

# Pacific Journal of Mathematics

**SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN  
DENSITY**

R. L. DUNCAN

## SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY

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Let  $S$  denote the set of all infinite increasing sequences of positive integers. For all  $A = \{a_n\}$  and  $B = \{b_n\}$  in  $S$ , define the metric  $\rho(A, B) = 0$  if  $A = B$ , i.e., if  $a_n = b_n$  for all  $n$  and  $\rho(A, B) = 1/k$  otherwise, where  $k$  is the smallest value of  $n$  for which  $a_n \neq b_n$ . Similar metrics have been considered previously [1, 2].

Our purpose here is to discuss several continuity properties of the Schnirelmann density  $d(A) = \inf A(n)/n$ , where  $A(n)$  is the number of elements of  $A$  not exceeding  $n$ . In particular, we obtain a characterization of the set of all sequences having density zero as the set of points of continuity of  $d(A)$ .

**THEOREM 1.**  $d(A)$  is upper semicontinuous on  $S$ .

*Proof.* For each  $\varepsilon > 0$  there is a  $k$  such that  $A(k)/k < d(A) + \varepsilon$ . If  $\rho(A_n, A) = 1/k_n \rightarrow 0$ , then  $k_n \rightarrow \infty$  and there is an  $N$  such that  $k_n > k$  for all  $n > N$ . Hence  $d(A_n) \leq A_n(k)/k = A(k)/k < d(A) + \varepsilon$  for all  $n > N$  and the desired result follows.

Let  $L_a = \{A \in S \mid d(A) = a\}$  ( $0 \leq a \leq 1$ ) denote the level sets of  $d(A)$  and define  $M_a = \{A \in S \mid d(A) \geq a\}$ . Also, let  $\bar{L}_a$  denote the closure of  $L_a$ .

**THEOREM 2.**  $\bar{L}_a = M_a$ .

*Proof.* If  $\lim \rho(A_n, A) = 0$  and  $A_n \in L_a$  for all  $n$ , then  $d(A) \geq \limsup d(A_n) = a$  by Theorem 1 and  $\bar{L}_a \subset M_a$ .

Now let  $B = \{b_k\} \in M_a$  and  $d(B) = b \geq a > 0$ . Also, let  $A = \{a_k\}$ , where  $a_k = 1 + [k - 1/a]$ , and define  $B_k = \{b_1, \dots, b_k, a_{k+1}, a_{k+2}, \dots\}$ . Then  $an \leq A(n) < an + 1$  and it follows that  $d(A) = a$ . Also,  $k = B(b_k) \geq bb_k$  and  $b_k \leq k/b < [k/b] + 1 \leq [k/a] + 1 = a_{k+1}$ . Hence,  $B_k$  is an increasing sequence, and  $B_k(n)/n \geq a$  for all  $n$  and  $k$ . Thus  $d(B_k) = \lim B_k(n)/n = a$  and  $B_k \in L_a$  for all  $k$ . Since  $\lim \rho(B_k, B) = 0$ , we have  $M_a \subset \bar{L}_a$  for  $a > 0$ .

Finally, if  $a = 0$  we define  $B_k = \{b_1, \dots, b_k, b_{k+1}^2, b_{k+2}^2, \dots\}$ . Then it is obvious that  $B_k \in L_0$  and  $\lim \rho(B_k, B) = 0$ . Hence  $M_0 \subset \bar{L}_0$  and  $\bar{L}_a = M_a$  for  $0 \leq a \leq 1$ .

**COROLLARY.**  $\bar{L}_0 = S$ .

It follows from the above corollary and Theorem 1 that  $d(A)$  is continuous at  $A$  if and only if  $d(A) = 0$ . It also follows from Theorem

2 that  $L_a$  is closed if and only if  $a = 1$  and it is easily shown that  $L_a$  is never open. However, it is a consequence of the following theorem that  $L_a$  is a  $G_\delta$  set and that a description of the graph of  $d(A)$  can be given [3].

**THEOREM 3.**  $d(A)$  is a function of Baire class one.

*Proof.* Let

$$d_n(A) = \inf_{1 \leq k \leq n} \frac{A(k)}{k}.$$

Then  $d_n(A) = d_n(B)$  if  $\rho(A, B) < 1/n$  and it follows that  $d_n(A)$  is continuous (uniformly) on  $S$ . It remains to be shown that  $\lim d_n(A) = d(A)$  for all  $A \in S$ .

Now let  $k_n$  be the smallest value of  $k$  for which  $d_n(A) = A(k)/k$ . Then  $\lim d_n(A)$  exists since  $d_n(A) \geq d_{n+1}(A) \geq 0$  for all  $n$ . Also,  $d_n(A) = A(k_n)/k_n \geq d(A)$  for all  $n$ . Hence  $\lim d_n(A) \geq d(A)$ . Since the sequence  $\{k_n\}$  is nondecreasing and these numbers are integers, we have either  $k_n = k$  for all  $n \geq N$  or  $k_n \rightarrow \infty$ . In the first case it is clear that  $d_n(A) = d(A)$  for all  $n \geq N$  and  $\lim d_n(A) = d(A)$ . Suppose that  $k_n \rightarrow \infty$ . Then  $A(k)/k \geq A(k_n)/k_n = d_n(A)$  for all  $k \leq k_n$ . Hence  $A(k)/k \geq \lim d_n(A)$  for all  $k$  and  $\lim d_n(A) \leq d(A)$  since the sequence  $\{d_n(A)\}$  is nonincreasing. Thus  $\lim d_n(A) = d(A)$  in this case also.

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Efraim Pacillas Armendariz, <i>Closure properties in radical theory</i> .....	1
Friedrich-Wilhelm Bauer, <i>Postnikov-decompositions of functors</i> .....	9
Thomas Ru-Wen Chow, <i>The equivalence of group invariant positive definite functions</i> .....	25
Thomas Allan Cootz, <i>A maximum principle and geometric properties of level sets</i> .....	39
Rodolfo DeSapio, <i>Almost diffeomorphisms of manifolds</i> .....	47
R. L. Duncan, <i>Some continuity properties of the Schnirelmann density</i> .....	57
Ralph Jasper Faudree, Jr., <i>Automorphism groups of finite subgroups of division rings</i> .....	59
Thomas Alastair Gillespie, <i>An invariant subspace theorem of J. Feldman</i> .....	67
George Isaac Glauberman and John Griggs Thompson, <i>Weakly closed direct factors of Sylow subgroups</i> .....	73
Hiroshi Haruki, <i>On inequalities generalizing a Pythagorean functional equation and Jensen's functional equation</i> .....	85
David Wilson Henderson, <i>D-dimension. I. A new transfinite dimension</i> .....	91
David Wilson Henderson, <i>D-dimension. II. Separable spaces and compactifications</i> .....	109
Julien O. Hennefeld, <i>A note on the Arens products</i> .....	115
Richard Vincent Kadison, <i>Strong continuity of operator functions</i> .....	121
J. G. Kalbfleisch and Ralph Gordon Stanton, <i>Maximal and minimal coverings of <math>(k - 1)</math>-tuples by <math>k</math>-tuples</i> .....	131
Franklin Lowenthal, <i>On generating subgroups of the Moebius group by pairs of infinitesimal transformations</i> .....	141
Michael Barry Marcus, <i>Gaussian processes with stationary increments possessing discontinuous sample paths</i> .....	149
Zalman Rubinstein, <i>On a problem of Ilyeff</i> .....	159
Bernard Russo, <i>Unimodular contractions in Hilbert space</i> .....	163
David Lee Skoug, <i>Generalized Ilstow and Feynman integrals</i> .....	171
William Charles Waterhouse, <i>Dual groups of vector spaces</i> .....	193