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**AN INVARIANT SUBSPACE THEOREM OF J. FELDMAN**

THOMAS ALASTAIR GILLESPIE

# AN INVARIANT SUBSPACE THEOREM OF J. FELDMAN

T. A. GILLESPIE

**Theorem.** Let  $t$  be a quasi-nilpotent bounded linear operator on a complex normed space  $X$  of dimension greater than one. Suppose further that there is a sequence  $\{p_n(t)\}$  of polynomials in  $t$  and a nonzero compact operator  $s$  on  $X$  such that  $p_n(t) \rightarrow s$  (in norm) as  $n \rightarrow \infty$ . Then  $t$  has a proper closed invariant subspace.

In [3], Feldman proves this theorem in the case when  $X$  is a Hilbert space. By adapting the proof given by Bonsall [2, Theorem (20.1)] of the Bernstein-Robinson invariant subspace theorem [1], the result can be shown to hold when  $X$  is a normed space, the necessary changes in the proof given in [2] being suggested by [3]. For the sake of completeness, the proof below repeats the relevant arguments in [2]. We need the following notation and simple results.

(i) If  $E$  is a nonempty subset of  $X$  and  $x \in X$ , the distance from  $x$  to  $E$ ,  $d(x, E)$ , is defined by

$$d(x, E) = \inf \{ \|x - y\| : y \in E \} .$$

(ii) Given a sequence  $\{E_n\}$  of linear subspaces of  $X$ , define  $\liminf E_n = \{x \in X : \exists \text{ a sequence } \{x_n\} \text{ with } x_n \in E_n \text{ and } x_n \rightarrow x\}$ . It is clear that  $\liminf E_n$  is a closed linear subspace of  $X$  and

$$\liminf E_n = \{x \in X : d(x, E_n) \rightarrow 0 \text{ as } n \rightarrow \infty\} .$$

(iii) Given a finite dimensional subspace  $E$  of  $X$  and  $x \in X$ ,  $\exists u \in E$  such that  $\|x - u\| = d(x, E)$ . We call such a  $u$  a nearest point of  $E$  to  $x$ . Also, if  $F$  is a finite dimensional subspace of  $X$  such that  $F \supset E$ ,  $F \neq E$ ,  $\exists v \in F$  such that  $\|v\| = 1 = d(v, E)$ .

*Proof of theorem.* Let  $e \in X$ ,  $\|e\| = 1$ . Clearly we may assume that  $X$  has infinite dimension, and that  $e, te, t^2e, \dots$  are linearly independent. Let  $E_n$  be the linear span of  $\{e, te, \dots, t^{n-1}e\}$ , and choose  $e_n \in E_n$  such that

$$\|e_n\| = 1 = d(e_n, E_{n-1}) .$$

Since  $E_n$  is the linear span of  $\{E_{n-1}, t^{n-1}e\}$ , for each integer  $n$  there is a unique  $\alpha_n \in C$ ,  $\alpha_n \neq 0$ , such that

$$(1) \quad e_n - \alpha_n t^{n-1}e \in E_{n-1} .$$

Since  $tE_{n-1} \subset E_n$ , (1) gives

$$(2) \quad t^r e_n - \alpha_n t^{n+r-1} e \in E_{n+r-1}$$

for  $n \geq 1, r \geq 1$ . Also, replacing  $n$  by  $n + r$  in (1),

$$(3) \quad e_{n+r} - \alpha_{n+r} t^{n+r-1} e \in E_{n+r-1},$$

and hence, by (2) and (3),

$$(4) \quad t^r e_n - \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \in E_{n+r-1}$$

for  $n \geq 1, r \geq 1$ . We note that, since  $d(e_n, E_{n-1}) = 1$ , it follows from (4) that

$$d(t^r e_n, E_{n+r-1}) = \frac{|\alpha_n|}{|\alpha_{n+r}|} \quad (n, r \geq 1).$$

We show that there is a subsequence  $\{\alpha_{j(n)}/\alpha_{j(n)+1}\}$  of  $\{\alpha_n/\alpha_{n+1}\}$  such that  $\alpha_{j(n)}/\alpha_{j(n)+1} \rightarrow 0$  as  $n \rightarrow \infty$ . (This corresponds to the lemma in [3]). Suppose not. Then

$$\liminf_{n \rightarrow \infty} \left| \frac{\alpha_n}{\alpha_{n+1}} \right| = \lambda > 0,$$

and so there exists  $n_0$  such that

$$\left| \frac{\alpha_n}{\alpha_{n+1}} \right| > \lambda/2 \quad \text{if } n \geq n_0.$$

Since

$$\begin{aligned} \|t^r\| &\geq \|t^r e_n\| \geq d(t^r e_n, E_{n+r-1}) = \left| \frac{\alpha_n}{\alpha_{n+r}} \right|, \\ \|t^r\| &\geq \prod_{j=n}^{n+r-1} \left| \frac{\alpha_j}{\alpha_{j+1}} \right|. \end{aligned}$$

Taking  $n = n_0$ , this gives

$$\|t^r\| \geq (\lambda/2)^r \quad (r \geq 1),$$

and so

$$\lim_{r \rightarrow \infty} \|t^r\|^{1/r} \geq \lambda/2 > 0,$$

contradicting the quasi-nilpotence of  $t$ . Therefore we can find a subsequence  $\{j(n)\}$  such that

$$\frac{\alpha_{j(n)}}{\alpha_{j(n)+1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e. such that

$$(5) \quad d(te_{j(n)}, E_{j(n)}) \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Define linear mappings  $t_n: E_n \rightarrow E_n$  ( $n \geq 1$ ) by

$$t_n|_{E_{n-1}} = t|_{E_{n-1}}, \quad t_n(e_n) = u_n ,$$

where  $u_n$  is a nearest point of  $E_n$  to  $te_n$ . We show that

$$(6) \quad \|tx - t_nx\| \leq d(te_n, E_n) \|x\| \quad (x \in E_n, n \geq 1) .$$

Let  $x \in E_n$ . Then  $x = y + \lambda e_n$  for some  $\lambda \in \mathbb{C}$ ,  $y \in E_{n-1}$ .

$$\|tx - t_nx\| = \|\lambda te_n - \lambda u_n\| = |\lambda| d(te_n, E_n) ,$$

and also

$$\|x\| \geq d(x, E_{n-1}) = d(\lambda e_n, E_{n-1}) = |\lambda| d(e_n, E_{n-1}) = |\lambda| .$$

Therefore

$$\|tx - t_nx\| \leq d(te_n, E_n) \|x\| \quad (x \in E_n, n \geq 1) .$$

From (5) and (6) we see that, if  $\{x_n\}$  is a bounded sequence with  $x_n \in E_{j(n)}$ , then

$$(7) \quad \|tx_n - t_{j(n)}x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

From (7) it follows that if  $\{H_{n_k}\}$  is a sequence of subspaces with  $H_{n_k} \subset E_{j(n_k)}$  and  $H_{n_k}$  invariant for  $t_{j(n_k)}$ , then  $\liminf H_{n_k}$  is invariant for  $t$ .

We prove next, by induction on  $k$ , that for each integer  $k$  there is a constant  $A_k$  such that

$$(8) \quad \|t^kx - t_n^kx\| \leq A_k d(te_n, E_n) \|x\| \quad (x \in E_n, n \geq 1) .$$

The case when  $k = 1$  is given by (6), ( $A_1 = 1$ ). Suppose that (8) holds for some  $k$ . Then, for  $x \in E_n$ ,

$$\begin{aligned} \|t_n^kx\| &\leq \|t^kx\| + A_k d(te_n, E_n) \|x\| \\ &\leq (\|t^k\| + A_k d(te_n, E_n)) \|x\| \\ &\leq (\|t^k\| + A_k \|t\|) \|x\| \\ &= B_k \|x\| , \quad \text{say} . \end{aligned}$$

Since  $t_n^k E_n \subset E_n$ , (6) gives

$$\begin{aligned} \|tt_n^kx - t_n^{k+1}x\| &\leq d(te_n, E_n) \|t_n^kx\| \\ &\leq B_k d(te_n, E_n) \|x\| . \end{aligned}$$

Therefore

$$\begin{aligned} \|t^{k+1}x - t_n^{k+1}x\| &\leq \|t^{k+1}x - tt_n^kx\| + \|tt_n^kx - t_n^{k+1}x\| \\ &\leq \|t\| \|t^kx - t_n^kx\| + \|tt_n^kx - t_n^{k+1}x\| \\ &\leq (\|t\| A_k + B_k) d(te_n, E_n) \|x\| . \end{aligned}$$

Hence, by induction, (8) is proved.

It follows immediately from (8) that, given a polynomial  $p(t)$  in  $t$ , there is a constant  $M$  such that

$$\|p(t)x - p(t_n)x\| \leq Md(te_n, E_n)\|x\| \quad (x \in E_n, n \geq 1).$$

Hence we can find positive constants  $\{M_r\}_{r \geq 1}$  such that

$$(9) \quad \|p_r(t)x - p_r(t_n)x\| \leq M_r d(te_n, E_n)\|x\|$$

for  $x \in E_n$ ,  $n \geq 1$ ,  $r \geq 1$ .

Since  $st = ts$  and  $s \neq 0$ , we may assume that  $s^{-1}(0) = (0)$ , for otherwise  $s^{-1}(0)$  is a proper closed invariant subspace for  $t$ . Therefore  $se \neq 0$ , and we can choose  $\alpha$  with  $0 < \alpha < 1$  and  $\alpha\|s\| < \|se\|$ . Choose sequences  $\{E_n^i\}_{i=0}^{j(n)}$  of subspaces of  $E_{j(n)}$  such that

$$(0) = E_n^0 \subset E_n^1 \subset \dots \subset E_n^{j(n)} = E_{j(n)},$$

where  $\dim E_n^i = i$  and  $E_n^i$  is invariant for  $t_{j(n)}$ . Since  $d(e, E_n^0) = \|e\| = 1$  and  $d(e, E_n^{j(n)}) = 0$ , for each  $n$  there is a greatest  $i$ ,  $i_n$  say, such that  $d(e, E_n^{i_n}) > \alpha$ . Put  $F_n = E_n^{i_n}$ ,  $G_n = E_n^{i_n+1}$ . Then

$$d(e, F_n) > \alpha, \quad d(e, G_n) \leq \alpha \quad (n \geq 1),$$

and so

$$(10) \quad e \notin \liminf F_{n_k}$$

for any subsequence  $\{n_k\}$ . Let  $y_n, z_n$  be nearest points of  $G_n$  to  $e, se$  respectively, and let  $v_n \in G_n$  with  $\|v_n\| = 1 = d(v_n, F_n)$ . We can write

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ z_n &= x'_n + \beta'_n v_n, \end{aligned}$$

where  $x_n, x'_n \in F_n$  and  $\beta_n, \beta'_n \in \mathbf{C}$ . We have

$$\begin{aligned} |\beta_n| &= d(\beta_n v_n, F_n) = d(y_n, F_n) \leq \|y_n\| \\ &\leq \|y_n - e\| + \|e\| = d(e, G_n) + \|e\| \leq 2\|e\|. \end{aligned}$$

Similarly

$$|\beta'_n| \leq 2\|se\|.$$

Also, for  $n \geq 1$ ,

$$(11) \quad \begin{aligned} \|sy_n\| &\geq \|se\| - \|sy_n - se\| \geq \|se\| - \|s\|\|y_n - e\| \\ &= \|se\| - \|s\|d(e, G_n) \geq \|se\| - \alpha\|s\| > 0. \end{aligned}$$

By the compactness of  $s$  and the boundedness of  $\{\|y_n\|\}$ ,  $\{|\beta_n|\}$ ,  $\{|\beta'_n|\}$ , we can find a subsequence  $\{n_k\}$  such that

$$\beta_{n_k} \rightarrow \beta, \quad \beta'_{n_k} \rightarrow \beta', \quad sy_{n_k} \rightarrow y \quad \text{as } k \rightarrow \infty.$$

We show that  $y \in \liminf G_{n_k}$ . Let  $\varepsilon > 0$ .  $\exists n_0$  such that

$$\|s - p_{n_0}(t)\| < \frac{\varepsilon}{4\|e\|}.$$

By (5),  $\exists k_0$  such that

$$d(te_{j(n_k)}, E_{j(n_k)}) < \frac{\varepsilon}{4M_{n_0}\|e\|} \quad \text{if } k \geq k_0.$$

Since  $\|y_n\| \leq 2\|e\|$  ( $n \geq 1$ ), by (9)

$$\|p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k}\| \leq M_{n_0}d(te_{j(n_k)}, E_{j(n_k)}) \cdot 2\|e\|$$

for  $k \geq 1$ . Therefore  $k \geq k_0$  implies that

$$\begin{aligned} \|sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k}\| &\leq \|sy_{n_k} - p_{n_0}(t)y_{n_k}\| \\ &\quad + \|p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k}\| \\ &\leq \|s - p_{n_0}(t)\|\|y_{n_k}\| \\ &\quad + 2M_{n_0}\|e\|d(te_{j(n_k)}, E_{j(n_k)}) \\ &\leq \frac{\varepsilon}{4\|e\|} \cdot 2\|e\| + 2M_{n_0}\|e\| \cdot \frac{\varepsilon}{4M_{n_0}\|e\|} = \varepsilon. \end{aligned}$$

Since  $sy_{n_k} \rightarrow y$ ,  $\exists k_1 \geq k_0$  such that  $\|sy_{n_k} - y\| < \varepsilon$  if  $k \geq k_1$ . Thus if  $k \geq k_1$ ,

$$\|y - p_{n_0}(t_{j(n_k)})y_{n_k}\| \leq \|y - sy_{n_k}\| + \|sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k}\| < \varepsilon + \varepsilon = 2\varepsilon.$$

But  $p_{n_0}(t_{j(n_k)})y_{n_k} \in G_{n_k}$  since  $G_{n_k}$  is invariant for  $t_{j(n_k)}$ , and so

$$d(y, G_{n_k}) \leq \|y - p_{n_0}(t_{j(n_k)})y_{n_k}\| < 2\varepsilon \quad \text{if } k \geq k_1.$$

Therefore  $d(y, G_{n_k}) \rightarrow 0$  as  $k \rightarrow \infty$ , and  $y \in \liminf G_{n_k}$ .

Now by (11)  $y \neq 0$ , and so  $\liminf G_{n_k}$  will be a proper closed invariant subspace for  $t$  unless  $\liminf G_{n_k} = X$ . Thus we may suppose that  $\liminf G_{n_k} = X$ , and hence that  $e, se \in \liminf G_{n_k}$ , i.e.

$$d(e, G_{n_k}) = \|e - y_{n_k}\| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and

$$d(se, G_{n_k}) = \|se - z_{n_k}\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore

$$x_{n_k} + \beta_{n_k}v_{n_k} \rightarrow e \quad \text{and} \quad x'_{n_k} + \beta'_{n_k}v_{n_k} \rightarrow se \quad \text{as } k \rightarrow \infty.$$

Hence

$$\beta'_{n_k}x_{n_k} - \beta_{n_k}x'_{n_k} \rightarrow \beta'e - \beta se \quad \text{as } k \rightarrow \infty,$$

and so

$$\beta'e - \beta se \in \liminf F_{n_k}.$$

If  $\beta = 0$  then  $x_{n_k} \rightarrow e$  and  $e \in \liminf F_{n_k}$ , contradicting (10). So  $\beta \neq 0$ . If  $\beta'e - \beta se = 0$  then  $(\beta'/\beta)e = se \neq 0$  and so  $\beta' \neq 0$ . Then  $s \neq (\beta'/\beta)\mathcal{I}$  since  $s$  is compact and  $X$  has infinite dimension ( $\mathcal{I}$  being the identity operator on  $X$ ). Therefore

$$0 \neq e \in \left( s - \frac{\beta'}{\beta}\mathcal{I} \right)^{-1}(0)$$

and  $\{s - (\beta'/\beta)\mathcal{I}\}^{-1}(0)$  is a proper closed invariant subspace for  $t$ . Finally, if  $\beta'e - \beta se \neq 0$  then  $\liminf F_{n_k} \neq (0)$ , and so, by (10),  $\liminf F_{n_k}$  is a proper closed invariant subspace for  $t$ .

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