AN INVARIANT SUBSPACE THEOREM OF J. FELDMAN

THOMAS ALASTAIR GILLESPIE
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T. A. GILLESPIE

Theorem. Let $t$ be a quasi-nilpotent bounded linear operator on a complex normed space $X$ of dimension greater than one. Suppose further that there is a sequence $(p_n(t))$ of polynomials in $t$ and a nonzero compact operator $s$ on $X$ such that $p_n(t) \to s$ (in norm) as $n \to \infty$. Then $t$ has a proper closed invariant subspace.

In [3], Feldman proves this theorem in the case when $X$ is a Hilbert space. By adapting the proof given by Bonsall [2, Theorem (20.1)] of the Bernstein-Robinson invariant subspace theorem [1], the result can be shown to hold when $X$ is a normed space, the necessary changes in the proof given in [2] being suggested by [3]. For the sake of completeness, the proof below repeats the relevant arguments in [2]. We need the following notation and simple results.

(i) If $E$ is a nonempty subset of $X$ and $x \in X$, the distance from $x$ to $E$, $d(x, E)$, is defined by

$$d(x, E) = \inf \{\|x - y\| : y \in E\}.$$

(ii) Given a sequence $\{E_n\}$ of linear subspaces of $X$, define

$$\liminf E_n = \{x \in X : \exists \text{ a sequence } \{x_n\} \text{ with } x_n \in E_n \text{ and } x_n \to x\}.$$

It is clear that $\liminf E_n$ is a closed linear subspace of $X$ and

$$\liminf E_n = \{x \in X : d(x, E_n) \to 0 \text{ as } n \to \infty\}.$$

(iii) Given a finite dimensional subspace $E$ of $X$ and $x \in X$, $\exists u \in E$ such that $\|x - u\| = d(x, E)$. We call such a $u$ a nearest point of $E$ to $x$. Also, if $F$ is a finite dimensional subspace of $X$ such that $F \supset E$, $F \neq E$, $\exists v \in F$ such that $\|v\| = 1 = d(v, E)$.

Proof of theorem. Let $e \in X$, $\|e\| = 1$. Clearly we may assume that $X$ has infinite dimension, and that $e, te, t^2e, \cdots$ are linearly independent. Let $E_n$ be the linear span of $\{e, te, \ldots, t^{n-1}e\}$, and choose $e_n \in E_n$ such that

$$\|e_n\| = 1 = d(e_n, E_{n-1}).$$

Since $E_n$ is the linear span of $\{E_{n-1}, t^{n-1}e\}$, for each integer $n$ there is a unique $\alpha_n \in C$, $\alpha_n \neq 0$, such that

$$e_n - \alpha_n t^{n-1}e \in E_{n-1}.$$

Since $tE_{n-1} \subset E_n$, (1) gives
(2) \[ t^r e_n = \alpha_n t^{n+r-1} e \in E_{n+r-1} \]

for \( n \geq 1, r \geq 1 \). Also, replacing \( n \) by \( n + r \) in (1),

(3) \[ e_{n+r} = \alpha_{n+r} t^{n+r-1} e \in E_{n+r-1} \]

and hence, by (2) and (3),

(4) \[ t^r e_n = \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \in E_{n+r-1} \]

for \( n \geq 1, r \geq 1 \). We note that, since \( d(e_n, E_{n-1}) = 1 \), it follows from (4) that

\[ d(t^r e_n, E_{n+r-1}) = \left| \frac{\alpha_n}{\alpha_{n+r}} \right| \quad (n, r \geq 1) . \]

We show that there is a subsequence \( \{ \alpha_{j(n)}/\alpha_{j(n)+1} \} \) of \( \{ \alpha_n/\alpha_{n+1} \} \) such that \( \alpha_{j(n)}/\alpha_{j(n)+1} \to 0 \) as \( n \to \infty \). (This corresponds to the lemma in [3]). Suppose not. Then

\[ \liminf_{n \to \infty} \left| \frac{\alpha_n}{\alpha_{n+1}} \right| = \lambda > 0 , \]

and so there exists \( n_0 \) such that

\[ \left| \frac{\alpha_n}{\alpha_{n+1}} \right| > \lambda/2 \text{ if } n \geq n_0 . \]

Since

\[ \| t^r \| \geq \| t^r e_n \| \geq d(t^r e_n, E_{n+r-1}) = \left| \frac{\alpha_n}{\alpha_{n+r}} \right| , \]

\[ \| t^r \| \geq \prod_{j=n}^{n+r-1} \left| \frac{\alpha_j}{\alpha_{j+1}} \right| . \]

Taking \( n = n_c \), this gives

\[ \| t^r \| \geq (\lambda/2)^r \quad (r \geq 1) , \]

and so

\[ \lim_{r \to \infty} \| t^r \|^{1/r} \geq \lambda/2 > 0 , \]

contradicting the quasi-nilpotence of \( t \). Therefore we can find a subsequence \( \{ j(n) \} \) such that

\[ \frac{\alpha_{j(n)}}{\alpha_{j(n)+1}} \to 0 \text{ as } n \to \infty , \]

i.e. such that
(5) \[ d(te_{j(n)}, E_{j(n)}) \to 0 \quad \text{as} \quad n \to \infty. \]

Define linear mappings \( t_n : E_n \to E_n (n \geq 1) \) by
\[
t_n | E_{n-1} = t | E_{n-1}, \quad t_n(e_n) = u_n,
\]
where \( u_n \) is a nearest point of \( E_n \) to \( te_n \). We show that

(6) \[ ||tx - t_nx|| \leq d(te_n, E_n)||x|| \quad (x \in E_n, \ n \geq 1). \]

Let \( x \in E_n \). Then \( x = y + \lambda e_n \) for some \( \lambda \in C, \ y \in E_{n-1} \).

\[ ||tx - t_nx|| = ||\lambda te_n - \lambda u_n|| = |\lambda|d(te_n, E_n), \]

and also
\[ ||x|| \geq d(x, E_{n-1}) = d(\lambda e_n, E_{n-1}) = |\lambda|d(e_n, E_{n-1}) = |\lambda|. \]

Therefore
\[ ||tx - t_nx|| \leq d(te_n, E_n)||x|| \quad (x \in E_n, \ n \geq 1). \]

From (5) and (6) we see that, if \( \{x_n\} \) is a bounded sequence with \( x_n \in E_{j(n)} \), then

(7) \[ ||tx_n - t_{j(n)}x_n|| \to 0 \quad \text{as} \quad n \to \infty. \]

From (7) it follows that if \( \{H_{n_k}\} \) is a sequence of subspaces with \( H_{n_k} \subset E_{j(n_k)} \) and \( H_{n_k} \) invariant for \( t_{j(n_k)} \), then \( \liminf H_{n_k} \) is invariant for \( t \).

We prove next, by induction on \( k \), that for each integer \( k \) there is a constant \( A_k \) such that

(8) \[ ||t_kx - t_k^x|| \leq A_kd(te_n, E_n)||x|| \quad (x \in E_n, \ n \geq 1). \]

The case when \( k = 1 \) is given by (6), \((A_1 = 1)\). Suppose that (8) holds for some \( k \). Then, for \( x \in E_n \),
\[
||t_k^x|| \leq ||t^x|| + A_kd(te_n, E_n)||x|| \\
\leq (||t^x|| + A_kd(te_n, E_n))||x|| \\
\leq (||t^x|| + A_k||t||)||x|| \\
= B_k||x||, \quad \text{say}.
\]

Since \( t_k^xE_n \subset E_n \), (6) gives
\[
||tt_k^x - t_k^{x+1}|| \leq d(te_n, E_n)||t_k^x|| \\
\leq B_kd(te_n, E_n)||x||.
\]

Therefore
\[
||t_k^{x+1} - t_k^x|| \leq ||t_k^{x+1} - tt_k^x|| + ||tt_k^x - t_k^{x+1}|| \\
\leq ||t|| ||t^x - t_k^x|| + ||tt_k^x - t_k^{x+1}|| \\
\leq (||t||A_k + B_k)d(te_n, E_n)||x||.
\]
Hence, by induction, (8) is proved.

It follows immediately from (8) that, given a polynomial \( p(t) \) in \( t \), there is a constant \( M \) such that

\[
\| p(t)x - p(t_n)x \| \leq Md(te_n, E_n) \| x \| \quad (x \in E_n, \ n \geq 1).
\]

Hence we can find positive constants \( \{M_r\}_{r=1}^\infty \) such that

\[
(9) \quad \| p_r(t)x - p_r(t_n)x \| \leq M_rd(te_n, E_n) \| x \|
\]

for \( x \in E_n, \ n \geq 1, \ r \geq 1 \).

Since \( st = ts \) and \( s \neq 0 \), we may assume that \( s^{-1}(0) = (0) \), for otherwise \( s^{-1}(0) \) is a proper closed invariant subspace for \( t \). Therefore \( se \neq 0 \), and we can choose \( \alpha \) with \( 0 < \alpha < 1 \) and \( \alpha \| s \| < \| se \| \).

Choose sequences \( \{E^{i_j}_n\}_{i_j=0}^{i_n} \) of subspaces of \( E_{j(n)} \) such that

\[
(0) = E^0_n \subset E^1_n \subset \cdots \subset E^{i_n}_n = E_{j(n)},
\]

where \( \dim E^i_n = i \) and \( E^i_n \) is invariant for \( t_{j(n)} \). Since \( d(e, E^0_n) = \| e \| = 1 \) and \( d(e, E^{i_n}_n) = 0 \), for each \( n \) there is a greatest \( i, i_n \) say, such that \( d(e, E^{i_n}_n) > \alpha \). Put \( F_n = E^{i_n}_n, G_n = E^{i_n+1}_n \). Then

\[
d(e, F_n) > \alpha, \quad d(e, G_n) \leq \alpha \quad (n \geq 1),
\]

and so

\[
(10) \quad e \in \lim \inf F^k_n
\]

for any subsequence \( \{n_k\} \). Let \( y_n, z_n \) be nearest points of \( G_n \) to \( e, se \) respectively, and let \( v_n \in G_n \) with \( \| v_n \| = 1 = d(v_n, F_n) \). We can write

\[
y_n = x_n + \beta_n v_n, \quad z_n = x'_n + \beta'_n v_n,
\]

where \( x_n, x'_n \in F_n \) and \( \beta_n, \beta'_n \in \mathcal{C} \). We have

\[
| \beta_n | = d(\beta_n v_n, F_n) = d(y_n, F_n) \leq \| y_n \| \\
\leq \| y_n - e \| + \| e \| = d(e, G_n) + \| e \| \leq 2 \| e \|.
\]

Similarly

\[
| \beta'_n | \leq 2 \| se \|.
\]

Also, for \( n \geq 1, \)

\[
(11) \quad \| sy_n \| \geq \| se \| - \| sy_n - se \| \geq \| se \| - \| s \| \| y_n - e \| \\
= \| se \| - \| s \| d(e, G_n) \geq \| se \| - \alpha \| s \| > 0.
\]

By the compactness of \( s \) and the boundedness of \( \{|y_n|\}, \{|\beta_n|\}, \{|\beta'_n|\} \), we can find a subsequence \( \{n_k\} \) such that

\[
\beta_{n_k} \rightarrow \beta, \quad \beta'_{n_k} \rightarrow \beta', \quad sy_{n_k} \rightarrow y \text{ as } k \rightarrow \infty.
\]
We show that \( y \in \lim \inf G_{n_k} \). Let \( \varepsilon > 0 \). \( \exists \ n_0 \) such that

\[
\left\| s - p_{n_0}(t) \right\| < \frac{\varepsilon}{4\| e \|}.
\]

By (5), \( \exists \ k_0 \) such that

\[
d(t_{j(n_k)}, E_{j(n_k)}) < \frac{\varepsilon}{4M_{n_0}\| e \|} \quad \text{if} \quad k \geq k_0.
\]

Since \( \| y_n \| \leq 2\| e \| \) \((n \geq 1)\), by (9)

\[
\| p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \leq M_{n_0}d(t_{j(n_k)}, E_{j(n_k)}) \cdot 2\| e \|
\]

for \( k \geq 1 \). Therefore \( k \geq k_0 \) implies that

\[
\| sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \leq \| sy_{n_k} - p_{n_0}(t)y_{n_k} \| + \| p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \\
\leq \| s - p_{n_0}(t) \| \| y_{n_k} \| + 2M_{n_0}\| e \| d(t_{j(n_k)}, E_{j(n_k)}) \\
\leq \frac{\varepsilon}{4\| e \|} \cdot 2\| e \| + 2M_{n_0}\| e \| \cdot \frac{\varepsilon}{4M_{n_0}\| e \|} = \varepsilon.
\]

Since \( sy_{n_k} \to y \), \( \exists \ k_1 \geq k_0 \) such that \( \| sy_{n_k} - y \| < \varepsilon \) if \( k \geq k_1 \). Thus if \( k \geq k_1 \),

\[
\| y - p_{n_0}(t_{j(n_k)})y_{n_k} \| \leq \| y - sy_{n_k} \| + \| sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| < \varepsilon + \varepsilon = 2\varepsilon.
\]

But \( p_{n_0}(t_{j(n_k)})y_{n_k} \in G_{n_k} \) since \( G_{n_k} \) is invariant for \( t_{j(n_k)} \), and so

\[
d(y, G_{n_k}) \leq \| y - p_{n_0}(t_{j(n_k)})y_{n_k} \| < 2\varepsilon \quad \text{if} \quad k \geq k_1.
\]

Therefore \( d(y, G_{n_k}) \to 0 \) as \( k \to \infty \), and \( y \in \lim \inf G_{n_k} \).

Now by (11) \( y \neq 0 \), and so \( \lim \inf G_{n_k} \) will be a proper closed invariant subspace for \( t \) unless \( \lim \inf G_{n_k} = X \). Thus we may suppose that \( \lim \inf G_{n_k} = X \), and hence that \( e, se \in \lim \inf G_{n_k} \), i.e.

\[
d(e, G_{n_k}) = \| e - y_{n_k} \| \to 0 \quad \text{as} \quad k \to \infty
\]

and

\[
d(se, G_{n_k}) = \| se - z_{n_k} \| \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore

\[
\beta_{n_k} x_{n_k} \to e \quad \text{and} \quad \beta'_{n_k} x'_{n_k} \to se \quad \text{as} \quad k \to \infty.
\]

Hence

\[
\beta'_{n_k} x_{n_k} - \beta_{n_k} x'_{n_k} \to \beta'e - \beta se \quad \text{as} \quad k \to \infty.
\]
and so
\[ \beta' e - \beta se \in \lim \inf F_{n_k}. \]
If \( \beta = 0 \) then \( x_{n_k} \to e \) and \( e \in \lim \inf F_{n_k} \), contradicting (10). So \( \beta \neq 0 \).
If \( \beta' e - \beta se = 0 \) then \( (\beta'/\beta)e = se \neq 0 \) and so \( \beta' \neq 0 \). Then \( s \neq (\beta'/\beta)\mathcal{F} \) since \( s \) is compact and \( X \) has infinite dimension (\( \mathcal{F} \) being the identity operator on \( X \)). Therefore
\[
0 \neq e \in \left( s - \frac{\beta'}{\beta} \mathcal{F} \right)^{-1}(0)
\]
and \( \{s - (\beta'/\beta)\mathcal{F}\}^{-1}(0) \) is a proper closed invariant subspace for \( t \).
Finally, if \( \beta' e - \beta se \neq 0 \) then \( \lim \inf F_{n_k} \neq (0) \), and so, by (10), \( \lim \inf F_{n_k} \) is a proper closed invariant subspace for \( t \).

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