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**THE EQUIVALENCE OF GROUP INVARIANT POSITIVE
DEFINITE FUNCTIONS**

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Let G be a separable locally compact group; ρ , a positive definite function; $M(G)$, the set of all finite Radon measures; and

$$\mathfrak{N}_\rho = \left\{ \alpha \in M(G) \mid B_\rho(\alpha, \alpha) \equiv \int_{G \times G} \int \rho(t^{-1}s) \alpha(ds) \alpha(dt) = 0 \right\}.$$

Let H_ρ be the Hilbert space obtained by completing $M(G)/\mathfrak{N}_\rho$. Similarly define H_σ as the Hilbert space corresponding to another positive definite function σ . ρ and σ are said to be equivalent (symbolically $\rho \sim \sigma$) if there is an equivalence operator T from H_ρ to H_σ which is induced by the identity operator on $M(G)$; i.e. a linear homeomorphism from H_ρ onto H_σ such that $1 - T^*T$ is Hilbert-Schmidt. Theorem 1 and Theorem 2 give necessary and sufficient conditions for $\rho \sim \sigma$ in terms of the unitary representations of G induced by ρ and σ . We discuss group invariant positive definite functions on $X \times X$ where X is a homogeneous space, and generalize Theorem 1 and 2 accordingly. The notion of equivalence operators comes exactly from Gaussian stochastic processes (cf. J. Feldman [4]). Some statistical applications will be discussed in a separate paper later in the year.

I. Preliminaries. Let H be a separable Hilbert space; \mathfrak{A} , a von Neumann algebra of bounded operators in H . It has been proved by von Neumann that H can be decomposed into a direct integral of Hilbert spaces so that \mathfrak{A} is the (central) direct integral decomposition into factors (cf. von Neumann [13]). Let A and A_1 be two separable metric spaces; μ and μ_1 be two finite regular positive Borel measures on A and A_1 respectively. Let

$$H = \int_A^\oplus H(\lambda) \mu(d\lambda)$$

and

$$H_1 = \int_{A_1}^\oplus H(\lambda_1) \mu_1(d\lambda_1)$$

be two direct integral Hilbert spaces; \mathfrak{A} and \mathfrak{A}_1 be two von Neumann algebras which have central decompositions

$$\mathfrak{A} = \int_A^\oplus \mathfrak{A}(\lambda) \mu(d\lambda) \quad \text{and} \quad \mathfrak{A}_1 = \int_{A_1}^\oplus \mathfrak{A}_1(\lambda_1) \mu_1(d\lambda_1)$$

in H and H_1 respectively. It can be proved (cf. J. T. Schwartz [11]) that \mathfrak{A} and \mathfrak{A}_1 are spatially isomorphic if and only if there exist a pair \tilde{A}, \tilde{A}_1 of Borel subsets of A and A_1 respectively such that $\mu(A - \tilde{A}) = 0$ and $\mu_1(A_1 - \tilde{A}_1) = 0$ and a Borel isomorphism $\varphi: \tilde{A} \rightarrow \tilde{A}_1$ such that $\mathfrak{A}(\lambda)$ and $\mathfrak{A}(\varphi(\lambda))$ are spatially isomorphic and such that $\mu \circ \varphi^{-1}$ is equivalent to $\mu_1|_{A_1}$. Since direct integral decomposition is uniquely determined up to a set of measure zero, it can be assumed for our purpose $A = \tilde{A}$ and $A_1 = \tilde{A}_1$. Hence the central decomposition is unique by the identification of \tilde{A} and $\varphi(\tilde{A})$.

For the general theory, we refer to Dixmier's book (cf. Dixmier [1]).

II. The equivalence of positive definite functions on groups.

Let G be a separable locally compact group; $M(G)$, the set of all finite Radon measures on G .

1. DEFINITION. A continuous function ρ is said to be positive definite on G if for any sequence of g_i , $i = 1, 2, \dots, n$ and any sequence of complex numbers c_i , $i = 1, 2, \dots, n$, the following is always satisfied

$$(2.1) \quad \sum_{i,j=1}^n \rho(g_j^{-1}g_i) c_i \bar{c}_j \geq 0.$$

It can be easily verified that ρ is positive definite on G if and only if

$$(2.2) \quad \int_{G \times G} \rho(g^{-1}h) \mu(dh) \bar{\mu}(dg) \geq 0$$

for all $\mu \in M(G)$.

2. The decomposition of a positive definite function: Consider the functional B_γ on $M(G) \times M(G)$ defined by

$$(2.3) \quad B_\gamma(\alpha, \beta) \equiv \int_{G \times G} \gamma(s^{-1}t) \alpha(dt) \bar{\beta}(ds)$$

where γ is a positive definite function on G , and $\alpha, \beta \in M(G)$. It is clear that B_γ is sesqui-linear and

$$(2.4) \quad \mathfrak{A}_\gamma = \{\alpha \in M(G) \mid B_\gamma(\alpha, \alpha) = 0\}$$

is invariant under the left action of the group G . Let H_γ be the Hilbert space obtained by completing the quotient space $M(G)/\mathfrak{A}_\gamma$ with the inner product given by (2.3). On $M(G)$, consider the linear transform defined by

$$(2.5) \quad \tilde{U}_s \alpha(A) = \alpha(s^{-1}A)$$

where α is any element in $M(G)$ and A is any measurable subset of G ; and s , any element of G . \mathfrak{N}_γ is invariant under \tilde{U}_s . Let U_s be the unitary transformation on H_γ which is densely defined (on $M(G)/\mathfrak{N}_\gamma$) as the quotient transformation of \tilde{U}_s . Then the pair (U, H_γ) forms a unitary representation of G .

Let \mathcal{U} be the von Neumann algebra generated by $\{U_s, s \in G\}$, that is $\{U_s, s \in G\}''$, the double commutant of $\{U_s, s \in G\}$. Let ξ be the element of H_γ corresponding to δ_e , the Dirac point mass at the identity. Then

$$(2.6) \quad (U_s \xi, \xi) = \int_{G \times G} \gamma(s_1^{-1}t) \delta(s^{-1}t) \delta(s_1) dt ds_1 = \gamma(s) .$$

For any $t \in G$, let

$$(2.7) \quad \xi_t = U_t \xi .$$

Since the smallest closed linear manifold containing $\{\xi_t, t \in G\}$ is H_γ , $\mathcal{U}\xi$ is dense in H_γ ; so ξ is cyclic. According to the theory of the central decomposition, there are a separable metric space Λ and a Radon measure μ on Λ such that

$$(2.8) \quad H_\gamma = \int_{\Lambda}^{\oplus} H_\gamma(\lambda) \mu(d\lambda)$$

and

$$(2.9) \quad \mathcal{U} = \int_{\Lambda}^{\oplus} \mathcal{U}(\lambda) \mu(d\lambda)$$

where the decomposition (2.9) is a central decomposition. It is also easy to see that

$$(2.10) \quad U_s = \int_{\Lambda}^{\oplus} U_s(\lambda) \mu(d\lambda)$$

where μ -almost all of the $U_s(\lambda)$ are unitary on $H_\gamma(\lambda)$ and that

$$(2.11) \quad \xi = \int_{\Lambda}^{\oplus} \xi(\lambda) \mu(d\lambda)$$

where μ -almost all of $\xi(\lambda)$ are cyclic in their respective Hilbert spaces. μ -almost all of functions $s \rightarrow (U_s(\lambda)\xi(\lambda), \xi(\lambda))_i$ are positive definite. So γ becomes an integral of positive definite functions as follows:

$$(2.12) \quad \gamma(s) = \int_{\Lambda} \gamma_\lambda(s) \mu(d\lambda)$$

where

$$(2.13) \quad \gamma_\lambda(s) = (U_s(\lambda)\xi(\lambda), \xi(\lambda))_\lambda ,$$

and

$$(2.13') \quad \gamma_\lambda(e) = (\xi(\lambda), \xi(\lambda))_\lambda = 1$$

μ -almost all λ .

3. DEFINITION. We call the measure μ normalized according to (2.12), (2.13) and (2.13') the central Radon measure of γ (with respect to

$$\int_A^\oplus H_\gamma(\lambda) \mu(d\lambda) .$$

4. The equivalence operator.

a. Definition. Let G be a separable locally compact group. Two positive definite functions ρ and σ are said to be equivalent if the identity operator on $M(G)$ induces an operator T on H_ρ to H_σ such that T is an equivalence operator (cf. introduction).

b. Let G be as in definition a; ρ and σ be two positive definite functions on G ; H_ρ and H_σ be the corresponding Hilbert spaces as defined in § II. 2. Let (U, H_ρ) and (V, H_σ) be the unitary representations of G induced by ρ and σ respectively. The Dirac point mass δ_e at the identity of G gives rise to cyclic vectors ξ and η in H_ρ and H_σ respectively. Let $\mathcal{U} = \{U_s, s \in G\}''$ and $\mathcal{V} = \{V_s, s \in G\}''$; and

$$(2.14a) \quad \mathcal{U} = \int_A^\oplus \mathcal{U}(\lambda) \mu(d\lambda)$$

$$(2.14b) \quad \mathcal{V} = \int_{A_1}^\oplus \mathcal{V}(\lambda_1) \mu_1(d\lambda_1)$$

be their central decompositions.

It has been remarked in § I that if \mathcal{U} and \mathcal{V} are spatially isomorphic, without real loss of generality we may assume A and A_1 are identical. (2.14) thus can be rewritten as

$$(2.14b') \quad \mathcal{V} = \int_A^\oplus \mathcal{U}(\lambda) \mathcal{V}(d\lambda)$$

if \mathcal{U} and \mathcal{V} are spatially isomorphic; that is if U and V are unitarily equivalent.

THEOREM 1. Let G be a separable locally compact group; ρ and σ two positive definite functions on G . If ρ and σ are equivalent (symbolically $\rho \sim \sigma$), then

(a) U and V are unitarily equivalent; and

(b) With (a) permitting the existence of the central decompositions (2.14a) and (2.14b'), μ and ν then satisfy the following condi-

tions, which we shall call the conditions (i), (ii) and (iii) in the rest of the discussion:

- (i) μ and ν have identical nonatomic parts;
- (ii) they have the same set of atoms which is countable; and
- (iii) if \mathfrak{A} is the set of all atoms, then $\mu(a) = \nu(a)$ unless $H_\rho(a)$ is finite dimensional, and

$$\sum_{a \in \mathfrak{A}} d(a) \left(1 - \frac{\mu(a)}{\nu(a)}\right)^2 < \infty$$

where $d(a)$ is the dimension of $H_\rho(a)$ if $H_\rho(a)$ is finite dimensional and ∞ otherwise.

Proof. We shall divide the proof into four steps:

Step 1. To show: If $\rho \sim \sigma$, then U and V are unitarily equivalent. Let $\xi \in H_\rho$ and $\eta \in H_\sigma$ be the elements corresponding to δ_e so that

$$(2.15a) \quad \rho(s) = (U_s \xi, \xi)_\rho$$

and

$$(2.15b) \quad \sigma(s) = (V_s \eta, \eta)_\sigma.$$

It $\rho \sim \sigma$, then it is immediately seen that $\mathfrak{N}_\rho \equiv \mathfrak{N}_\sigma$. Hence $M(G)/\mathfrak{N}_\rho \equiv M(G)/\mathfrak{N}_\sigma$. Let T be the equivalence operator from H_ρ to H_σ induced by the identity operator on $M(G)$. We have for all $s \in G$

$$(2.16) \quad T U_s = V_s T.$$

Moreover, the center $\mathfrak{K}_{\mathcal{U}}$ of \mathcal{U} is carried over by T to, the center $\mathfrak{K}_{\mathcal{V}}$ of \mathcal{V} . Since T is invertible, T^* is well-defined on all of H_σ . From (2.16)

$$(2.17) \quad U_s T^* = T^* V_s$$

combining (2.16) and (2.17), we obtain

$$(2.18) \quad (T^* T) U_s = U_s (T^* T).$$

Hence $T^* T$ commutes with U_s for all $s \in G$. Consequently $T^* T$ is in the center $\mathfrak{K}_{\mathcal{U}}$ of \mathcal{U} ; i.e. $T^* T$ is a diagonal operator (cf. Dixmier [1])

$$(2.19) \quad T^* T = \int_{\mathcal{A}}^{\oplus} a(\lambda) I(\lambda) \mu(\lambda)$$

where $a(\lambda)$ is a nonnegative function in $L^\infty(\mathcal{A}, \mu)$, and $I(\lambda)$ is the identity operator on $H_\rho(\lambda)$. Let $S = (T^* T)^{1/2}$, and R be the unitary operator: $H_\rho \rightarrow H_\sigma$ satisfying

$$(2.20) \quad T = RS .$$

Then, for all $s \in G$.

$$V_s = TU_sT^{-1} = RU_sR^* .$$

Hence (V, H_σ) and (U, H_ρ) are unitarily equivalent.

Step 2. To show: μ and ν have the same set of atoms if $\rho \sim \sigma$. If $\rho \sim \sigma$, by step 1, U and V are unitarily equivalent so that \mathcal{U} and \mathcal{V} are spatially isomorphic. As we remarked in the last part of § I, $\mu \sim \nu$ by the assumption of $\mathcal{A} = \mathcal{A}_1$. Since atoms are points, μ and ν therefore have the same set of atoms. This completes the step 2.

Before we work on step 3, we shall introduce more notations. Let μ_a and μ_c be the atomic and nonatomic parts of μ respectively; ν_a and ν_c be those of ν . Let

$$(2.21a) \quad H_{\rho,c} \equiv \int_{\mathcal{A}}^{\oplus} H_{\rho}(\lambda) \mu_c(d\lambda)$$

and

$$(2.21b) \quad H_{\rho,a} \equiv \int_{\mathcal{A}}^{\oplus} H_{\rho}(\lambda) \mu_a(d\lambda) .$$

Then

$$(2.22a) \quad H_{\rho} = H_{\rho,c} \oplus H_{\rho,a}$$

$$(2.22b) \quad H_{\sigma} = H_{\sigma,c} \oplus H_{\sigma,a} .$$

It is easy to see that $T: H_{\rho,c} \rightarrow H_{\sigma,c}$ and $T: H_{\rho,a} \rightarrow H_{\sigma,a}$. If T is an equivalence operator, then so are the restrictions $T|_{H_{\rho,c}}$ and $T|_{H_{\rho,a}}$. It is also true that $\xi = \xi_c \oplus \xi_a$ and $\eta = \eta_a \oplus \eta_c$ where $\xi_a \in H_{\rho,a}$, $\xi_c \in H_{\rho,c}$, $\eta_a \in H_{\sigma,a}$ and $\eta_c \in H_{\sigma,c}$ are cyclic in their respective Hilbert spaces. Moreover,

$$(2.23a) \quad (T|_{H_{\rho,c}})\xi_c = \eta_c$$

$$(2.23b) \quad (T|_{H_{\rho,a}})\xi_a = \eta_a .$$

So ρ and σ decompose into sum of two positive definite functions respectively:

$$(2.24a) \quad \rho \equiv \rho_c + \rho_a \equiv (U_s \xi_c, \xi_c) + (U_s \xi_a, \xi_a)$$

$$(2.24b) \quad \sigma \equiv \sigma_c + \sigma_a \equiv (V_s \eta_c, \eta_c) + (V_s \eta_a, \eta_a) .$$

Step 3. To show: If $\rho \sim \sigma$, then μ and ν have identical nonatomic parts; i.e. $\mu_c(D) = \nu_c(E)$ for all measurable subsets E of \mathcal{A} . From above discussion, we may assume μ and ν having only nonatomic

parts by passing from the relation $\rho \sim \sigma$ to $\rho_e \sim \sigma_e$. Suppose there exists a measurable subset \tilde{E} such that $\mu(\tilde{E}) \neq \nu(\tilde{E})$. Given a sufficiently small $\varepsilon > 0$, there exists a measurable subset E of \tilde{E} of positive measure such that

$$\left| 1 - \frac{d\nu}{d\mu}(\lambda) \right| > \varepsilon$$

for all $\lambda \in E$. Since μ is nonatomic, we can partition E into a disjoint union of infinitely many measurable subsets of positive measures $\{E_1, E_2, \dots\}$. Since T is an equivalence operator, $\mu(E_i) \neq 0$ implies $\nu(E_i) \neq 0$ for all i . Normalizing

$$\left\{ \int_{E_i}^{\oplus} \xi(\lambda) \mu(d\lambda) \right\}$$

we obtain an orthonormal set $\{z_i\}$. Since by (2.19), the definition of central Radon measures and $\mu \sim \nu$,

$$\begin{aligned} \int_A \|x(\lambda)\|^2 d\nu &= (T^*Tx, x) = \int_A a(\lambda) \|x(\lambda)\|^2 d\mu \\ &= \int_A a(\lambda) \left(\frac{d\mu}{d\nu} \right) \|x(\lambda)\|^2 d\nu \end{aligned}$$

for all $x = \int_A^{\oplus} x(\lambda) d\mu \in H_p$, it follows that

$$a(\lambda) \left(\frac{d\mu}{d\nu} \right) (\lambda) = 1 \quad \nu - \text{a.e.}$$

$$\text{i.e. } T^*T = \int_A^{\oplus} \{(d\nu)/(d\mu)\} I(x) d\mu.$$

Hence

$$\sum_{i=1}^{\infty} \|(1 - T^*T)z_i\|^2 \geq \sum_{i=1}^n \|(1 - T^*T)z_i\|^2 \geq n\varepsilon^2.$$

This estimate increases to infinity as n goes to infinity. This contradicts to the fact that T is Hilbert-Schmidt.

Step 4. To show: If $\rho \sim \sigma$, then

$$\sum_{\lambda \in \mathfrak{U}} d(\lambda) \left(1 - \frac{\nu(\lambda)}{\mu(\lambda)} \right)^2 < \infty.$$

As we remarked in step 3, we may reduce to the case where μ and ν have only atomic parts. The set \mathfrak{U} is at most countable, for μ and ν are finite. Then

$$(2.25) \quad H_\rho = \bigoplus_{\lambda \in \mathfrak{U}} H_\rho(\lambda)$$

and

$$(2.26) \quad \|x\|^2 = \sum_{\lambda \in \mathfrak{U}} \mu(\lambda) \|x(\lambda)\|_\lambda^2$$

for all $x \in H_\rho$ and $x = \bigoplus_{\lambda \in \mathfrak{U}} x(\lambda)$ where $x(\lambda) \in H_\rho(\lambda)$. Let $\{\varphi_i\}$ be any orthonormal set in H_ρ . If $\rho \sim \sigma$, then

$$T^*T = \bigoplus_{\lambda \in \mathfrak{U}} a(\lambda)I(\lambda).$$

From a theorem of K. Fan (cf. Fan [3]), it follows that

$$\begin{aligned} \max_{\text{all O.N. } \{\varphi_i\}} \sum_i \|(1 - T^*T)\varphi_i\|^2 &= \max_i \|(1 - T^*T)^2\varphi_i\|^2 \\ &= \text{Tr}(1 - T^*T)^2 = \sum_\lambda \text{Tr}(1 - T^*T(\lambda))^2. \end{aligned}$$

If $H_\rho(\lambda)$ is infinite dimensional, then $\text{Tr}(1 - T^*T(\lambda))^2$ is finite only when $a(\lambda) = 1$. Hence $\mu(\lambda) = \nu(\lambda)$ if $\rho \sim \sigma$ and $H_\rho(\lambda)$ is infinite dimensional. When $H_\rho(\lambda)$ is finite dimensional, then $\text{Tr}(1 - T^*T(\lambda))^2 = d(a)(1 - a(\lambda))^2$. Hence

$$(2.27) \quad \begin{aligned} \text{Tr}(1 - T^*T)^2 &= \sum d(\lambda)(1 - a(\lambda))^2 \\ &= \sum d(\lambda) \left(1 - \frac{\nu(\lambda)}{\mu(\lambda)}\right)^2. \end{aligned}$$

$1 - T^*T$ is Hilbert-Schmidt, therefore (2.27) is finite. We have proved Theorem 1.

Theorem 1 has a converse which we shall state in the following:

THEOREM 2. *If U and V of the last theorem are unitarily equivalent, and if the corresponding central Radon measures, μ and ν satisfy conditions (i), (ii) and (iii), then $\rho \sim \sigma$.*

Proof is immediate.

III. Positive definite functions on homogeneous spaces. Let G be a separable locally compact group; H , a closed subgroup of G ; X , the space of the right cosets; x_0 , be the point of X which corresponds to the identity coset H ; and finally let $M(X)$ be the set of all finite Radon measures on X . Then positive definite functions on $X \times X$ can be properly defined in the following way:

1. **DEFINITION.** A continuous function $\hat{\rho}$ on $X \times X$ is said to be positive definite if for all α in $M(X)$

$$(3.1) \quad \iint \hat{\rho}(x, y) \alpha(dx) \bar{\alpha}(dy) \geq 0 ,$$

or alternatively,

$$(3.1') \quad \sum_{i,j=1}^n \hat{\rho}(x_i, x_j) \alpha_i \bar{\alpha}_j \geq 0$$

for any sequence of points $x_i, i = 1, 2, \dots, n$ and any sequence of complex numbers $\alpha_i, i = 1, 2, \dots, n$.

2. DEFINITION. A positive definite function $\hat{\rho}$ is said to be G -invariant if, for any g in G ,

$$(3.2) \quad \hat{\rho}(gx, gy) = \hat{\rho}(x, y) .$$

3. Group representations and group invariant positive definite function: Let $\hat{\rho}$ be a positive definite function on $X \times X$ which is also group-invariant. Let

$$(3.3) \quad N_{\hat{\rho}} = \left\{ \alpha \in M(X) \mid \iint \hat{\rho}(x, y) \alpha(dx) \bar{\alpha}(dy) = 0 \right\} .$$

Then by completing $M(X)/N_{\hat{\rho}}$, we obtain again a Hilbert space $H_{\hat{\rho}}$ with an inner product given by

$$(3.4) \quad B_{\hat{\rho}}(\alpha, \beta) \equiv \iint \hat{\rho}(x, y) \alpha(dx) \bar{\beta}(dy) .$$

With the group-invariance property, $N_{\hat{\rho}}$ is invariant under the action of G , i.e., if $\alpha \in N_{\hat{\rho}}$, then the left translates α_g of α also are in $N_{\hat{\rho}}$. This translation gives rise to a unitary transformation $U_g, g \in G$ on $H_{\hat{\rho}}$ in a similar way as in the group case (cf. § II. 2). Moreover, $(U, H_{\hat{\rho}})$ is a unitary representation of G .

3. DEFINITION. We call $(U, H_{\hat{\rho}})$ the canonical unitary representation of G associated with $\hat{\rho}$.

4. DEFINITION. Two positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ on $X \times X$ are said to be equivalent if the identity operator on $M(X)$ induces an equivalence operator $\hat{T}: H_{\hat{\rho}} \rightarrow H_{\hat{\sigma}}$.

5. Relation between positive definite functions on groups and those on homogeneous spaces: By using the same technique as used in § II, we may obtain the necessary and sufficient conditions for $\hat{\rho}$ and $\hat{\sigma}$ to be equivalent. However, we shall establish them through investigation of the relation between the positive definite functions on groups and those on the homogeneous spaces.

Let $(U, H_{\hat{\rho}})$ be the unitary representation of G associated with a G -invariant positive definite function $\hat{\rho}$ on $X \times X$. Let $\xi \in H_{\hat{\rho}}$ be the element corresponding to δ_{x_0} , the Dirac point mass at x_0 . Then for any $s \in G$,

$$(3.5) \quad (U_{s\xi}, \xi) = \int_{X \times X} \hat{\rho}(x, y) \delta_{sx_0}(dx) \delta_{x_0}(dy) = \hat{\rho}(sx_0, x_0) .$$

(3.5) defines a positive definite function σ on the group G which satisfies:

$$(3.6) \quad \rho(s) = \hat{\rho}(sx_0, x_0) .$$

Since x_0 remains fixed under the action of H , ρ satisfies

$$(3.7) \quad \rho(s) = \rho(h_1 s h_2)$$

for all h_1 and h_2 in H . We thus have the following lemma.

LEMMA 3.1. *For any G -invariant positive definite function $\hat{\rho}$ on $X \times X$, there corresponds a positive definite function ρ on G which is constant on double cosets of H ; i.e., (3.7) is satisfied.*

We shall prove the following lemma before we can establish a converse form of Lemma 3.1.

LEMMA 3.2. *Let ρ be a positive definite function on G . Let*

$$(3.8) \quad K = \{k \in G \mid \rho(k) = \rho(e)\} ,$$

Then K is a closed subgroup of G .

Proof. Since $\rho(e)$ is positive and finite, it can be assumed that $\rho(e) = 1$. According to the theory of group representations (cf. Naimark, [10] Godement [5, 6]) there exists a vector ξ in some Hilbert space L such that

$$(3.9) \quad \rho(s) = (U_s \xi, \xi)$$

where (\cdot, \cdot) is the inner product in the Hilbert space L and (U, L) is a unitary representation of the group. Let $k \in K$. Then it can be proved that

$$(3.10) \quad U_k \xi = \xi .$$

So for any $h, k \in K$

$$U_h \xi = U_k \xi = \xi .$$

We have $U_{kh} \xi = U_k U_h \xi = U_k \xi = \xi$; i.e., if $h, k \in K$, then $hk \in K$. If

$h \in K$, $\rho(h) = \bar{\rho}(h^{-1}) = 1 = \rho(h^{-1})$. Hence $h^{-1} \in K$ if $h \in K$. Moreover, since ρ is continuous, we see that K is closed.

LEMMA 3.3. *For any positive definite function ρ on G , there is a largest closed subgroup K such that ρ is constant on double cosets of K . As a consequence, it gives rise to a group-invariant positive definite function $\hat{\rho}$ on $(G/K) \times (G/K)$.*

Proof. Let $\{e\}$ be the subgroup consisting of only the identity of G . Then it is clear that ρ is constant on double cosets of $\{e\}$. Let

$$\begin{aligned} H &= \{h \mid \rho(gh) = \rho(hg) = \rho(g) \text{ for all } g \in G\} \\ K &= \{k \mid \rho(k) = \rho(e) = 1\}. \end{aligned}$$

If $h \in H$, choosing $g = e$, then $\rho(h) = \rho(e)$. So $H \subset K$. We now prove that ρ is constant on double cosets of K . Let $k \in K$, then as in Lemma 3.2, $U_k \xi = \xi$ and

$$\begin{aligned} \rho(gk) &= (U_g U_k \xi, \xi) = (U_g \xi, \xi) = \rho(g) = (U_g \xi, U_{k^{-1}} \xi) \\ &= (U_k U_g \xi, \xi) = \rho(kg); \end{aligned}$$

i.e., $\rho(g) = \rho(KgK)$.

So K is the largest closed subgroup such that ρ is constant on double cosets of K . (3.6) defines a G -invariant positive definite function as it can be easily verified. Combining the preceding three lemmas, we have the following theorem.

THEOREM 3. *Let G be a separable locally compact group. Each positive-definite function ρ on G gives rise by (3.6) to a G -invariant positive definite function $\hat{\rho}$ on $(G/H) \times (G/H)$ where H is the set of all x in G such that $\rho(x) = \rho(e)$. Conversely, to any G -invariant positive definite function $\hat{\rho}$ on $(G/H) \times (G/H)$ there corresponds by (3.6) a positive definite function ρ on G such that $H = \{x: \rho(x) = \rho(e)\}$.*

5. Let G and H be the same as before; τ , the canonical mapping from G to $G/H = X$. Then a subset E of X is measurable if and only if $\tau^{-1}(E)$ is measurable in G (cf. Mackey [11]). According to the theory of decomposition of measure (cf. Halmos [7, 8], von Neumann [12], Dieudonné [2], Mackey [9]), for any finite Radon measure α on G , there is a measure $\hat{\alpha}$ in X such that for all measurable $E \subset X$

$$(3.11) \quad \hat{\alpha}(E) = \alpha(\tau^{-1}(E))$$

and

$$(3.12) \quad \int_x f(x) \int_g g(s) d\alpha_x(s) d\hat{\alpha}(x) = \int_g f(\tau(s)) g(s) d\alpha(s)$$

where α_x is a finite Radon measure on G which only lives on the coset x ; f , a function in $L^1(X, \tilde{\alpha})$; and g , a bounded measurable function on G . Conversely, $\tilde{\alpha}$ certainly defines a measure on G . If ρ is a positive definite function on G which is constant on H , then $\rho(t^{-1}s) = \hat{\rho}(sx_0, tx_0) = \hat{\rho}(x, y)$. By generalizing (3.11) and (3.12) to the two dimensional product measures (cf. Mackey [9]), it follows that

$$(3.13) \quad \int_{G \times G} \rho(t^{-1}s) \alpha(ds) \tilde{\alpha}(dt) \\ = \int_{X \times X} \hat{\rho}(x, y) \tilde{\alpha}(dx) \tilde{\alpha}(dy) \int_{G \times G} \alpha_x(ds') \alpha_y(dt').$$

Hence $\alpha \in \mathfrak{N}_\rho \subset M(G)$ if $\alpha_x(G) = 0$. If we let $x_\alpha \in H_\rho$ be the element corresponding to α in $M(G)$ and $x_\alpha \in H_\rho$ be the element corresponding to $\alpha_x(G)\tilde{\alpha}$ in $M(X)$, we conclude from the above discussion that $\varphi: H_\rho \rightarrow H_\rho$ satisfying $\varphi(x_\alpha) = x_\alpha$ is a spatial isomorphism which commutes with the transformation induced by translations. There is a similar mapping $\psi: H_\sigma \rightarrow H_\sigma$ satisfying $\psi(y_\alpha) = y_\alpha$ where $y_\alpha \in H_\sigma$, $y_\alpha \in H_\sigma$ are the corresponding elements of α and $\alpha_x(G)\tilde{\alpha}$ respectively. Furthermore, if $T: H_\rho \rightarrow H_\sigma$ and $\tilde{T}: H_\rho \rightarrow H_\sigma$ are the mapping induced by the identity mappings on $M(G)$ and $M(X)$ respectively, then it is clear that

$$(3.14) \quad Tx_\alpha = y_\alpha$$

$$(3.15) \quad \tilde{T}x_\alpha = y_\alpha$$

and the following diagram commutes

$$\begin{array}{ccc} H_\rho & \xrightarrow{T} & H_\sigma \\ \downarrow \varphi & & \downarrow \psi \\ H_\rho & \xrightarrow{\tilde{T}} & H_\sigma \end{array}$$

Hence \tilde{T} is a linear homeomorphism if and only if so is T . If \tilde{T} is such, the similar commutative diagram for \tilde{T}^*, T^* holds

$$\begin{array}{ccc} H_\rho & \xrightarrow{T^*} & H_\sigma \\ \downarrow \varphi & & \downarrow \psi \\ H_\rho & \xrightarrow{\tilde{T}^*} & H_\sigma \end{array}$$

Therefore,

$$(3.16) \quad \varphi T^* T = \tilde{T}^* \psi T = \tilde{T}^* \tilde{T} \varphi$$

and

$$(3.17) \quad \|(I - T^* T)x\|^2 = \|\varphi(I - T^* T)x\|^2 = \|(I - \tilde{T}^* \tilde{T})\varphi x\|^2.$$

We immediately conclude that \tilde{T} is an equivalence operator if and only if so is T , and arrive the following theorem.

THEOREM 4. *Two group-invariant positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ on $X = G/H$ are equivalent if and only if the corresponding positive definite functions ρ and σ on G are equivalent.*

6. Let X be a separable metric space; G , a locally compact transformation group of X . Let $M(X)$ be the set of all finite Radon measures on X . The positive definite functions and group invariance property can be similarly defined as in Definitions III.1 and III.2. Suppose that X has dense orbits. Let $x_0 \in X$ such that $X_0 = Gx_0$ is dense in X . Then $X_0 = G/H_0$, where $H_0 = \{h \in G \mid hx_0 = x_0\}$. We now embed G/H_0 in X .

COROLLARY 4.1. *Let X, G be as above; and suppose that X has dense orbits. Then two G -invariant positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ are equivalent if and only if the conditions in Theorem 1 are satisfied.*

Proof. The elements ξ in H_ρ and η in H_σ corresponding to the Dirac point mass at x_0 are cyclic, by the continuity of positive definite functions. Hence applying Theorem 4 and Theorem 1, we assert the corollary.

COROLLARY 4.2. *Let X be a separable metric space; G , a locally compact group acting ergodically on X . Suppose that the ergodic measure μ satisfies $\mu(0) > 0$ for any open set $0 \subset X$. Then $\hat{\rho} \sim \hat{\sigma}$ if and only if (a) and (b) of Theorem 1 are satisfied.*

Proof. It is known that if G acts ergodically on X and satisfies the above hypothesis, then X has dense orbits; in fact

$$\mu\{x \mid \bar{G}x \neq X\} = 0.$$

Hence applying Corollary 4.1, we prove the corollary.

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