Pacific Journal of Mathematics

WEAKLY CLOSED DIRECT FACTORS OF SYLOW SUBGROUPS

GEORGE ISAAC GLAUBERMAN AND JOHN GRIGGS THOMPSON

Vol. 26, No. 1 November 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 1, 1968

WEAKLY CLOSED DIRECT FACTORS OF SYLOW SUBGROUPS

G. GLAUBERMAN AND J. G. THOMPSON

In many finite classical linear groups and permutation groups, certain Sylow subgroups have weakly closed direct factors. In this paper we establish a sufficient condition for this to occur in arbitrary finite groups.

The purpose of this paper is to prove the following result:

THEOREM A. *Let p be an odd prime, and let P be a Sylow p-subgroup of a finite group G. Suppose Q and R are subgroups of G such that $P = Q \times R$. Assume that no indecomposable factor of R is isomorphic to a subgroup of Q. Then P contains a weakly closed direct factor that is isomorphic to R.*

Our notation is taken from [3]. In addition, for every finite p-group P, we let

$$d(P) = \max. \{|A| \mid A \text{ is an Abelian subgroup of } P\}$$

and

$$J(P) = \langle A \mid A \text{ an Abelian subgroup of } P \text{ and } |A| = d(P)\rangle .$$

The following lemma is a special case of a result of Wielandt (Satz 6 of [9]).

LEMMA 1. *Let A and B be subgroups of a finite group G such that $G = AB$. Suppose p is a prime, A_p is a normal p-subgroup of A, and B_p is a normal p-subgroup of B. Then $\langle A_p, B_p\rangle$ is a p-group.*

Proof. By Sylow's Theorem, $\langle (A_p)^g, B_p\rangle$ is a p-group for some $g \in G$. Take $a \in A$ and $b \in B$ such that $ab = g$. Then $(A_p)^g = ((A_p)^a)^b = (A_p)^b$. Also, $(B_p)^{b^{-1}} = B_p$. Thus

$$\langle A_p, B_p\rangle = \langle (A_p)^a, (B_p)^{b^{-1}}\rangle = \langle (A_p)^g, B_p\rangle^{b^{-1}} ,$$

which is a p-group.

An automorphism α of a group G is said to be *central* if $g^\alpha g^{-1} \in Z(G)$ for all $g \in G$. We say that an element (or a subgroup) of Aut G *fixes* a subgroup H of G if it (or its elements) map H onto H.

THEOREM 1. *Let π be a set of primes and G be a finite π-group.*

Suppose $G = H \times K$ and no indecomposable factor of H is isomorphic to an indecomposable factor of K. Let $A = \operatorname{Aut} G$ and let C be the group of central automorphisms of G. Then G has the following properties:

(a) *If $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$, then $G = H^* \times K = H \times K^*$.*

(b) *The groups $H \times Z(K)$, $Z(H) \times K$, H', and K' are characteristic subgroups of G.*

(c) *There exists a normal, nilpotent π-subgroup D of A that is contained in C and permutes transitively the pairs (H^*, K^*) such that*

$$H^* \cong H,\ K^* \cong K,\ \text{and}\ G = H^* \times K^* .$$

(d) *If B is a π'-subgroup of A then there exists a pair (H^*, K^*) such that*

$$H^* \cong H, K^* \cong K, G = H^* \times K^* ,$$

and B fixes H^* and K^*. Moreover, if B fixes H, we may take $H^* = H$.

Proof. (a) Represent H and K as products of indecomposable factors, say, $H = H_1 \times \cdots \times H_r$ and $K = K_1 \times \cdots \times K_s$. Then $G = H \times K = H_1 \times \cdots \times H_r \times K_1 \times \cdots \times K_s$. Since $H^* \cong H$ and $K^* \cong K$, we have a similar representation

$$G = H^* \times K^* = H_1^* \times \cdots \times H_r^* \times K_1^* \times \cdots \times K_s^* .$$

Obviously, there exists a one-to-one correspondence ϕ between the factors F of the first representation and those of the second representation. By the Krull-Schmidt Theorem [7, p. 81], ϕ may be chosen to have the properties that $\phi(F) \cong F$ for each F and

$$G = \phi(H_1) \times \cdots \times \phi(H_r) \times K_1 \times \cdots \times K_s .$$

Clearly, for every H_i, $\phi(H_i)$ is some H_j^*. Hence $G = H^* \times K$. By symmetry, $G = H \times K^*$.

(b) Let $\alpha \in A$. Then $G = H^\alpha \times K^\alpha$. By (a), $G = H^\alpha \times K$. Thus

$$(C(K))^\alpha = (H \times Z(K))^\alpha \subseteqq H^\alpha Z(G) \subseteqq C(K) .$$

Hence $H \times Z(K)$ is a characteristic subgroup of G. Since $H' = (H \times Z(K))'$, H' is also a characteristic subgroup of G. By symmetry, $Z(H) \times K$ and K' are characteristic in G.

(c) For each $\alpha \in C$, define $\alpha - 1$ by $g^{\alpha-1} = g^{-1}g^\alpha$ for all $g \in G$. Since $\alpha \in C$, $\alpha - 1$ is an endomorphism of G and $G^{\alpha-1} \subseteqq Z(G)$. Thus $g^{\alpha-1} = g^\alpha g^{-1}$ for all $g \in G$.

Let D_H be the group of all $\alpha \in C$ for which $g^\alpha = g$ for all $g \in H$ and $g^{\alpha-1} \in \boldsymbol{Z}(H)$ for all $g \in G$. Then

$$(1) \qquad H^{\alpha-1} = 1 \quad \text{and} \quad G^{\alpha-1} \subseteqq \boldsymbol{Z}(H), \quad \text{for} \quad \alpha \in D_H .$$

Define D_K similarly.

Suppose $\alpha \in D_H$. Let $\eta = \alpha - 1$. Take $g \in G$, and let $h = g^\eta$. By (1), it is clear by induction that

$$g^{\alpha^i} = gh^i \quad \text{for} \quad i = 1, 2, 3, \cdots .$$

Thus
(2) the order of α, the exponent of $G^{\alpha-1}$, and the exponent of $G/\mathrm{Ker}\,(\alpha - 1)$ are equal.
We also observe from (1) that if $\alpha, \beta \in D_H$, then $\alpha\beta = \beta\alpha$. Thus
(3) D_H is an Abelian π-group.

Suppose $\alpha \in D_H$, $\beta \in D_K$, and α and β have relatively prime orders. Let $g \in G$, and let $h = g^{\alpha-1}$ and $k = g^{\beta-1}$. Then $h \in \boldsymbol{Z}(H)$ and $k \in \boldsymbol{Z}(K)$. By (2), the order of h divides the order of α. Since an analogue of (2) also holds for elements of D_K, $h \in \mathrm{Ker}\,(\beta - 1)$. Similarly, $k \in \mathrm{Ker}\,(\alpha - 1)$. Hence

$$g^{\alpha\beta} = (g^\alpha)^\beta = (gh)^\beta = g^\beta h^\beta = g^\beta h = gkh = ghk$$

and

$$g^{\beta\alpha} = (g^\beta)^\alpha = (gk)^\alpha = g^\alpha k^\alpha = g^\alpha k = ghk = g^{\alpha\beta} .$$

Thus $\alpha\beta = \beta\alpha$. In particular, if p and q are distinct primes,
(4) the Sylow p-subgroup of D_H centralizes the Sylow q-subgroup of D_K.

Suppose $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$. By (a),

$$G = H \times K = H \times K^* = H^* \times K .$$

Define a mapping $\eta\colon G \to G$ as follows: For each $k \in K$, take $h' \in H$ and $k^* \in K^*$ such that $k = h'k^*$. Let $k^\eta = h'$. For $h \in H$ and $k \in K$, let

$$(hk)^\eta = k^\eta .$$

Then η is an endomorphism of G. Since K and K^* centralize H, $G^\eta = K^\eta \subseteqq \boldsymbol{Z}(H) \subseteqq \boldsymbol{Z}(G)$. Hence the mapping $\alpha\colon G \to G$ given by $g^\alpha = (g^\eta)^{-1}g$ is an endomorphism of G. Since $H^\alpha = H$ and $K^\alpha = K^*$, α is an automorphism of G. Clearly, $\alpha \in D_H$. Thus D_H permutes transitively all the direct factors of G that are isomorphic to K. Similarly D_K permutes transitively all the direct factors of G that are isomorphic to H.

Let A_H be the set of all $\alpha \in A$ such that $H^\alpha = H$. Define A_K similarly. Then

$$D_H \lhd A_H \quad \text{and} \quad D_K \lhd A_K \,. \tag{5}$$

Let $\alpha \in A$. Then $H^\alpha \cong H$, $K^\alpha \cong K$, and $G = H^\alpha \times K^\alpha$. Hence there exists $\beta \in D_H$ such that $K^\beta = K^\alpha$. Therefore $K^{\alpha\beta^{-1}} = K$, and $\alpha\beta^{-1} \in A_K$. Thus $\alpha \in A_K A_H$. So

$$A = A_K A_H = A_H A_K \,. \tag{6}$$

Let $I = A_H \cap A_K$, and take $\alpha \in A_H$. As in the previous paragraph, there exists $\beta \in D_H$ such that $K^\alpha = K^\beta$. Thus $\alpha\beta^{-1} \in A_H \cap A_K = I$. So $A_H = ID_H = D_H I$. Similarly, $A_K = ID_K = D_K I$.

Let p be a prime. By (5), $O_p(D_H)$ is a normal subgroup of A_H and $O_p(D_K)$ is a normal subgroup of A_K. Let $D_p = \langle O_p(D_H), O_p(D_K)\rangle$. By (5), (6), and Lemma 1, D_p is a p-group. By (3) and (4), every p'-element in D_H or D_K centralizes D_p. Since D_p normalizes itself, D_H and D_K normalize D_p. Since I normalizes D_H and D_K, I normalizes D_p. Hence

$$N(D_p) \supseteq \langle D_H, D_K, I\rangle = \langle D_H I, D_K I\rangle = A_H A_K = A \,.$$

Let D be the subgroup of C generated by the groups D_p for all primes p. Then $D_H \subseteq D$ and $D_K \subseteq D$, by (3). Suppose $H^* \cong H$, $K^* \cong K$, and $G = H^* \times K^*$. Then there exists $\alpha \in D_K$ and $\beta \in D_H$ such that $H^{*\alpha} = H$ and $((K^*)^\alpha)^\beta = K$. Now $\alpha\beta \in D$, $H^{*\alpha\beta} = H$, and $K^{*\alpha\beta} = K$. This completes the proof of (c).

(d) Retain the notation of (c). Then $I = A_H \cap A_K$ and $A = ID$. Since $D \subseteq BD \subseteq A = ID$, $BD = (BD \cap I)D$. Note that D is nilpotent and $|B|$ and $|D|$ are relatively prime. By Schur's Theorem [10, p. 162], $BD \cap I$ splits over $D \cap I$. Let B^* be a complement of $D \cap I$ in $BD \cap I$. Thus B^* is a complement of D in BD. By the Schur-Zassenhaus Theorem [10, p. 162], B^* is conjugate to B in BD. Take $\alpha \in BD$ such that $B = \alpha^{-1}B^*\alpha$. Since $B^* \subseteq A_H \cap A_K$, B fixes H^α and K^α.

If B fixes H, then $B \subseteq A_H = ID_H$. An argument similar to the previous one shows that $\alpha B\alpha^{-1} \subseteq I$ for some $\alpha \in BD_H$. Then B fixes H^α and K^α, and $H^\alpha = H$. This completes the proof of Theorem 1.

LEMMA 2. *Let p be a prime and P be a p-subgroup of a finite group G. Suppose H is a p'-subgroup of G that normalizes P. Then:*

(a) $P = [P, H]C_P(H)$;

(b) $[[P, H], H] = [P, H]$; *and*

(c) *if P is Abelian, then* $P = [P, H] \times C_P(H)$.

Proof. This result is well known. Parts (a) and (b) appear as Corollary 3 of Theorem 1 of [4]. Part (c) follows directly from part (a) and from the lemma on page 172 of [10].

LEMMA 3. *Let p be a prime and P be a p-subgroup of a finite group G. Suppose H is a p'-subgroup that normalizes P. Assume that*

(a) *P is Abelian and H centralizes $\Omega_1(P)$*
or that

(b) *P has no Abelian direct factors and H centralizes $P/Z(P)$. Then H centralizes P.*

Proof. (a) By Lemma 2, $P=[P,H]\times C_P(H)$. Hence $\Omega_1([P,H])=1$. Therefore, $[P,H]=1$, i.e., H centralizes P.

(b) Let $Q=[P,H]$. Then $Q\subseteqq Z(P)$, so Q is Abelian. By Lemma 2, $P=QC_P(H), Q=[Q,H]$, and $Q\cap C_P(H)=[Q,H]\cap C_Q(H)=1$. Since $Q\subseteqq Z(P), C_P(H)\lhd P$. Hence $P=Q\times C_P(H)$. By (b), $Q=1$.

LEMMA 4. *Let P and Q be normal Abelian p-subgroups of a finite group G. Suppose that $Q\subseteqq P$ and that some Sylow p-subgroup of G normalizes some complement of Q in P. Then G normalizes some complement R of Q in P.*

Proof. By constructing a semi-direct product if necessary, we may assume that G is a splitting extension of P by a group E that is isomorphic to $G/C(P)$. Let S be a Sylow p-subgroup of E. Then S normalizes some complement R^* of O in P. Now, SP is a Sylow p-subgroup of G and SR^* is a complement of Q in SP. Thus SP splits over Q. By a theorem of Gaschütz [6, p. 246], G splits over Q. Let C be a complement of Q in G, and let $R=C\cap P$.

The following result is a special case of a theorem of Wielandt (Satz 12, page 193, of [8]).

LEMMA 5. *Suppose p is a prime and P is a Sylow p-subgroup of a finite group G. Let $n=|N(P)/P|$. Let V be the transfer of G into P/P'.*

(a) *If $a\in P\cap Z(N(P))$ and $a^p=1$, then $V(a)=a^nP'$.*

Furthermore, suppose $P'\subseteqq Q\subseteqq P$ and suppose W is the transfer of G into P/Q. Then:

(b) *If $A\subseteqq P\cap Z(N(P))$ and $A\cap Q=1$, then $A\cap G'=A\cap$* Ker $W=1$.

(c) *If $Q\lhd N(P)$, then $\Omega_1(Q\cap Z(P))\subseteqq$* Ker W.

Proof. (a) Let $r=|G:P|$, and let $Px_i, i=1,2,\cdots,r$, be the distinct cosets of P in G. We may assume that

$$x_1,\cdots,x_n\in N(P);\ Px_ia=Px_i(1\leqq i\leqq s);$$
$$Px_ia\neq Px_i(s+1\leqq i\leqq r)\ ,$$

where $s \geqq n$. Since $a^p = 1$, Lemma 14.4.1, page 206, of [6] yields

$$V(a) = P' \prod_{i \leqq i \leqq s} x_i a x_i^{-1} .$$

Since $a \in \boldsymbol{Z}(\boldsymbol{N}(P))$,

$$V(a) = P'a^n \prod_{n < i \leqq s} x_i a x_i^{-1} . \tag{7}$$

Suppose $x \in P$ and $n < i \leqq s$. Then $(Px_i)x = Px_j$ for some j. Since

$$Px_j a = Px_i x a = Px_i a x = Px_i x = Px_j$$

and since $x_i \notin \boldsymbol{N}(P)$, $n < j \leqq s$. Thus P permutes the cosets Px_i, $n < i \leqq s$, by right multiplication. We may assume that $Px_{n+1}, \cdots, Px_t$ are representatives of the distinct orbits of P. For $i = n + 1, \cdots, t$, let P_i be the subgroup of P fixing Px_i, and let $y_{i1}, \cdots, y_{im_i}$ be representatives of the distinct left cosets of P_i in P. Then the orbit of Px_i is $Px_i y_{ij}$, $1 \leqq j \leqq m_i$.

Suppose $n + 1 \leqq i \leqq t$. Since $x_i \notin \boldsymbol{N}(P)$, $Px_i P \neq Px_i$ Thus $P_i \subset P$ and

$$m_i \equiv |P : P_i| \equiv 0, \quad \text{modulo} \quad p . \tag{8}$$

We may assume that, for $k = n + 1, \cdots, s$, every x_k has the form $x_i y_{ij}$ for some (unique) i and j. By (7) and (8),

$$\begin{aligned} V(a) &= P'a^n \prod_{n < i \leqq t} \prod_{1 \leqq j \leqq m_i} x_i y_{ij} a y_{ij}^{-1} x_i^{-1} \\ &= P'a^n \prod_{n < i \leqq t} (x_i a x_i^{-1})^{m_i} = P'a^n , \end{aligned}$$

as desired.

(b) Suppose $a \in A$ and $a^p = 1$. Now, W is simply the composition of V with the natural mapping of P/P' into P/Q. Hence $W(a) = a^n Q$, by (a). Since p does not divide n and since $a \notin Q$, $W(a) \neq Q$. Thus $A \cap \operatorname{Ker} W$ has no elements of order p, so $A \cap \operatorname{Ker} W = 1$. Since $G' \subseteqq \operatorname{Ker} W$, $A \cap G' = 1$.

(c) Let $B = \Omega_1(Q \cap \boldsymbol{Z}(P))$ and $N = \boldsymbol{N}(P)$. Since $N/\boldsymbol{C}_N(B)$ is a p'-group,

$$B = [B, N] \times \boldsymbol{C}_B(N) ,$$

by Lemma 2. Obviously, $[B, N] \subseteqq G' \subseteqq \operatorname{Ker} W$. Let $a \in \boldsymbol{C}_B(N)$. From (a),

$$W(a) = (a^n P')Q = a^n Q = Q ,$$

so $a \in \operatorname{Ker} W$. Thus $B \subseteqq \operatorname{Ker} W$. This completes the proof of Lemma 5.

We now require the following proposition, which is the main result of [5]:

THEOREM 2. *Let p be an odd prime, and let P be a Sylow p-subgroup of a finite group G. Suppose $x \in P \cap \boldsymbol{Z}(N(\boldsymbol{J}(P)))$. Then $g^{-1}xg = x$ whenever $g \in G$ and $g^{-1}xg \in P$.*

THEOREM 3. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G. Suppose Q and R are normal subgroups of $N(P)$ and $P = Q \times R$. Assume that $R \subseteqq \boldsymbol{O}_p(G)$ and that no indecomposable direct factor of R is isomorphic to a subgroup of Q. Then R' is a normal subgroup of G, and there exists a normal subgroup R^* of G such that $P = Q \times R^*$. Moreover, if p is odd and R/R' is a normal subgroup of $N_{G/R'}(\boldsymbol{J}(P/R'))$, we may take $R^* = R$.*

Proof. Let $Q_1 = \boldsymbol{O}_p(G) \cap Q$. Since $R \subseteqq \boldsymbol{O}_p(G) \subseteqq P = R \times Q$, $\boldsymbol{O}_p(G) = R \times Q_1$. Now, no indecomposable factor of R is isomorphic to an indecomposable factor of Q_1. By Theorem 1, $R\boldsymbol{Z}(Q_1)$ and R' are characteristic subgroups of $\boldsymbol{O}_p(G)$ and are therefore normal subgroups of G.

Let $T = R\boldsymbol{Z}(Q_1) = \boldsymbol{Z}(Q_1) \times R$. Represent R as a direct product of an Abelian subgroup R_a and a subgroup R_b having no Abelian direct factors. By Theorem 1, we may assume that R_a and R_b are normalized by a complement of P in $N(P)$ and are therefore normal in $N(P)$. If $R_a \neq 1$, let p^e be the minimum of the exponents of the indecomposable factors of R_a. If $R_a = 1$, let $p^e = p|T|$. Then let

$$T_0 = \langle x^{p^{e-1}} \mid x \in T \rangle .$$

Now $T_0 \lhd G$ and

$$\Omega_1(R_a) \subseteqq T_0 \subseteqq R . \tag{9}$$

Since Q centralizes R, Q centralizes T_0 and $T/\boldsymbol{Z}(T)$. Let

$$C = \boldsymbol{C}_G(T/\boldsymbol{Z}(T)) \cap \boldsymbol{C}_G(T_0) \quad \text{and} \quad H = CT .$$

Then C and H are normal in G and $P = QR \subseteqq CT = H$.

Let K be a complement of P in $N_H(P)$. Since $H/C \cong T/(C \cap T)$, $K \subseteqq C$. Thus $[T, K] \subseteqq \boldsymbol{Z}(T)$ and K centralizes T_0. Therefore $[R_b, K] \subseteqq \boldsymbol{Z}(R_b)$ and, by (9), K centralizes $\Omega_1(R_a)$. By Lemma 3, K centralizes R_a and R_b. So K centralizes R.

Let $\bar{H} = H/R'$, $\bar{R} = R/R'$, $\bar{K} = KR/\bar{R}'$, and so forth. Then $\bar{R} \subseteqq \boldsymbol{Z}(\bar{P})$ and $N_{\bar{H}}(\bar{P}) = \bar{P}\bar{K}$, so

$$N_{\bar{H}}(\bar{P}) \quad \text{centralizes } \bar{R} . \tag{10}$$

Let W be the transfer of $\bar{H}$ into $\bar{P}/\bar{Q}$. By Lemma 5(b),

(11) $$\bar{R} \cap \bar{H}' \subseteqq \bar{R} \cap \operatorname{Ker} W = 1 .$$

By the Frattini argument,

(12) $$G = HN(P) .$$

Suppose p is odd and $\bar{R} \lhd N_{\bar{G}}(J(\bar{P}))$. Then by (11)

$$[\bar{R}, N_{\bar{H}}(J(\bar{P}))] \subseteqq \bar{R} \cap \bar{H}' = 1 .$$

Thus by Theorem 2 no element of $\bar{R}$ is conjugate to any other element of $\bar{P}$. Since $\bar{R} \subseteqq O_p(\bar{G}) \subseteqq \bar{P}$, we must have $\bar{R} \subseteqq Z(\bar{H})$. Therefore, $R \lhd H$. By (12) R is normal in G, as claimed.

Let us return to the general case. Now, $\bar{P} = \bar{Q} \times \bar{R}$. By (11), $\bar{R} \cap \operatorname{Ker} W = 1$. Since

$$|\operatorname{Image}(W)| \leqq |\bar{P}/\bar{Q}| = |\bar{R}| ,$$

$\bar{R}$ is a complement to $\operatorname{Ker} W$ in $\bar{H}$. Hence $\bar{R}$ is a complement to $\bar{T} \cap \operatorname{Ker} W$ in $\bar{T}$. Since W depends only on $\bar{H}$ and $\bar{Q}$ and since $N(P)$ normalizes H and Q, $N(P)$ normalizes $\operatorname{Ker} W$. By (12), $\bar{G}$ normalizes $\operatorname{Ker} W$. Hence $\bar{T} \cap \operatorname{Ker} W \lhd \bar{G}$. Now $\bar{T}' = \bar{R}' = 1$ and $\bar{P}$ normalizes $\bar{R}$. By Lemma 4, there exists a complement $\bar{R}^*$ of $\bar{T} \cap \operatorname{Ker} W$ in $\bar{T}$ such that $\bar{R}^* \lhd \bar{G}$. Let R^* be the subgroup of T that contains R' and satisfies $R^*/R' = \bar{R}^*$.

By Lemma 5, $\Omega_1(Z(\bar{Q})) \subseteqq \operatorname{Ker} W$. Since $\Omega_1(Z(Q))R'/R' \subseteqq \Omega_1(Z(\bar{Q}))$, (11) yields

$$\Omega_1(Z(Q)) \cap R^* \subseteqq \Omega_1(Z(Q)) \cap R' \subseteqq Q \cap R = 1 .$$

Hence $Q \cap R^*$ is normal in Q but intersects $Z(Q)$ in 1, so $Q \cap R^* = 1$. Consequently, $|QR^*| = |Q||R^*| = |Q||R| = |P|$. Since $Q, R^* \lhd P$, $P = Q \times R^*$. This completes the proof of Theorem 3.

We now require the following concepts and results of Alperin and Gorenstein (§ 2 of [2] and § 5 of [1]):

DEFINITION. Let G be a finite group and p be a prime. Let $\mathscr{H}$ be the set of all nonidentity p-subgroups of G. A *conjugacy functor* $\boldsymbol{W}$ on $\mathscr{H}$ is a mapping from $\mathscr{H}$ into $\mathscr{H}$ that satisfies the following two conditions for each H in $\mathscr{H}$:

(a) $\boldsymbol{W}(H) \subseteqq H$;

(b) $\boldsymbol{W}(H^x) = \boldsymbol{W}(H)^x$ for all $x \in G$.

THEOREM 4. *Let p be a prime and P be a nonidentity Sylow p-subgroup of a finite group G. Let $\boldsymbol{W}$ be a conjugacy functor on the set of nonidentity p-subgroups of G. Then there exists a class* *of nonidentity subgroups of P, called well-placed subgroups, having the following properties:*

(1) *If H is a well-placed subgroup then $N(H) \cap P$ is a Sylow p-subgroup of $N(H)$, and $W(N(H) \cap P)$ is a well-placed subgroup.*

(2) *Suppose $R \subseteqq P, g \in G$, and $R^g \subseteqq P$. Then there exists a sequence of well-placed subgroups $H_1, \cdots, H_n$ and elements $x_1, \cdots, x_n$ of G such that*

(a) $g = x_1 \cdots x_n$,

(b) $x_i \in N(H_i), 1 \leqq i \leqq n$, *and*

(c) $R \subseteqq H_1$ *and* $R^{x_1 \cdots x_i} \subseteqq H_{i+1}, 1 \leqq i \leqq n-1$.

Theorem 4 easily yields the following result:

COROLLARY. *Let p be a prime and P be a Sylow p-subgroup of a finite group G. Suppose $Q \subseteqq P$ and Q is not weakly closed in P with respect to G. Then there exists $H \subseteqq P$ and $g \in N(H)$ such that H is well-placed, $Q \subseteqq H$, and $Q^g \neq Q$.*

THEOREM 5. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G. Suppose $P = Q \times R$ and no indecomposable direct factor of R is isomorphic to a subgroup of Q. Let J be the subgroup of P that contains R' and satisfies $J/R' = J(P/R')$. Then*

(a) *There exists $R^* \lhd N(J)$ such that $P = Q \times R^*$.*

(b) *If p is odd and R^* satisfies* (a), *R^* is weakly closed in P with respect to G.*

Proof. (a) Let K be a complement of P in $N(P)$. By Theorem 1, we may assume that K normalizes Q and R. Hence $Q, R \lhd N(P)$. Since $R/R' \subseteqq Z(P/R')$,

$$R \subseteqq J \subseteqq O_p(N(J)) .$$

Thus, (a) follows from Theorem 3.

(b) Assume p is odd and R^* satisfies (a) but is not weakly closed in P. We may assume that $R = R^*$. By a theorem of Burnside [6, p. 46], there exists a subgroup P_0 of P such that $P_0 \supseteqq R$ and $R \ntriangleleft N(P_0)$. Since

$$R \subseteqq P_0 \subseteqq P = R \times Q, \qquad P_0 = R \times (P_0 \cap Q) .$$

By Theorem 1 and our hypothesis on Q and on R, $R' \lhd N(P_0)$. Therefore, R is not weakly closed in P with respect to $N(R')$. Since $P \subseteqq N(J) \subseteqq N(R')$, we may assume that $R' \lhd G$.

We define a conjugacy functor W on the set of nonidentity subgroups H of G as follows:

$$W(H) = H, \quad \text{if} \quad R' \not\subseteq H\,;$$

and

$$R' \subseteq W(H) \quad \text{and} \quad W(H)/R' = J(H/R'), \quad \text{if} \quad R' \subseteq H\,.$$

By the Corollary of Theorem 4, there exists a well-placed subgroup H of G having the properties that $H \supseteq R$ and $R \ntriangleleft N(H)$. Choose H such that $P \cap N(H)$ has maximal order subject to these conditions. Let $P_1 = P \cap N(H)$. Since H is well-placed, P_1 is a Sylow p-subgroup of $N(H)$. By Theorem 3, $R/R' \ntriangleleft N_{G/R'}(J(P_1/R'))$. Hence $P_1 \subset P$ by (a). But $J(P_1/R') = W(P_1)/R'$. Thus $R \subseteq P_1$ and $R \ntriangleleft N(W(P_1))$. Since H is well placed and $P_1 \subset P$, $W(P_1)$ is well placed and

$$P_1 \subset P \cap N(P_1) \subseteq P \cap N(W(P_1))\,.$$

But this contradicts the choice of H. Thus we have proved Theorem 5. Theorem A obviously follows from Theorem 5.

REMARK. Let A^n and S^n be the alternating and symmetric groups of degree n, for $n = 4, 6$. Since Theorem 2 holds for $p = 2$ when S^4 is not involved in G [5], Theorem A holds for $p = 2$ when S^4 is not involved in $N(R')/R'$.

Let $H = A^6$, and let R be an indecomposable 2-group of order greater than eight. Take a transposition τ in S^6 and a subgroup R_0 of index two in R. Consider R as an operator group on H by defining $h^r = h$ when $r \in R_0$ and $h^r = \tau^{-1}h\tau$ when $r \in R$ and $r \notin R_0$. Let G be the semi-direct product of H by R, and embed H and R in G in the natural manner. Then $C_H(R)$ contains a Sylow 2-subgroup Q of H. Let $P = Q \times R$. Then P is a Sylow 2-subgroup of G and R is not isomorphic to any subgroup of Q, but P has no weakly closed direct factor isomorphic to R.

REFERENCES

1. J. Alperin, *Sylow intersections and fusion*, J. Algebra **6** (1967), 222-241.

2. J. Alperin and D. Gorenstein, *Transfer and fusion in finite groups*, J. Algebra **6** (1967), 242-255.

3. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775-1029.

4. G. Glauberman, *Fixed points in groups with operator groups*, Math. Z. **84** (1964), 120-125.

5. ———, *Weakly closed elements of Sylow subgroups*, Math. Z. (to appear).

6. M. Hall, *The Theory of Groups*, Macmillan, New York, 1959.

7. J. Rotman, *The Theory of Groups: An Introduction*, Allyn and Bacon, Boston, 1965.

8. H. Wielandt, *p-Sylowgruppen and p-Faktorgruppen*, J. reine angew. Math. **182** (1940), 180-193.

9. ———, *Uber das Product paarweise vertauschbarer nilpotenter Gruppen*, Math. Z. **55** (1951), 1–7.
10. H. Zassenhaus, *The Theory of Groups*, Second English Edition, Chelsea, New York, 1958.

Received May 12, 1967.

UNIVERSITY OF CHICAGO

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 26, No. 1 November, 1968