

# Pacific Journal of Mathematics

**THEOREMS ON BREWER SUMS**

STANLEY F. ROBINSON

## THEOREMS ON BREWER SUMS

S. F. ROBINSON

Let  $V_m(x, Q)$  be the polynomial determined by the recurrence relation

$$(1.1) \quad V_{m+2}(x, Q) = x \cdot V_{m+1}(x, Q) - Q \cdot V_m(x, Q)$$

( $m = 1, 2, \dots$ ),  $Q$  an integer, with  $V_1(x, Q) = x$  and  $V_2(x, Q) = x^2 - 2Q$ . In a recent paper, B. W. Brewer has defined the sum

$$(1.2) \quad A_m(Q) = \sum_{x=0}^{p-1} \chi(V_m(x, Q))$$

where  $\chi(s)$  denotes the Legendre symbol  $(s/p)$  with  $p$  and odd prime.

The purpose of this paper is to consider the evaluation of  $A_{2n}(Q)$  when  $n$  is odd. The principle result obtained is the expression of  $A_{2n}(Q)$  as the sum of  $\chi(Q) \cdot A_n(1)$  and one half the character sum  $\psi_{2e}(1)$ .  $\psi_{2e}(1)$  can in turn be expressed in terms of the Gaussian cyclotomic numbers  $(i, j)$ . The values of  $A_6(Q)$  and  $A_{10}(Q)$  follow immediately from this result utilizing values for  $A_3(1) = A_3$  and  $A_5(1) = A_5$  computed by B. W. Brewer and A. L. Whiteman.

2. The character sums  $\Omega_m(Q)$  and  $\theta_m(Q)$  and Brewer's lemma. Let  $p$  be an odd prime and  $\lambda$  a generating element of the multiplicative group of  $GF(p^2)$ . Then  $\lambda^{p+1} = g$  is a primitive root of  $GF(p)$ . Set  $Q = g^r = \lambda^{r(p+1)}$ ,  $0 \leq r < p - 1$ . In order to facilitate the evaluation of  $A_m(Q)$ , Brewer defines the following two sums:

$$(2.1) \quad \begin{aligned} \Omega_m(Q) &= \sum_{s=1}^{p-1} \chi(\lambda^{ms(p+1)} + Q^m \lambda^{ms(p+1)}) \\ &= \sum_{s=1}^{p-1} \chi(g^{ms} + g^{mr} g^{-ms}) \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} \theta_m(Q) &= \sum_{t=1}^{p+1} \chi(\lambda^{m(t(p-1)+r)} + Q^m \lambda^{-m(t(p-1)+r)}) \\ &= \sum_{t=1}^{p+1} \chi(\lambda^{m(t(p-1)+r)} + \lambda^{mr(p+1)} \lambda^{-m(t(p-1)+r)}). \end{aligned}$$

Brewer relates the sums  $\Omega_m(Q)$  and  $\theta_m(Q)$  to the sum  $A_m(Q)$  by the equation [2, Lemma 2].

$$(2.3) \quad 2A_m(Q) = \theta_m(Q) + \Omega_m(Q).$$

(Compare also [1, Lemma 2] and [14, Lemma 1].)  
 The following theorem is fundamental [2, Th. 1].

**THEOREM 2.1.** *Let  $p$  be an odd prime,  $A_m(Q)$  be defined as in (1.2). If  $\chi(Q') = \chi(Q)$  and  $Q' \equiv n^2Q \pmod{p}$ , then  $A_m(Q') = \chi(n)^m A_m(Q)$ , ( $m = 1, 2, \dots$ ).*

**3. The Jacobsthal sum.** Closely related to the Brewer sum are the character sums of Jacobsthal

$$(3.1) \quad \phi_e(n) = \sum_{h=1}^{p-1} \chi(h)\chi(h^e + n)$$

and the related sum

$$(3.2) \quad \psi_e(n) = \sum_{h=1}^{p-1} \chi(h^e + n).$$

We note

$$A_2(Q) = \sum_{x=0}^{p-1} \chi(x^2 - 2Q) = \psi_2(-2Q) + \chi(-2Q)$$

and

$$A_3(Q) = \sum_{x=0}^{p-1} \chi(x^3 - 3Qx) = \phi_2(-3Q).$$

In general if  $m$  is even and  $g$  a primitive root of  $p$

$$(3.3) \quad \begin{aligned} \Omega_m(Q) &= \sum_{s=1}^{p-1} \chi(g^{ms} + Q^m g^{-ms}) \\ &= \psi_{2m}(Q^m), \end{aligned}$$

while if  $m$  is odd

$$(3.4) \quad \begin{aligned} \Omega_m(Q) &= \sum_{s=1}^{p-1} \chi(g^{ms} + Q^m g^{-ms}) \\ &= \sum_{s=1}^{p-1} \chi(g^{-ms}) \cdot \chi(g^{2ms} + Q^m) \\ &= \sum_{s=1}^{p-1} \chi(g^{-(m+1)s}) \cdot \chi(g^s) \cdot \chi(g^{2ms} + Q^m) \\ &= \phi_{2m}(Q^m). \end{aligned}$$

The following results concerning Jacobsthal sums will be applied: if  $p \nmid x$  [10, Equation 3.8]

$$(3.5) \quad \psi(nx^e) = \chi(x)^e \cdot \psi_e(n),$$

the reduction formula [10, Equation 3.9] and [7, Equation 6]

$$(3.6) \quad \phi_e(n) + \psi_e(n) = \psi_{2e}(n)$$

and [7, Formula 10] if  $e$  is odd

$$(3.7) \quad \psi_{2e}(n) = \phi_e(n) + \chi(n) \cdot \phi_e(n')$$

where  $n \cdot n' \equiv 1 \pmod{p}$ .

4. Cyclotomy. Let  $p$  be an odd prime and  $g$  a primitive root of  $p$ . Let  $e$  be a divisor of  $p - 1$ ,  $p - 1 = e \cdot f$ . The cyclotomic number of order  $e$ ,  $(i, j)$  is the number of solutions of  $1 + g^{es+i} \equiv g^{et+j} \pmod{p}$ ,  $s, t = 0, 1, \dots, f - 1$ .

If we write  $2ef = p - 1$ ,

$$\begin{aligned} \psi_{2e}(1) &= \sum_{s=0}^{p-1} \chi(g^{2es} + 1) \\ &= 2e \sum_{s=0}^{f-1} \chi(g^{2es} + 1) \\ \chi(g^{2es} + 1) &= \begin{cases} 1 & g^{2es} + 1 \equiv g^{2et+a} \pmod{p} \text{ } a \text{ even} \\ -1 & g^{2es} + 1 \equiv g^{2et+a} \pmod{p} \text{ } a \text{ odd.} \end{cases} \end{aligned}$$

Thus in this case  $\psi_{2e}(1)$  can be expressed in terms of the cyclotomic numbers of order  $2e$

$$(4.1) \quad \psi_{2e}(1) = \frac{1}{2e} \sum_{i=0}^{2e-1} (-1)^i (2e)^2(0, i).$$

In the theory of cyclotomy, the Jacobi sum and the related Lagrange resolvent play a fundamental role. In what follows we will use some of the properties of the Jacobi sum.

Let  $\beta = \exp(2\pi i/e)$ ,  $e \cdot f + 1 = p$ . The Jacobi sum is defined by the equation

$$(4.2) \quad \psi(\beta^m, \beta^n) = \sum_{\substack{a+b=1 \pmod{p} \\ 1 \leq a, b \leq p-1}} \beta^{m \operatorname{ind} a + n \operatorname{ind} b}$$

The following equalities for the Jacobi sum can be derived: [12, Formula 2.4]

$$(4.3) \quad \psi(\beta^m, \beta^n) = \psi(\beta^n, \beta^m) = (-1)^{nf} \psi(\beta^{-m-n}, \beta^n).$$

Placing  $n = 0$  in (4.2) we have [12, Formula 2.5]

$$(4.4) \quad \psi(\beta^m, \beta^0) = \begin{cases} p - 2 & m = 0 \\ -1 & 1 \leq m \leq e - 1 \end{cases}$$

and the important formula

$$(4.5) \quad \psi(\beta^m, \beta^n) \cdot \psi(\beta^{-m}, \beta^{-n}) = p$$

provided  $e$  does not divide  $m, n$  or  $m + n$ .

Since  $\psi(\beta^m, \beta^n)$  is periodic in both  $m$  and  $n$  with respect to  $e$ , it may be expanded into a double finite fourier series [11, Formula 2.6]

$$(4.6) \quad \psi(\beta^m, \beta^n) = (-1)^{mf} \sum_{h,k=0}^{e-1} (h, k) \beta^{mh+nk} .$$

We may also write (4.6) in the inverted form [11, Formula 2.7]

$$(4.7) \quad e^2(h, k) = \sum_{m,n=0}^{e-1} (-1)^{mf} \psi(\beta^m, \beta^n) \beta^{-mh-nk} .$$

In (4.6) replace  $m$  by  $vn$ , where  $v$  is an integer, collecting the exponents of  $\beta$  in the same residue class modulo  $e$ , we get an alternate form of the finite fourier series expansion [11, Equation 2.8]

$$(4.8) \quad \psi(\beta^{vn}, \beta^n) = (-1)^{vnf} \sum_{i=0}^{e-1} B(i, v) \beta^{ni}$$

where the fourier coefficients  $B(i, v)$  are the Dickson-Hurwitz sums

$$(4.9) \quad B(i, v) = \sum_{h=0}^{e-1} (h, i - vh) .$$

The inverted form of (4.8) is

$$(4.10) \quad e \cdot B(i, v) = \sum_{n=0}^{e-1} (-1)^{vnf} \psi(\beta^{vn}, \beta^n) \beta^{-ni} .$$

If  $e \cdot f = p - 1$ , Whiteman [10, Formula 5.8] expresses the Jacobsthal sum in terms of the cyclotomic function  $B(i, v)$

$$(4.11) \quad eB(v, 1) = \begin{cases} p - 1 + \phi_e(4g^v) & e \text{ odd} \\ p - 1 + \psi_e(4g^v) & e \text{ even} . \end{cases}$$

Thus if  $e \cdot f = p - 1$  with  $e$  odd, from (3.7) and (4.11) we can write

$$(4.12) \quad \begin{aligned} \psi_{2e}(1) &= 2\phi_e(1) \\ &= 2(e \cdot B(i, 1) - (p - 1)) \\ &= 2e(B(i, 1) - f) \end{aligned}$$

where  $i$  is selected so that  $4g^i \equiv 1 \pmod{p}$ .

5. The evaluation of  $\mathcal{A}_{2n}(Q)$  for odd values of  $n$ . In this section we will develop our principle result in the evaluation of  $\mathcal{A}_{2n}(Q)$  for odd values of  $n$ . We will consider  $\Omega_{2n}(Q)$  and  $\theta_{2n}(Q)$  separately and combine the results by use of equation (2.3).

**THEOREM 5.1.** *If  $d$  is the g.c.d. of  $m$  and  $p + 1$  and  $Q = \lambda^{r(p+1)}$  then*

$$(5.1) \quad \phi(Q) = \theta_d(Q^{m/d}) .$$

*Proof.* This theorem is a direct result of the fact that if the

g.c.d. of  $a$  and  $M$  is  $d$ , and  $\{r_1, r_2, \dots, r_M\}$  is a complete residue system modulo  $M$ , then the set  $\{a \cdot r_1, a \cdot r_2, \dots, a \cdot r_M\}$  contains the same elements modulo  $M$  as the set  $\{d \cdot r_1, d \cdot r_2, \dots, d \cdot r_M\}$ . Now if the g.c.d. of  $m$  and  $p + 1$  is  $d$ , then

$$\begin{aligned} \theta_m(Q) &= \sum_{i=1}^{p+1} \chi(\lambda^{m(i(p-1)+r)} + \lambda^{mr(p+1)}\lambda^{-m(i(p-1)+r)}) \\ &= \sum_{i=1}^{p+1} \chi(\lambda^{d(i(p-1)+(mr/d)} + \lambda^{d(m/d)r(p+1)}\lambda^{-d(i(p-1)+(mr/d)}) \\ &= \theta_d(Q^{m/d}). \end{aligned}$$

We note  $d < m$  unless  $p \equiv -1 \pmod{m}$ . This exceptional case is considered in the following two theorems.

**THEOREM 5.2.** *If  $p = (4m) \cdot f - 1$*

$$(5.2) \quad \theta_m(Q) = \theta_{2m}(Q) = 0.$$

*Proof.*  $\{2m \cdot 1, 2m \cdot 2, \dots, 2m \cdot (p+1)/2m\}$  has the same elements as  $\{2m \cdot 1 + 2mf, 2m \cdot 2 + 2mf, \dots, 2m \cdot (p+1)/2m + 2mf\}$  modulo  $p+1$ . Also  $\{m \cdot 1, m \cdot 2, \dots, m \cdot (p+1)/m\}$  has the same elements modulo  $p+1$  as  $\{m \cdot 1 + 2mf, m \cdot 2 + 2mf, \dots, m \cdot (p+1)/m + 2mf\}$ . Since  $\chi(\lambda^{2mf(p-1)}) = \chi(\lambda^{(p^2-1)/2}) = -1$  when  $p \equiv 3 \pmod{4}$ , we have

$$\begin{aligned} \theta_{2m}(Q) &= 2m \sum_{i=1}^{(p+1)/2m} \chi(\lambda^{2m(i(p-1)+r)} + Q^{2m}\lambda^{-2m(i(p-1)+r)}) \\ &= 2m \sum_{i=1}^{(p+1)/2m} \chi(\lambda^{2m(i(p-1)+r)+(p^2-1)/2} + Q^{2m}\lambda^{-2m(i(p-1)+r)+(p^2-1)/2}) \\ &= 2m\chi\left(\lambda^{\frac{p^2-1}{2}}\right) \sum_{i=1}^{(p+1)/2m} \chi(\lambda^{2m(i(p-1)+r)} + Q^{2m}\lambda^{-2m(i(p-1)+r)}) \\ &= -\theta_{2m}(Q) \end{aligned}$$

and

$$\begin{aligned} \theta_m(Q) &= m \sum_{i=1}^{(p+1)/m} \chi(\lambda^{m(i(p-1)+r)} + Q^m\lambda^{-m(i(p-1)+r)}) \\ &= m \sum_{i=1}^{(p+1)/m} \chi(\lambda^{m(i(p-1)+r)+(p^2-1)/2} + Q^m\lambda^{-m(i(p-1)+r)+(p^2-1)/2}) \\ &= -\theta_m(Q). \end{aligned}$$

**THEOREM 5.3.** *If  $p = (2f+1)m - 1$  with  $m \equiv 2 \pmod{4}$  and  $Q = \lambda^{r(p+1)}$ , then*

$$(5.3) \quad \theta_m(Q) = \theta_{m/2}(Q^2).$$

*Proof.* Since now  $p \equiv 1 \pmod{4}$ ,  $\chi(\lambda^{p^2-1/2}) = 1$ . Let  $F = 2f + 1$ . The set  $\{m \cdot 1, m \cdot 2, \dots, m \cdot (p+1)/m\} \cup \{m \cdot 1 + (m \cdot F/2), m \cdot 2 + m \cdot F/2,$

$\dots, m \cdot p + 1/m + mF/2\}$  has the same elements as the set  $\{m/2 \cdot 1, m/2 \cdot 2, \dots, m/2 \cdot 2(p+1)/m\}$  modulo  $p+1$ . Thus

$$\begin{aligned} \theta_m &= m \sum_{i=1}^{(p+1)/m} \chi(\lambda^{m(i(p-1)+r)} + \lambda^{mr(p+1)} \lambda^{-m(i(p-1)+r)}) \\ &= \frac{m}{2} \sum_{i=1}^{(p+1)/m} \chi(\lambda^{m(i(p-1)+r)} + \lambda^{mr(p+1)} \lambda^{-m(i(p-1)+r)}) \\ &\quad + \frac{m}{2} \chi(\lambda^{p^2-1/2}) \sum_{i=1}^{(p+1)/m} \chi(\lambda^{m(i(p-1)+r)} + \lambda^{mr(p+1)} \lambda^{-m(i(p-1)+r)}) \\ &= \frac{m}{2} \sum_{i=1}^{(2(p+1)/m)} \chi(\lambda^{m/2(i(p-1)+2r)} + \lambda^{m/22r(p+1)} \lambda^{-(m/2)(i(p-1)+r)}) \\ &= \theta_{m/2}(Q^2). \end{aligned}$$

**THEOREM 5.4.** *If  $n$  is odd and  $p$  is an odd prime, then*

$$(5.4) \quad \theta_{2n}(Q) = \theta_n(Q^2).$$

*Proof.* If  $p \equiv -1 \pmod{n}$ , the result follows from Theorems 5.2 and 5.3. If  $p \not\equiv -1 \pmod{n}$ , let the g.c.d. of  $n$  and  $p+1$  be  $d$ . Then the g.c.d. of  $2n$  and  $p+1$  is  $2d$  and  $p \equiv -1 \pmod{d}$ . Thus  $\theta_{2n}(Q) = \theta_{2d}(Q^{n/d}) = \theta_d(Q^{2n/d}) = \theta_n(Q^2)$ .

**THEOREM 5.5.** *If  $n$  is odd and the g.c.d. of  $p-1$  and  $n$  is  $e$ , then*

$$(5.5) \quad \Omega_{2n}(Q) = \psi_{2e}(1) + \Omega_n(Q^2).$$

*Proof.* From equations (3.3), (3.4), and (3.6) we can write

$$(5.6) \quad \begin{aligned} \Omega_{2n}(Q) &= \psi_{4n}(Q^{2n}) = \psi_{2n}(Q^{2n}) + \phi_{2n}(Q^{2n}) \\ &= \psi_{2n}(Q^{2n}) + \Omega_n(Q^2). \end{aligned}$$

By equation (3.5)  $\psi_{2n}(Q^{2n}) = \chi(Q)^{2n} \cdot \psi_{2n}(1) = \psi_{2n}(1)$ . Now applying the reasoning used in Theorem 5.1 with  $g$  a primitive root of  $p$

$$(5.7) \quad \begin{aligned} \psi_{2n}(1) &= \sum_{h=1}^{p-1} \chi(h^{2n} + 1) \\ &= \sum_{r=1}^{p-1} \chi(g^{2nr} + 1) \\ &= \sum_{r=1}^{p-1} \chi(g^{2er} + 1) \\ &= \psi_{2e}(1). \end{aligned}$$

The result now follows from (5.6) and (5.7).

We can now state the basic tool for the evaluation of  $A_{2n}(Q)$  when  $n$  is odd.

Combining the results of Theorem 5.4 and 5.5 along with Equation 2.3, we have

**THEOREM 5.6.** *Assume  $n$  is odd. Let  $e$  be the g.c.d. of  $n$  and  $p - 1$ , then*

$$(5.8) \quad A_{2n}(Q) = A_n(Q^2) + \frac{1}{2}\psi_{2e}(1).$$

Applying Theorem 2.1 we can write

$$(5.9) \quad A_{2n}(Q) = \chi(Q) \cdot A_n(1) + \frac{1}{2}\psi_{2e}(1).$$

**COROLLARY.** *If  $n$  is an odd prime,  $p \nmid Q$  and  $p \not\equiv 1 \pmod{n}$*

$$(5.10) \quad A_{2n}(Q) = \chi(Q) \cdot A_n(1) - 1.$$

**6. The evaluation of  $A_6(Q)$ .** The values of  $A_6(Q)$  depend upon the decompositions  $p = x^2 + 4y^2$  and  $p = A^2 + 3B^2$  with the signs selected so that  $x \equiv 1 \pmod{4}$ ,  $2y \equiv x \pmod{3}$  and  $A \equiv 1 \pmod{3}$ .

$p \not\equiv 1 \pmod{6}$ . By the corollary to Theorem 5.6,

$$A_6(Q) = \chi(Q) \cdot A_3(1) - 1.$$

Brewer [1] and Whiteman [14] have evaluated  $A_3(1) = A_3$  with the results

$$(6.1) \quad A_3(1) = \begin{cases} 0 & p \equiv 3 \pmod{4} \\ 4y & p \equiv 5 \pmod{12} \\ 2x & p \equiv 1 \pmod{12} \quad 3 \mid x \\ -2x & p \equiv 1 \pmod{12} \quad 3 \nmid x. \end{cases}$$

Thus we have for  $A_6(Q)$  when  $p \not\equiv 1 \pmod{6}$ .

$$(6.2) \quad A_6(Q) = \begin{cases} \chi(Q) \cdot 4y - 1 & p \equiv 5 \pmod{12} \\ -1 & p \equiv 11 \pmod{12} \end{cases}$$

$p \equiv 1 \pmod{6}$ . By Equation (5.9)

$$(6.3) \quad A_6(Q) = \chi(Q) \cdot A_3 + \frac{1}{2}\psi_6(1).$$

In this case (4.1) becomes

$$(6.4) \quad \psi_6(1) = \frac{1}{6} \sum_{i=0}^2 \{36(0, 2i) - 36(0, 2i + 1)\}.$$

The values for the  $36(o, h)$  can be determined from tables such as those given by Hall [4, p. 981] or computed directly using Equation (4.7) which becomes



$$(6.5) \quad 36(o, k) = \sum_{m, n=0}^5 (-1)^{m+f} \psi(\beta^m, \beta^n) \beta^{-nk}$$

where  $\beta = \exp(2\pi i)/6$ . We can write

$$(6.6) \quad \psi_6(1) = \frac{1}{6} \sum_{m, n=0}^5 (-1)^{m+f} \psi(\beta^m, \beta^n) \sum_{j=0}^5 (-1)^j \beta^{-nj}.$$

The right side of (6.6) reduces to

$$(6.7) \quad \begin{aligned} & \frac{1}{6} \sum_{m=0}^5 (-1)^{m+f} \psi(\beta^m, \beta^3) \sum_{j=0}^5 (-1)^j \beta^{-3j} \\ &= \psi(\beta^0, \beta^3) + (-1)^f \psi(\beta^1, \beta^3) \\ & \quad + \psi(\beta^2, \beta^3) + (-1)^f \psi(\beta^3, \beta^3) + \psi(\beta^4, \beta^3) + (-1)^f \psi(\beta^5, \beta^3). \end{aligned}$$

By (4.3) and (4.4) we have

$$(6.8) \quad \begin{aligned} (-1)^f \psi(\beta^3, \beta^3) &= \psi(\beta^0, \beta^3) = -1 \\ \psi(\beta^1, \beta^3) &= (-1)^f \psi(\beta^2, \beta^1) \\ \psi(\beta^2, \beta^3) &= \psi(\beta^2, \beta^1) \\ \psi(\beta^4, \beta^3) &= \psi(\beta^5, \beta^4) \\ \psi(\beta^5, \beta^3) &= (-1)^f \psi(\beta^5, \beta^4). \end{aligned}$$

Set

$$\psi(\beta^2, \beta^1) = -A + B\sqrt{-3}$$

Then  $\psi(\beta^5, \beta^4) = -A - B\sqrt{-3}$ .

Dickson [2, p. 410] proved if  $\psi(\beta^2, \beta^1) = -A + B\sqrt{-3}$ , then  $A \equiv 1 \pmod{3}$ . We can now write

$$(6.9) \quad \begin{aligned} \psi_6(1) &= -2 - 2A + 2B\sqrt{-3} - 2A - 2B\sqrt{-3} \\ &= -2 - 4A \end{aligned}$$

and by (4.5)

$$(6.10) \quad p = \psi(\beta^1, \beta^2) \cdot \psi(\beta^5, \beta^4) = A^2 + 3B^2.$$

Combining (6.1), (6.3), and (6.9) we have

$$(6.11) \quad A_6(Q) = \begin{cases} -1 - 2A & p \equiv 7 \pmod{12} \\ -1 - 2A + 2x \cdot \chi(Q) & p \equiv 1 \pmod{12} \quad 3 \mid x \\ -1 - 2A - 2x \cdot \chi(Q) & p \equiv 1 \pmod{12} \quad 3 \nmid x. \end{cases}$$

Using Equation (3.7), Equation (6.9) can be written in the form

$$(6.12) \quad \psi_6(1) = 2\phi_3(1) = -2 - 4A.$$

Thus we have

$$(6.13) \quad \phi_3(1) = -1 - 2A.$$

Which corresponds to the result of Von Schrutka [8].

7. The evaluation of  $A_{10}(Q)$ . If  $p = x^2 + 4y^2 = u^2 + 5v^2$ ,  $A_{10}(Q)$  is expressed as a linear combination of  $u, X, U, V$  and  $W$ , where  $X, U, V$ , and  $W$  are solutions of the pair of diophantine equations

$$16p = X^2 + 50U^2 + 50V^2 + 125W^2 \quad \text{and} \quad XW = V^2 - 4UV - U^2.$$

Signs are selected so that  $X \equiv 1 \pmod{5}$ , and  $u \equiv x \pmod{5}$  where  $x \equiv 1 \pmod{4}$ .

$p \not\equiv 1 \pmod{10}$ . Brewer [2] has evaluated  $A_5(Q)$ ,  $Q \equiv m^2 \pmod{p}$ , with the results

$$(7.1) \quad A_5(Q) = \begin{cases} -4u \cdot \chi(m) & p \equiv 1 \pmod{20} \quad 5 \nmid x \\ 4u \cdot \chi(m) & p \equiv 9 \pmod{20} \quad 5 \nmid x \\ 0 & \text{otherwise.} \end{cases}$$

These results together with the corollary to Theorem 5.6 gives us for  $p \not\equiv 1 \pmod{10}$

$$(7.2) \quad A_{10}(Q) = \begin{cases} -1 + 4u \cdot \chi(Q) & p \equiv 9 \pmod{20} \quad 5 \nmid x \\ -1 & \text{otherwise.} \end{cases}$$

$p \equiv 1 \pmod{10}$ . Say  $p = 10f + 1$ . By Equation (5.9)

$$(7.3) \quad A_{10}(Q) = \chi(Q) \cdot A_5 + \frac{1}{2} \psi_{10}(1).$$

Whiteman [10] has expressed the cyclotomic numbers of order ten as linear combinations of  $p, X, U, V$ , and  $W$ , where  $X, U, V$ , and  $W$  are solutions of the pair of diophantine equations  $16p = X^2 + 50U^2 + 50V^2 + 125W^2$  and  $XW = V^2 - 4UV - U^2$  with  $X \equiv 1 \pmod{5}$ . However, rather than evaluating  $\psi_{10}(1)$  directly from the cyclotomic numbers as was done with  $\psi_5(1)$  in the case of  $A_5(Q)$ , it is more expeditions to use (4.12). Thus (7.3) becomes

$$(7.4) \quad A_{10}(Q) = \chi(Q) \cdot A_5(1) + 5(B(i, 1) - 2f)$$

where  $i$  is selected so that  $4g^i \equiv 1 \pmod{p}$ . Let  $g^r \equiv 2 \pmod{p}$  then  $g^{2r} \equiv 4 \pmod{p}$ . Thus  $i$  is selected so that  $2r + i \equiv 0 \pmod{p-1}$ . Since  $(B(i, 1))$  is periodic with respect to  $e$ , we may write  $i \equiv -2r \pmod{e}$ . Whiteman [11, p. 101] gives the following values for the  $B(i, 1)$ :

$$(7.5) \quad \begin{aligned} 5B(0, 1) &= p - 2 + X \\ 20B(1, 1) &= 4p - 8 - X + 10U + 20V + 25W \\ 20B(2, 1) &= 4p - 8 - X + 20U - 10V - 25W \\ 20B(3, 1) &= 4p - 8 - X - 20U + 10V - 25W \\ 20B(4, 1) &= 4p - 8 - X - 10U - 20V + 25W. \end{aligned}$$

Thus if  $p \equiv 1 \pmod{10}$  we have the following values

$$(7.6) \quad A_{10}(Q) = \begin{cases} \chi(Q) \cdot A_5(1) - 1 + X & r \equiv 0 \pmod{5} \\ \chi(Q) \cdot A_5(1) + \frac{1}{4}(-4 - X - 20U + 10V - 25W) & r \equiv 1 \pmod{5} \\ \chi(Q) \cdot A_5(1) + \frac{1}{4}(-4 - X + 10U + 20V + 25W) & r \equiv 2 \pmod{5} \\ \chi(Q) \cdot A_5(1) + \frac{1}{4}(-4 - X - 10U - 20V + 25W) & r \equiv 3 \pmod{5} \\ \chi(Q) \cdot A_5(1) + \frac{1}{4}(-4 - X + 20U - 10V - 25W) & r \equiv 4 \pmod{5} \end{cases}$$

with the value of  $A_5(1)$  from (7.1)

$$A_5(1) = \begin{cases} -4u & p \equiv 1 \pmod{20} \text{ and } 5 \nmid x \\ 0 & \text{otherwise.} \end{cases}$$

#### REFERENCES

1. B. W. Brewer, *On certain character sums*, Trans. Amer. Math. Soc. **99** (1961), 241-245.
2. ———, *On primers of the form  $u^2 + 5v^2$* . Proc. Amer. Math. Soc. **17** (1966), 502-509.
3. L. E. Dickson, *Cyclotomy, higher congruences and Warings problem*, Amer. J. Math. **57** (1935), 391-424.
4. M. Hall, Jr., *A survey of difference sets*, Proc. Amer. Math. Soc. **7** (1956), 975-986.
5. H. Hasse, *Vorlesungen Über Zahlentheorie*, Berlin, Springer-Verlag, 1950.
6. E. Jacobsthal, *Über die Darstellung der Primzahlen der Form  $4n + 1$  als summe zweier Quadrate*, J. Reine Angew. Math. **132** (1907), 238-245.
7. E. Lehmer, *On the number of solutions of  $u^k + D = w^2 \pmod{p}$* , Pacific J. Math. **5** (1955), 103-118.
8. L. Von Schrutka, *Eine Beweis für die zerlegbarkeit der primzahlen Von der Form  $6n + 1$  in enifaches und ein driefaches Quadrat*, J. Reine Angew. Math. **140** (1911), 252-265.
9. A. L. Whiteman, *Theorems analogous to Jacobsthal's theorem*, Duke Math. J. **16** (1949), 619-626.
10. ———, *Cyclotomy and Jacobsthal sums*, Amer. J. Math. **74** (1952) 89-99.
11. ———, *The cyclotomic numbers of order ten*, Proceedings of the symposia in Applied Mathematics, Vol. X, Combinatorial Analysis, 1960, 95-111.
12. ———, *The cyclotomic numbers of order twelve*, Acta Arith. **6** (1960), 53-75.
13. ———, *A theorem of Brewer on character sums*, Duke Math. J. **30** (1963), 545-552.
14. ———, *Theorems on Brewer and Jacobsthal sums. I*, Proceedings of symposia in Pure Mathematics, Vol. XIII, Number Theory, 1965, 49-55.
15. ———, *Theorems on Brewer and Jacobsthal sums, II*, Michigan Math. J. **12** (1965) 65-80.

Recieved August 8, 1966, and in revised form November 1, 1967. This research was supported in part by National Science Foundation Grant GP 3464.

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN

Stanford University  
Stanford, California

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

J. P. JANS

University of Washington  
Seattle, Washington 98105

RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Philip Marshall Anselone and Theodore Windle Palmer, <i>Collectively compact sets of linear operators</i> .....	417
Philip Marshall Anselone and Theodore Windle Palmer, <i>Spectral analysis of collectively compact, strongly convergent operator sequences</i> .....	423
Edward A. Bender, <i>Characteristic polynomials of symmetric matrices</i> .....	433
Robert Morgan Brooks, <i>The structure space of a commutative locally convex algebra</i> .....	443
Jacob Feldman and Frederick Paul Greenleaf, <i>Existence of Borel transversals in groups</i> .....	455
Thomas Muirhead Flett, <i>Mean values of power series</i> .....	463
Richard Vernon Fuller, <i>Relations among continuous and various non-continuous functions</i> .....	495
Philip Hartman, <i>Convex sets and the bounded slope condition</i> .....	511
Marcel Herzog, <i>On finite groups containing a CCT-subgroup with a cyclic Sylow subgroup</i> .....	523
James Secord Howland, <i>On the essential spectrum of Schroedinger operators with singular potentials</i> .....	533
Thomas William Hungerford, <i>On the structure of principal ideal rings</i> .....	543
Paul Joseph Kelly and Ernst Gabor Straus, <i>Curvature in Hilbert geometries. II</i> .....	549
Malempati Madhusudana Rao, <i>Linear functionals on Orlicz spaces: General theory</i> .....	553
Stanley F. Robinson, <i>Theorems on Brewer sums</i> .....	587
Ralph Tyrrell Rockafellar, <i>A general correspondence between dual minimax problems and convex programs</i> .....	597
Richard Benjamin Sher, <i>Defining subsets of <math>E^3</math> by cubes</i> .....	613
Howard Jacob Weiner, <i>Invariant measures and Cesàro summability</i> .....	621