isometric to the inverse limit of this system. An important consequence of this is the following fact. If $\{\xi_n\}_{n=0}^{\infty}$ is a sequence where $\xi_n \in A_n$ and $\Pi_n^m \xi_m = \xi_n$ whenever $n \leqq m$, then there exists $x \in A$ such that $\Pi_n x = \xi_n$ for each $n \geqq 0$. For details of this construction and the basic facts about such systems, the reader is referred to [9].

We state without proof two theorems, the first is just Theorem 4.2 of [2] in our terminology, the second is immediate.

THEOREM 2.1. *Suppose* $\{a_1, \cdots, a_m\}$ *is a family of elements of* A *such that* $(\Pi_n a_1, \cdots, \Pi_n a_m) = A_n$ *for each* $n \geqq 0$. *Then* $(a_1, \cdots, a_m) = A$.

THEOREM 2.2. *If* $\{\xi_1, \cdots, \xi_m\}$ *is a finite family in* A_n *and* $\hat{\xi}_1, \cdots, \hat{\xi}_m$ *have no common zeros on* $\mathscr{M}(A_n)$ *(the structure space of* A_n*), then* $(\xi_1, \cdots, \xi_m) = A_n$.

The spectrum $\mathscr{M}_0$ of A has the following structure. $\mathscr{M}_0 = \bigcup\{\mathscr{M}^k : k = 0, 1, 2, \cdots\}$, where each $\mathscr{M}^k$ is homeomorphic to $\mathscr{M}(A_k)$, the structure space of A_k. The homeomorphism σ_k of $\mathscr{M}(A_k)$ into $\mathscr{M}_0$ is defined by $[\sigma_k(M^k)](x) = M^k(\Pi_k x)$ for each $M^k \in \mathscr{M}(A_k)$ and $x \in A$.

THEOREM 2.3. *If A is a commutative F-algebra with identity, then A satisfies the condition (hH).*

Proof. We fix a family $\{x_1, \cdots, x_n\}$ in A satisfying $h(x_1, \cdots, x_n) = \phi$ and show $H(x_1, \cdots, x_n) = \phi$. We need only show that for each $k \geqq 0$ the family $\{\Pi_k x_1, \cdots, \Pi_k x_n\}$ generates the improper ideal A_k. We fix $k \geqq 0$. For each i we have $(\Pi_k x_i)(M^k) = x_i(\sigma_k M^k)$. Therefore, the family $\{(\Pi_k x_1)\hat{}, \cdots, (\Pi_k x_n)\hat{}\}$ has a common zero on $\mathscr{M}(A_k)$ if, and only if, the intersection of $\mathscr{M}^k$ and $h(x_1, \cdots, x_n)$ is nonempty. We have assumed that $h(x_1, \cdots, x_n)$ is empty. Thus, Theorem 2.2 implies $(\Pi_k x_1, \cdots, \Pi_k x_n) = A_k$ for each $k \geqq 0$, and we obtain $(x_1, \cdots, x_n) = A$.

Thus, F-algebras always satisfy the conditions of Theorem 1.4 and $\mathscr{M} = w\mathscr{L}$. We next extend Theorem 2.1 to pin down further the space $\mathscr{M}$. We note that Theorem 2.4 is immediate for Banach algebras (since $\mathscr{M}_0 = \mathscr{M}$) and false for commutative LMC algebras in general (cf. Example 1.1 above).

THEOREM 2.4. *If I_1 and I_2 are closed ideals in A and if $h(I_1) \cap h(I_2) = \phi$, then $I_1 + I_2 = A$.*

Proof. We shall construct two sequences in $\Pi_n A_n$, show that they yield elements of I_1 and I_2 whose sum is 1. We let $F_1 = h(I_1)$ and $F_2 = h(I_2)$. Since the tail of a sequence is the important thing in determining whether it corresponds to an element of A we assume that $F_1 \cap \mathscr{M}^0 \neq \phi$ and $F_2 \cap \mathscr{M}^0 \neq \phi$. If not we begin the construction with the first integer k so that both F_1 and F_2 meet $\mathscr{M}^k$ and define the first k terms by the maps Π_i^k, $i = 0, \cdots, k-1$.

Since $F_1 \cap F_2 = \phi$, $\sigma_n^{-1}(F_1 \cap \mathscr{M}^n) \cap \sigma_n^{-1}(F_2 \cap \mathscr{M}^n) = \phi$ in $\mathscr{M}(A_n)$ for each $n \geqq 0$. We first note that for each $n \geqq 0$ $\Pi_n(I_1)^-$ and $\Pi_n(I_2)^-$ are closed ideals in A_n and $\Pi_n(I_1)^- + \Pi_n(I_2)^- = A_n$. If not, then there exists $M^n \in (A_n)$ such that $\Pi_n(I_1)^-, \Pi_n(I_2)^- \subseteqq M^n$. Then $I_1, I_2 \subseteqq M = \sigma_n(M^n)$, and $M \in F_1 \cap F_2$, a contradiction. By Lemma 7.8 of [9] I_j is the inverse limit of the sequence $\{\Pi_n(I_j)^-\}$ with the restricted homomorphism, for $j = 1, 2$, and for each pair $n, m, n \leqq m$, $\Pi_n^m[\Pi_m(I_j)^-]$ is dense in $\Pi_n(I_j)^-$, since the former contains $\Pi_n^m[\Pi_m(I_j)] = \Pi_n(I_j)$ which is dense in $\Pi_n(I_j)^-$.

We first choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers such that the series $\sum_{n=1}^{\infty} \varepsilon_n$ converges. Since $\Pi_0(I_1)^- + \Pi_0(I_2)^- = A_0$ we choose

$\xi_0^j \in \Pi_0(I_j)^-, j = 1, 2$ such that $\xi_0^1 + \xi_0^2 = 1$.

We next choose ζ_1^j in $\Pi_1(I_j)^-, j = 1, 2$, such that $\zeta_1^1 + \zeta_1^2 = 1$, then choose η_1^j in $\Pi_1(I_j)^-, j = 1, 2$, such that

$$\| \Pi_0^1\eta_1^j - \xi_0^j \| < \min\left(\varepsilon_1/4, \varepsilon_1/4 \max_{j=1,2} \| \xi_1^j \|\right) .$$

This is possible because $\Pi_0^1[\Pi_1(I_j)^-]$ is dense in $\Pi_0(I_j)^-, j = 1, 2$. We let $\xi_1^j = \eta_1^j + \zeta_1^j(1 - \eta_1^1 - \eta_1^2), i = 1, 2$. Then

$$\xi_1^j \in \Pi_1(I_j)^-, j = 1, 2; \xi_1^1 + \xi_1^2 = 1 ,$$

and $\| \Pi_0^1\xi_1^j - \xi_0^j \| < \varepsilon_1$, for each j.

Proceeding inductively we choose for each $n \geqq 1, j = 1, 2, \xi_n^j \in \Pi_n(I_j)^-$ such that $\| \Pi_{n-1}^n\xi_n^j - \xi_{n-1}^j \| < \varepsilon_n$, and $\xi_n^1 + \xi_n^2 = 1$. Then for $k = 0, 1, \cdots, n - 1$ we have

$$\| \Pi_k^n\xi_n^j - \Pi_k^{n-1}\xi_{n-1}^j \| < \varepsilon_n . \tag{3.1}$$

From this point on the construction is identical to that given in the proof of Theorem 4.2 of [2]. We sketch the important steps.

We first fix $n \geqq 0$ and let $x_j(n)_k = \Pi_n^k\xi_k^j$ for each $k \geqq n, j = 1, 2$. $\{x_j(n)_k\}_{k=n}^\infty$ is a sequence in $\Pi_n(I_j)^-$ and satisfies

(i) $\Pi_n^{n+1}(x_j(n + 1)_k) = x_j(x)_k$ for each $k \geqq n + 1, j = 1, 2$;

(ii) $x_1(n)_k + x_2(n)_k = 1$,

(iii) $\| x_j(n)_k - x_j(n)_{k+p} \| < \varepsilon_{k+1} + \cdots + \varepsilon_{k+p}$.

Thus the sequences are Cauchy for each n, j and converge to elements $x_j(n)$ in $\Pi_n(I_j)^-$ for each $n \geqq 0, j = 1, 2$. There exist $x_1 \in I_1, x_2 \in I_2$ such that $\Pi_n(x_j) = x_j(n)$ for each $n \geqq 0, j = 1, 2$. Thus, $x_1 + x_2 = 1$.

COROLLARY 2.4.1. *If F_1 and F_2 are disjoint hk-closed subsets of $\mathscr{M}_0$, then* $\mathrm{Cl}_{\mathscr{M}} F_1 \cap \mathrm{Cl}_{\mathscr{M}} F_2 = \phi$.

Proof. Letting $I_1 = kF_1$ and $I_2 = kF_2$ yields $I_1 + I_2 = A$. Apply Theorem 1.5.

COROLLARY 2.4.2. *If A is a commutative F-algebra with identity, then $\mathscr{M} = W(\mathscr{M}_0, hk)$. Moreover, if A is regular, then A is normal and $\mathscr{M} = W(\mathscr{M}_0) = \beta\mathscr{M}_0$.*

Proof. The first statement follows from Corollary 2.4.1, Theorem 1.5, and Corollary 1.7. The second follows from Corollaries 1.7 and 2.4.1.

We note that Rosenfeld [11] has indicated a proof of part of Corollary 2.4.2 (A regular implies A normal) using Silov's theorem. This theorem also yields a proof of Corollary 2.4.1, since $F_1 \cup F_2$ is

hk-closed in $\mathscr{M}_0$ and is $\mathscr{M}_0(B)$, where $B = A/(kF_1 \cap kF_2)$. However, since the application of this theorem yields an element a of A such that $\hat{a} \equiv 0$ on F_1 and $\hat{a} \equiv 1$ on F_2, we can conclude only that $kF_1 + kF_2 = A$. Thus, it does not appear that the proof of Theorem 2.4 can be simplified by the use of this tool.

THEOREM 2.5. *Let I be a closed ideal in A and $B = A/I$. Then B is a commutative F-algebra with identity, $\mathscr{M}_0(B)$ is homeomorphic to $h(I)$ with respect to both the w^*- and hk-topologies, and $\mathscr{M}(B) =$ $\mathrm{Cl}_{\mathscr{M}(A)}\, h(I)$.*

Proof. The first conclusion follows from the open mapping theorem for F-spaces (cf. [8, Lemma 11.3]) and the fact that the natural map Π of A onto B is continuous and open. The range of $\Pi^*: \mathscr{M}_0(B) \to \mathscr{M}_0(A)$ is easily seen to be $h(I)$ and it is also immediate that Π^* is a w^*- homeomorphism. For convenience we let $F = h(I)$ for the remainder of the proof.

We show that for each $E \subseteq F$, $\Pi^{*-1}[hk(E)] = h'k'[\Pi^{*-1}(E)]$, where h' and k' are the h- and k- operators for B. $M' \in \Pi^{*-1}[hk(E)]$ if, and only if, $M \in hk(E)$ $(M = \Pi^* M')$ if, and only if, $M(x) = 0$ for each $x \in kE$. And $x \in kE$ if, and only if, $M_1(x) = 0$ for each $M_1 \in E$ if, and only if, $M_1'(\Pi x) = 0$ for each $M_1' \in \Pi^{*-1}(E)$. So $x \in kE$ if, and only if, $\Pi x \in k'[\Pi^{*-1}(E)]$. Thus, from above, $M(x) = 0$ for each $x \in kE$ if, and only if, $M'(\Pi x) = 0$ for each $\Pi x \in k'[\Pi^{*-1}(E)]$, if, and only if, $M' \in h'k'[\Pi^{*-1}(E)]$. The equality is established and it is immediate that Π^* is a homeomorphism with respect to the hk-topologies in $\mathscr{M}_0(B)$ and F.

For each $x \in A$ we have $\Pi^*[h'(\Pi x)] = h(x) \cap F$. Thus, there is a lattice isomorphism of $\mathscr{L}' = \{h(\xi_1, \cdots, \xi_n): \{\xi_1, \cdots, \xi_n\} \subseteq B\}$ onto $\mathscr{L}_F = \{E \subseteq F: E = B \cap F$ for some $B \in \mathscr{L}\}$, and there is induced a homeomorphism of $w\mathscr{L}'$ onto $w\mathscr{L}_F$. Therefore, $\mathscr{M}(B)$ is homeomorphic to $w\mathscr{L}_F$. For each $M \in \mathrm{Cl}_{\mathscr{M}(A)} F$ we define $\tau(M) = \{E \in \mathscr{L}_F: M \in \mathrm{Cl}_{\mathscr{M}} E\}$. $M \to \tau(M)$ is a one-to-one mapping of $\mathrm{Cl}_{\mathscr{M}(A)} F$ onto $w\mathscr{L}_F$. From the easily verified equation $H(x) \cap \mathrm{Cl}_{\mathscr{M}(A)} F = C[h(x) \cap F]$ it follows that τ is a homeomorphism.

BIBLIOGRAPHY

1. R. Arens, *A generalization of normed rings*, Pacific J. Math. **2** (1952), 455-471.
2. ———, *Dense inverse limit rings*, Michigan Math. J. **5** (1958), 169-182.
3. R. M. Brooks, *On Wallman compactifications*, Fund. Math. LX (1967), 157-173.
4. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.
5. E. Hewitt, *Rings of real-valued continuous functions*, I, Trans. Amer. Math. Soc. **64** (1948), 45-99.

6. S. Kakutani, *Rings of analytic functions*, Lectures on functions of a complex variable, Ann Arbor, 1955.
7. J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.
8. I. Namioka *et al*, *Linear Topological Spaces*, Van Nostrand, New York, 1963.
9. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952).
10. M. A. Naimark, *Normed Rings*, P. Noordhoff, Ltd., Groningen, The Netherlands, 1960.
11 M. Rosenfeld, *Commutative F-algebras*, Pacific J. Math. **16** (1966), 159-166
12. P. Samuel, *Ultrafilters and compactification of uniform spaces*, Trans. Amer. Math. Soc. **64** (1948), 100-132.
13. H. Wallman, *Lattices and topological spaces*, Ann of Math. (2) **39** (1938), 112-126.

Received September 22, 1966. The research for this paper was supported in part by NSF Grant G 24295.

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 25, No. 3 November, 1968