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For $m \geqq k$, an (m, k) *system* is a set of k-tuples (k-subsets) of $1, 2, \cdots, m$. A *minimal* (m, k) *system* is an (m, k) system with the property that every $(k-1)$-tuple of the m elements appears in at least one k-tuple of the system, but no system with fewer k-tuples has this property. The numbers of k-tuples in a minimal (m, k) system will be denoted by $N_k(m)$. A *maximal* (m, k) is an (m, k) system with the property that no $(k-1)$-tuple appears in more than one k-tuple of the system, but no system with more k-tuples has this property. The number of k-tuples in a maximal (m, k) system is $D_k(m)$. In this paper we shall be concerned with evaluating N_k and D_k and investigating the properties of extremal (m, k) systems for $k = 2, 3$, and 4.

In § 2 we dispose of the trivial case $k = 2$, and give inductive upper and lower bounds for D_k and N_k. In § 3 and § 4 we discuss systems of triples and quadruples.

2. Bounds for N_k and D_k. It is easy to see that

$$D_2(2m) = N_2(2m) = D_2(2m+1) = N_2(2m-1) = m .$$

In the first three cases, the m pairs

$$(1, 2), (3, 4), \cdots, (2m-1, 2m)$$

form extremal systems. In the fourth case, replace the last pair by $(2m-1, 1)$.

Theorems 1 and 2 below provide inductive bounds for N_k and D_k with $k \geqq 3$. The bounds in Theorem 1 are best possible in most of the cases we consider, but those of Theorem 2 are not usually as good.

THEOREM 1. $D_k(m) \leqq mD_{k-1}(m-1)/k$; $N_k(m) \geqq mN_{k-1}(m-1)/k$.

Proof. Consider the k-tuples $(a_1, a_2, \cdots, a_k)$ which contain a specified element a_1 in a maximal (m, k) system. Since no $(k-1)$-tuple is repeated, the $(k-1)$-tuples $(a_2, \cdots, a_k)$ contain no repeated $(k-2)$-tuples. Thus a_1 occurs in at most $D_{k-1}(m-1)$ k-tuples. There are m elements in all, and they appear k to a block. Thus a maximal (m, k) system contains at most $mD_{k-1}(m-1)/k$ blocks. Similarly in a minimal (m, k) system each element occurs in at least $N_{k-1}(m-1)$ blocks, and the theorem follows.

THEOREM 2. $D_k(m) \geqq D_k(m+1) - D_{k-1}(m)$; $N_k(m) \leqq N_k(m-1) + N_{k-1}(m-1)$.

Proof. Let S_1 be a mininal $(m-1, k)$ system, and let T_1 be a minimal $(m-1, k-1)$ system. We add to S_1 a new element x and k-tuples $(x, a_1, \cdots, a_{k-1})$ where $(a_1, \cdots, a_{k-1})$ runs through all $(k-1)$-tuples of T_1. The resulting (m, k) system contains $N_k(m-1) + N_{k-1}(m-1)$ blocks, and every $(k-1)$-tuple appears in at least one block. Thus $N_k(m) \leqq N_k(m-1) + N_{k-1}(m-1)$.

Similarly, if S_2 is a maximal $(m+1, k)$ system, delete all blocks of S_2 containing a specified element. By the argument in Theorem 1, an element occurs in at most $D_{k-1}(m)$ blocks of S_2, and there remain at least $D_k(m+1) - D_{k-1}(m)$ blocks with no repeated $(k-1)$-tuple. Thus $D_k(m) \geqq D_{k+1}(m+1) - D_{k-1}(m)$.

If $N_k(m) = D_k(m)$, an extremal (m, k) system has the property that every $(k-1)$-tuple belongs to exactly one k-tuple of the system. Then the k-tuples form a Steiner system $S(k-1, k, m)$. Properties of Steiner triple and quadruple systems are useful in discussing cases $k = 3$ and $k = 4$ in the next two sections.

3. Systems of triples.

For $k = 3$, the bounds given by Theorem 1 are the best possible in all but one case which is considered in Lemma 1. Fort and Hedlund [2] have evaluated $N_3(m)$, and their results are summarized in Theorem 3. $D_3(m)$ is evaluated in Theorem 4, and some properties of maximal $(m, 3)$ systems are obtained.

LEMMA 1. $D_3(m) \leqq (m^2 - m - 8)/6$ *for* $m \equiv 5 (\mathrm{mod}\ 6)$.

Proof. Suppose that $D_3(m) = (m^2 - m - 2)/6$, the largest value possible by Theorem 1. Since $\binom{m}{2} - 3(m^2 - m - 2)/6 = 1$ and no pair of elements appears in two triples, exactly one pair $(1, 2)$ does not occur in any triple of a maximal system. Then 1 occurs in triples with 3, $4 \cdots, m$, and for m odd this requires at least $(m-1)/2$ triples. Thus some pair $(1, x)$ with $x \geqq 3$ will be repeated. This is a contradiction, and consequently $D_3(m) \leqq (m^2 - m - 8)/6$.

THEOREM 3. (*Fort and Hedlund*).

$$N_3(m) = \begin{cases} m^2/6 & \text{if } m = 6k \\ m(m-1)/6 & \text{if } m = 6k+1 \text{ or } 6k+3 \\ (m^2+2)/6 & \text{if } m = 6k+2 \text{ or } 6k+4 \\ (m^2 - m + 4)/6 & \text{if } m = 6k+5\,. \end{cases}$$

Furthermore, minimal $(m, 3)$ *systems have the following properties*:

(i) *If* $m = 6k$, $3k$ *disjoint pairs* $(1, 2), (3, 4), \cdots, (6k-1, 6k)$ *occur twice and every other pair belongs to just one triple.*

(ii) *If* $m = 6k+2$ *or* $6k+4$, $m/2+1$ *pairs* $(1, 2), (1, 3), (1, 4), (5, 6), (7, 8), \cdots, (m-1, m)$ *occur twice, and every other pair belongs to just one triple.*

(iii) *If* $m = 6k+5$, *one pair occurs in three triples, and every other pair belongs to just one triple.*

Reiss [4] has proved that Steiner triple systems $S(2, 3, m)$ exist for $m = 6k+1$ or $6k+3$, and then $N_3(m) = m(m-1)/6$. Theorems 1 and 2 now give $N_3(m) = (m^2+2)/6$ for $m = 6k+2$ or $6k+4$. Of the two cases remaining, $m = 6k+5$ seems to be the more difficult; a construction for a minimal $(m, 3)$ system is given in [2]. Fort and Hedlund construct a minimal $(m, 3)$ system for $m \equiv 0 \pmod 6$ by modifying a minimal $(m-1, 3)$ system. It is perhaps worth nothing that a minimal $(m, 3)$ system may also be obtained from a Steiner triple system $S(2, 3, m-3)$. Adding modulo $2t+1$ to the $3t+1$ initial blocks

$$(i_1, (2t+1-i)_1, 0_2), (i_2, (2t+1-i)_2, 0_3), (i_3, (2t+1-i)_3, 0_1) \qquad (i = 1, 2, \cdots, t)$$

$$(0_1, 0_2, 0_3)$$

gives, by Bose's first module theorem [1], the $(3t+1)(2t+1)$ blocks of a triple system on $6t+3$ elements. The blocks generated by $(0_1, 0_2, 0_3)$ are disjoint and contain every element once. Delete these blocks from the system; add new elements x_1, x_2, x_3 and triples

$$(x_1, i_2, i_3), (i_1, x_2, i_3), (i_1, i_2, x_3) \qquad i = 0, 1, \cdots, 2t$$

$$(x_j, (2i)_j, (2i+1)_j) \qquad i = 0, 1, \cdots, t-1;\ j = 1, 2, 3$$

$$(x_2, x_3, (2t)_3), (x_1, x_2, (2t)_2), (x_3, x_1, (2t)_1)\ .$$

Every pair occurs at least once, and, the numer of triples is

$$(3t+1)(2t+1) - (2t+1) + 3(2t+1) + 3t + 3 = m^2/6$$

where $m = 6t+6$. Thus $N_3(m) = m^2/6$ for $m = 6k$.

THEOREM 4.

$$D_3(m) = \begin{cases} m(m-2)/6 & \textit{if } m = 6k \textit{ or } 6k+2 \\ m(m-1)/6 & \textit{if } m = 6k+1 \text{ or } 6k+3 \\ (m^2-2m-2)/6 & \textit{if } m = 6k+4 \\ (m^2-m-8)/6 & \textit{if } m = 6k+5\ . \end{cases}$$

Furthermore, maximal $(m, 3)$ systems have the following properties:

(i) *If $m = 6k$ or $6k + 2$, $m/2$ pairs $(1, 2), (3, 4), \cdots, (m - 1, m)$ do not occur in any triple, and every other pair appears once.*

(ii) *If $m = 6k + 4$, $3k + 3$ pairs $(1, 2), (1, 3), (1, 4), (5, 6), (7, 8), \cdots, (m - 1, m)$ do not appear in any triple, and every other appears once.*

(iii) *If $m = 6k + 5$, four pairs $(1, 2), (1, 3), (2, 4), (3, 4)$, do not appear in any triple, and every other pair appears once.*

Proof. The existence of $S(2, 3, m)$ for $m \equiv 1$ or $3 \pmod 6$ establishes $D_3(m) = m(m - 1)/6$ in these cases. Now Theorems 1 and 2 together imply that $D_3(m) = m(m - 2)/6$ for $m \equiv 1$ or $3 \pmod 6$.

In the remaining two cases, the values stated are the largest possible by Theorem 1 and Lemma 1. We need only show that there exist $(m, 3)$ systems with the given numbers of triples and no repeated pairs. In each case we take $m \equiv 5 \pmod 6$ and begin with a minimal $(m, 3)$ system with $(m^2 - m + 4)/6$ triples (see [2] for a construction). There exists a pair $(1, 2)$ which appears in three triples, and every other pair appears in one triple (Theorem 3 (iii)). Deleting two of the blocks containing $(1, 2)$, we obtain $(m^2 - m - 8)/6$ blocks with no repeated pairs, and thus

$$D_3(m) = (m^2 - m - 8)/6$$

for $m \equiv 5 \pmod 6$. Deleting all $(m + 1)/2$ triples containing the element 1, we obtain $(m^2 - 4m + 1)/6$ triples of $m - 1$ elements with no repeated pairs. Thus

$$D_3(m - 1) = (m^2 - 4m + 1)/6 = ((m - 1)^2 - 2(m - 1) - 2)/6$$

for $m - 1 = 4 \pmod 6$, as required.

Properties (i), (ii), and (iii) in Theorem 4 (and in Theorem 3) follow from the observation that the number of pairs involving a specified element in an $(m, 3)$ system must be even. Each element belongs to $m - 1$ different pairs. If m is even, $m - 1$ is odd, and thus each element belongs to an odd number of the pairs which do not occur in a maximal $(m, 3)$ system (and to an odd number of the repeated pairs in a minimal $(m, 3)$ system). For example, if $m \equiv 4 \pmod 6$, exactly $\binom{m}{2} - 3(m^2 - 2m - 2) = m/2 + 1$ pairs do not appear in any triple of a maximal $(m, 3)$ system. These pairs contain a total of $m + 2$ elements, and each of the m elements belongs to an odd number of pairs. Thus one element belongs to three of the pairs and every other element belongs to just one pair. If the elements are suitably named, the pairs which do not appear in any triple will be as stated in (ii). Parts

(i) and (iii) are proved similarly.

Results on extremal (m, k) systems have proved useful in a recent investigation of the covering properties of finite Abelian groups [5]. In this connection it is sometimes necessary to obtain all possible extremal (m, k) systems. This is a very difficult problem, and we wish only to remark that the construction suggested in Theorem 2 yields all possible maximal $(m, 3)$ systems for $m \equiv 0$ or 2(mod 6); for in the light of Theorem 4 (ii), it is always possible to add a new element x and triples $(x, 1, 2), (x, 3, 4), \cdots, (x, m-1, m)$ to obtain a Steiner triple system $S(2, 3, m+1)$. Thus every maximal $(m, 3)$ system with $m \equiv 0$ or 2(mod 6) can be obtained by deleting blocks from Steiner systems $S(2, 3, m+1)$.

4. Systems of quadruples. Hanani [4] has shown that Steiner quadruple systems $S(3, 4, m)$ exist for $m \equiv 2$ or 4(mod 6). Thus $N_4(m) = D_4(m) = m(m-1)(m-1)(m-2)/24$ for $m \equiv 2$ or 4(mod 6). Now Theorems 1 and 2 together establish the values of $D_4(m)$ for $m \equiv 1$ or 3(mod 6), and of $N_4(m)$ for $m \equiv 3$ or 5(mod 6). (It is necessary to adjust the bound provided by Theorem 1 for $N_4(m)$ with $m \equiv 3$ or 5 so that it is an integer). We have

THEOREM 5.

$$N_4(m) = \begin{cases} m(m-1)(m-2)/24 & \text{if } m = 6k+2 \text{ or } 6k+4 \\ (m^3 - 2m^2 + 3m + 6)/24 & \text{if } m = 6k+3 \text{ or } 6k+5 \end{cases}$$

$$D_4(m) = \begin{cases} m(m-1)(m-2)/24 & \text{if } m = 6k+2 \text{ or } 6k+4 \\ m(m-1)(m-3)/24 & \text{if } m = 6k+1 \text{ or } 6k+3\,. \end{cases}$$

In Theorem 6 below we investigate the properties of some extremal $(m, 4)$ systems. Note that Theorem 6 (i) implies that all maximal $(m, 4)$ systems with $m \equiv 1$ or 3(mod 6) are constructed by Theorem 2; for given a maximal $(m, 4)$ system, it is always possible to add quadruples (x, a, b, c), where (a, b, c) runs over all nonoccurring triples, to get a quadruple system $S(3, 4, m+1)$. Thus every maximal $(m, 4)$ system with $m \equiv 1$ or 3(mod 6) may be obtained by deleting blocks from a Steiner system $S(3, 4, m+1)$ as in Theorem 2.

THEOREM 6. (i) *In a maximal* $(m, 4)$ *system with* $m \equiv 1$ *or* 3(mod 6), *the* $m(m-1)/6$ *triples which do not occur in any quadruple form a Steiner triple system* $S(2, 3, m)$.

(ii) *In a minimal* $(m, 4)$ *system with* $m \equiv 3$ *or* 5(mod 6), *exactly* $(m^2 + m + 6)/6$ *triples appear in two quadruples, and every other* *triple appears in one quadruple. Furthermore* $(m+3)/2$ *pairs* $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(6, 7)$, $(8, 9)$, $\cdots$, $(m-1, m)$ *appear in three repeated triples each, while every other pair belongs to just one repeated triple.*

Proof. In an $(m, 4)$ system each pair appears in an even number of triples. The number of different triples containing a given pair is $m-2$, which is odd if m is odd. Thus a given pair belongs to an odd number of triples which do not occur in a maximal $(m, 4)$ system, and to an odd number of repeated triples in a minimal $(m, 4)$ system.

(i) In a maximal $(m, 4)$ system with $m \equiv 1$ or $3 \pmod 6$, there are $\binom{m}{3} - 4m(m-1)(m-3)/24 = m(m-1)/6$ repeated triples, and these contain $\binom{m}{2}$ pairs of elements. Since each pair of elements belongs to an odd number of these triples, each of the $\binom{m}{2}$ different pairs must belong to exactly one of these repeated triples. Thus the repeated triples from a Steiner triple system $S(2, 3, m)$.

(ii) The number of elements occurring in blocks of a minimal $(m, 4)$ system with $m \equiv 3$ or $5 \pmod 6$ is $4(m^3 - 2m^2 + 3m + 6)/24 = m(m^2 - 2m + 3)/6 + 1$. Every triple appears in some quadruple, so each elements occurs in at least $N_3(m-1) = (m^2 - 2m + 3)/6$ blocks. Thus one element 1 appears in $(m^2 - 2m + 9)/6$ blocks, and each of the remaining elements $2, 3, \cdots, m$ appears in $N_3(m-1)$ blocks. By Theorem 3 (ii), the repeated triples involving element $a \neq 1$ are

$$(a, a_1, a_2), (a, a_1, a_3), (a, a_1, a_4), (a, a_5, a_6), (a, a_7, a_8), \cdots, (a, a_{m-2}, a_{m-1})$$

where

$$\{a, a_1, \cdots, a_{m-1}\} = \{1, 2, \cdots, m\} .$$

Thus each element $a \neq 1$ belongs to exactly one pair (a, a_1) which occurs in three repeated triples (all different), and every other pair containing a appears in one repeated triple. Since every pair contains an element $a \neq 1$, no pair appears in more than three repeated triples, and no triple is repeated more than once. The number of repeated triples is

$$4(m^3 - 2m^2 + 3m + 6)/24 - m(m-1)(m-2)/6$$
$$= (m^2 + m + 6)/6 = \binom{m}{2} + m + 3 .$$

Thus exactly $(m+3)/2$ pairs belong to three repeated triples, and each remaining pair belongs to one repeated triple. But $(m+3)/2$ pairs contain $m+3$ elements, and each of $2, 3, \cdots, m$ occurs in one pair. Thus 1 occurs in four pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$ and every other element occurs in just one pair. The remaining pairs which occur in three repeated triples are $(6, 7), (8, 9), \cdots, (m-1, m)$, and the theorem follows.

We now discuss the case $m = 6k$. In Theorem 7 we show that the bounds provided by Theorem 1 for $D_4(6k)$ and $N_4(6k)$ are exact for infinitely many k values. It seems likely that this is so for all k values but we are unable to prove this at present.

We first show that $D_4(6) = 3$ and $N_4(6) = 6$. Since these are the extreme values allowed in Theorem 1, we need only construct extremal (6, 4) systems. It is easy to verify that the three quadruples

$$(1, 2, 3, 4), (1, 2, 5, 6), (3, 4, 5, 6)$$

form a maximal (6, 4) system, and the six quadruples

$$(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 3, 6)$$
$$(4, 5, 6, 1), (4, 5, 6, 2), (4, 5, 6, 3)$$

form a minimal (6, 4) system.

In the proof of theorem 7 we shall require the following Lemma due to Reiss [4]:

LEMMA (*Reiss*). *The $n(2n-1)$ pairs of $2n$ elements may be partitioned into $2n-1$ sets $S_1, S_2, \cdots, S_{2n-1}$, each containing n disjoint pairs.*

Proof. Put $l_{ij} = i + j - 1 (\mathrm{mod}\, 2n-1)$ where $1 \leqq i, j, l_{ij} \leqq 2n-1$, and define $l_{i,2n} = l_{ii}$ $(i = 1, 2, \cdots, 2n-1)$. Then take

$$S_q = \{(1, j), i < j, l_{ij} = q\} \qquad (q = 1, 2, \cdots, 2n-1)\ .$$

It is easy to check that these sets have the required property.

THEOREM 7.

$$N_4(m) = m(m^2 - 3m + 6)/24 \quad and \quad D_4(m) = m(m^2 - 3m - 6)/24$$
$$for\ m = 2^n \cdot 6\ (n = 0, 1, 2, \cdots)\ .$$

Proof. We have shown that $D_4(6)$ and $N_4(6)$ are as stated in the theorem. We assume the results for $m = 6t$ and prove them for $m = 12t$. Put $A_j = \{(i, j), 1 \leqq i \leqq 6t\}$ $(j = 1, 2)$, so that $A = A_1 \cup A_2$ contains $12t$ elements. Partition the pairs of elements of A_j into sets $S_1^j, S_2^j, \cdots, S_{6t-1}^j$ as in the Reiss Lemma $(j = 1, 2)$. For each $q = 1, 2, \cdots, 6t-1$, form the $(3t)^2$ quadruples

$$(a_1, 1)(a_2, 1)(b_1, 2)(b_2, 2)$$

where $(a_1, 1)(a_2, 1)$ runs through all $3t$ pairs of S_q^1, and $(b_1, 2)(b_2, 2)$ runs through all $3t$ pairs of S_q^2. This gives a set T of $(6t-1)(3t)^2$ quadruples with the property that very triple $(c_1, i)(c_2, j)(c_3, k)$ with elements from both A_1 and A_2 appears in exactly one quadruple of T.

Now add to T a minimal $(6t, 4)$ system on the elements of A_j $(j = 1, 2)$. Every triple appears in at least one quadruple, and thus

$$\begin{aligned} N_4(12t) &\leqq (6t-1)(3t)^2 + 2(6t)(36t^2 - 18t + 6)/24 \\ &= m(m^2 - 3m + 6)/24 \end{aligned}$$

where $m = 12t$. Since this is also the smallest value permitted by Theorem 1, Theorem 7 follows for $N_4(m)$.

Alternatively, add to T a maximal $(6t, 4)$ system on the elements of A_j $(j = 1, 2)$. No triple occurs in more than one quadruple, and thus

$$\begin{aligned} D_4(12t) &\geqq (6t-1)(3t)^2 + 2(6t)(36t^2 - 18t - 6)/24 \\ &= m(m^2 - 3m - 6)/24 \end{aligned}$$

where $m = 12t$. This is also the largest value permitted by Theorem 1, and the proof of Theorem 7 is complete.

It is possible to prove that the bounds given in Theorem 1 for $D_4(6k)$ and $N_4(6k)$ are exact for many other values of k as well, and it is likely that Theorem 7 holds for all $m \equiv 0 (\text{mod } 6)$. We now discuss some of the properties of $(6k, 4)$ systems.

First suppose that $m = 6k$ and $D_4(m) = m(m^2 - 3m - 6)/24$. By the argument in Theorem 1, each element in a maximal $(m, 4)$ system occurs in at most $D_3(m-1) = (m^2 - 3m - 6)/24$ triples. It follows that each element occurs in exactly $D_3(m-1)$ triples. By Theorem 4 (iii), the triples containing 1 which do not occur in any quadruple are (1, 2, 4), (1, 3, 4), (1, 2, 5), (1, 3, 5). Applying Theorem 4 (iii) again, the triples containing 2 which do not occur must be (2, 4, 1), (2, 5, 1) (2, 4, 6), and (2, 5, 6). Continuing, we find that the eight triples

(1, 2, 4), (1, 3, 4), (1, 2, 5), (1, 3, 5), (2, 4, 6), (2, 5, 6), (3, 4, 6), (3, 5, 6)

do not occur, and every other triple involving 1, 2, 3, 4, 5, or 6 appears in exactly one quadruple. It is thus possible to partition the $6k$ elements into k sets $A_1, A_2, \cdots, A_k$ of six elements each such that every triple containing elements of more than one set A_j occurs in some quadruple, and exactly eight triples whose elements all belong to A_j do not occur $(j = 1, 2, \cdots, k)$.

Next suppose that $N_4(m) = m(m^2 - 3m + 6)/6$ where $m = 6k$. Using Theorem 3 (iii) and arguments similar to those in the preceding paragraph, we find that in a minimal $(m, 4)$ system, $m/3$ disjoint triples

$$(1, 2, 3), (4, 5, 6), \cdots, (m-2, m-1, m)$$

occur in three quadruples each, and every other triple appears once. These structural properties are proving quite useful in our attempts to construct minimal and maximal $(m, 4)$ system.

At present very little is known about the remaining cases $N_4(6k+1)$ and $D_4(6k-1)$. The bounds provided by Theorems 1 and 2 differ by about $3k/2$ in each case, and we are unable to improve them in general. We give one result in Theorem 8 which shows that the bound given by Theorem 1 for $N_4(7)$ is not the best possible. However we have been unable to generalize the argument to $N_4(6k+1)$ for $k > 1$.

THEOREM 8. $N_4(7) = 12$.

Proof. Theorems 1 and 2 imply $11 \leqq N_4(7) \leqq 12$. Suppose that $N_4(7) = 11$. The 11 quadruples contain $44 = 6.7 + 2$ elements, and each element appears in at least $N_3(6) = 6$ blocks. Thus at least five elements appear in exactly six blocks. If 1 is such an element, the repeated triples containing 1 are $(1, 2, 3), (1, 4, 5)$, and $(1, 6, 7)$ by Theorem 3 (i). Every triple contains one such element, and thus all repeated triples are different (each triple appears in one quadruple or two). Furthermore no pair containing 1 belongs to more than one repeated triple. But there are $4.11 - \binom{7}{3} = 9$ repeated triples containing $3.9 = 27$ pairs, and there are $\binom{7}{2} = 21$ pairs in all. The only pair which can appear in more than one triple is the exceptional pair (a, b) where a and b appear in 7 blocks each. Thus the pair (a, b) occurs in 7 blocks, and some triple (a, b, c) is repeated twice. This is impossible, and it follows that $N_4(7) = 12$.

Note added in proof. Two papers by J. Schönheim on this problem have recently come to our attention, through the SIAM conference on Combinatorics in Santa Barbara (December, 1967). In the first of these (On coverings, Pacific J. Math. 14 (1964), 1405–1411) results for $N_k(m)$ similar to those of Theorems 1, 2 and 5 were given. In the second (On Maximal Systems of k-tuples, Studia Scientiarium Math. Hungarica 1 (1966), 363–368) will be found results for $D_k(m)$ similar to those of Theorems 1, 2, and 5, and the first part of Theorem 4.

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