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In this paper a sufficient condition is given on the covariance of a stationary Gaussian process for it to have sample paths that are unbounded in all intervals. This result is an extension of a result of Belyaev.

It is also shown that an obvious analogue of this result applies to Gaussian processes with stationary increments.

Let $X(t)$ be a real valued Gaussian process with stationary increments such that $E\{X(t)\} = 0$ and

$$(1) \qquad E\{(X(t + h) - X(t))^2\} = \sigma^2(h) \, ,$$

$\sigma(h) \to 0$ as $h \to 0$. Suppose that $\sigma^2(h)$ is concave in $[0, \delta]$ for some $\delta > 0$ and that

$$(2) \qquad \sigma^2(h) \geqq \frac{C}{|\log |h||} \, , \quad C > 0 \, ,$$

for $h \in [0, a]$, for some $a > 0$. Then it follows that at any point t_0, with probability one, the sample paths of $X(t)$ are discontinuous. This result was obtained by Belyaev [1] for stationary Gaussian processes; its extension to Gaussian processes with stationary increments is trivial.

Suppose $X(t)$ is a real valued stationary Gaussian process, normalized so that $E\{(X(t))^2\} = 1$. Denote its covariance by $\varphi(h)$; thus $\sigma^2(h) = 2(1 - \varphi(h))$. Let Φ be the class of all covariance functions $\varphi(h)$ such that $\varphi(0) = 1$, $\varphi(h)$ is convex in some interval $[0, \delta]$, $\delta > 0$ (δ need not be the same for all $\varphi \in \Phi$), and $2(1 - \varphi(h))$ satisfies Eq. (2) for $h \in [0, a]$, for some $a > 0$. Then Belyaev's result asserts that the stationary Gaussian processes defined by the covariance functions in Φ have discontinuous sample paths. The main result of this paper is Theorem 2, which is equivalent to the following statement: Let Γ be the set of all normalized covariance functions. Let Φ be as defined above. Then if $\gamma(h) \in \Gamma$, $\varphi(h) \in \Phi$, with probability one the stationary Gaussian process with covariance $\gamma(h) \cdot \varphi(h)$ has sample paths that are discontinuous in every interval. In other words, the class of stationary Gaussian processes with discontinuous sample paths is extended from Φ to $\Gamma\Phi$. A similar result is obtained for Gaussian processes with stationary increments.

Once it is shown that the sample paths of a stationary Gaussian

process are discontinuous with probability greater than zero, Belyaev's result [1]—that either the sample paths of a stationary Gaussian process are continuous with probability one or else with probability one they are unbounded in every interval—further describes the behavior of the paths.

An unsolved question is the exact nature of the set of covariance functions $\Gamma \Phi$. The set contains Φ since Γ contains the constant 1. Also, $\Gamma \Phi$ is larger than Φ, since we can take for $\gamma \in \Gamma$ a nowhere differentiable function (obtained from the theory of lacunary series). Therefore, a $\gamma \varphi$ can be found that is nowhere differentiable and hence not convex in some interval $[0, a]$, $a > 0$. This covariance function will not be contained in Φ. The problem of identifying the product of a characteristic function of a probability measure with all other characteristic functions has been posed in harmonic analysis, but not much seems to be known about it.

2.   The first lemma is probabilistic: in the rest of the paper we are concerned with evaluating determinants of covariance matrices.

LEMMA 1.   *Let* $X_1, \cdots, X_n$ *be linearly independent real valued Gaussian random variables with mean zero and finite variances. Let* D_i *be the determinant of the* i^{th} *principal minor of their covariance matrix* R. *Define* $\rho_i = D_{i-1}/D_i$ $i = 1, \cdots, n$. $D_0 \equiv 1$. *Then*

$$P\left\{\max_{1 \leqq i \leqq n} |X_i| \leqq a\right\} \leqq \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{a(\rho_i)^{1/2}} e^{-\xi^2/2} d\xi \ .$$

Proof.   We have

$$P\left\{\max_{1 \leqq i \leqq n} |X_i| \leqq a\right\}$$

(3)

$$= \frac{|A|^{1/2}}{(2\pi)^{n/2}} \int_{|x_i| \leqq a} \cdots \int \exp\left\{-\frac{1}{2} \sum_{i,j} A_{ij} x_i x_j\right\} dx_i \cdots dx_n \ ,$$

where $A = R^{-1}$, and $|A|$ denotes the determinant of A.

Since the random variables are square integrable functions on an n-dimensional Gaussian measure space, they can be considered as elements in a Hilbert Space. Their covariance corresponds to the inner product. Using the Gram-Schmidt orthogonalization procedure, we can extract an orthonormal sequence $\xi_i = \sum_j P_{ij} X_j$, where $P^T P = A$. The functions ξ_i have the following form:

$$\xi_i = (\rho_i)^{1/2} X_i + f_i(\xi_{i-1}, \cdots, \xi_1) \ , \quad i = 1, \cdots, n$$

(See [4], p. 37).   The change of variables $\xi_i = \sum_j P_{ij} X_j$ diagonalizes the

quadratic form in Eq. (3) and the integral is equal to

$$(4) \qquad \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{B_i} \exp\{-\xi_i^2/2\} d\xi_i \ .$$

The interval of integration B_i is given by the inequality

$$-(\rho_i)^{1/2} a + f(\xi_{i-1}, \cdots, \xi_1) \leqq \xi_i \leqq (\rho_i)^{1/2} a + f_i(\xi_{i-1}, \cdots, \xi_1) \ .$$

Because the density function of the Gaussian distribution is symmetric and decreases monotonically as $|\xi_i|$ increases, the integrals in Eq. (4) are increased if we take $|\xi_i| \leqq (\rho_i)^{1/2} a$. This observation proves the lemma.

Using Ostrowski's results [3] on matrices with dominant principal diagonals, an upper bound for D_{i-1}/D_i can be obtained when the determinants D_i are those of a covariance matrix formed by the increments of a stationary Gaussian process which itself has a covariance that is convex in $[0, \delta]$ for some $\delta > 0$. For the sake of completeness of this paper, Ostrowski's result will be restated.

Let $A = \{a_{\mu\nu}\}$ be a real valued matrix with positive diagonal elements such that the sum of the absolute values of all the off diagonal elements in a given row is less than the diagonal element in that row. Such a matrix A is called a matrix with dominant principal diagonal. For these matrices we define $\alpha_{\mu\nu} = |a_{\mu\nu}|$, $\mu, \nu = 1, \cdots, n$ and

$$(5) \qquad s_\mu = \sum_{\nu=1}^{n} \alpha_{\mu\nu} - \alpha_{\mu\mu} = \sigma_\mu \alpha_{\mu\mu} \ , \qquad\qquad 1 \leqq \mu, \nu \leqq n \ .$$

Note that $\sigma_\mu < 1$. The following lower bound is obtained for $|A|$:

$$(6) \qquad |A| \geqq \prod_{\mu=1}^{n} (\alpha_{\mu\mu} - s_\mu) \ .$$

Equation (6) shows that a matrix with dominant principle diagonal is strictly positive definite.

Let D_i be the determinant of $\{a_{\mu\nu}\} \mu, \nu = 1, \cdots, i$. In a manner similar to Eq. (5), we define

$$s_\mu^i = \sum_{\nu=1}^{i} \alpha_{\mu\nu} - \alpha_{\mu\mu} = \sigma_\mu^i \alpha_{\mu\mu} \ , \qquad\qquad 1 \leqq \mu \leqq i \ ,$$

and

$$t_i^i = \max \sigma_k^i \ , \qquad\qquad (k = 1, \cdots, i - 1) \ .$$

Then it follows that

$$(7) \qquad (\alpha_{ii} + t_i^i s_i^i) D_{i-1} \geqq D_i \geqq (\alpha_{ii} - t_i^i s_i^i) D_{i-1} \ .$$

Note that $s_i^i = \sum_{\nu=1}^{i-1} \alpha_{i\nu}$ is the sum of the absolute values of the elements

in the ith row of A that lie to the left of the diagonal element.

We shall now exhibit a collection of Gaussian random variables which have as their covariance matrix a matrix with dominant principal diagonal. Let $\gamma(h)$ be the covariance of a stationary Gaussian process and suppose that $\gamma(h)$ is convex for $h \in [0, \delta]$ for some $\delta > 0$, and that $\gamma(\delta - \varepsilon) > \gamma(\delta) > 0$ $(0 < \varepsilon \le \delta)$. Let $0 < t_1 < t < t_n \le \delta$ be a partition of $[0, \delta]$. The covariance matrix of $(X(t_j) - X(t_{j-1})), j = 1, \cdots, n$ is a matrix with dominant principal diagonal. This is easy to verify since all of the off diagonal terms of the n-dimensional covariance matrix of $X(t_j) - X(t_{j-1})$, $j = 1, \cdots, n$ are negative due to the convexity of $\gamma(h)$. Thus the absolute values of the off diagonal elements in a given row can be added. In the case under consideration, $s_i^i \le \alpha_{ii}/2 = \sigma^2(t_i - t_{i-1})/2$. Since $t_i^i \le 1$ we have

$$(8) \qquad \frac{D_{i-1}}{D_i} \le \frac{2}{\sigma^2(t_i - t_{i-1})} .$$

Also note that the functions D_{i-1}/D_i for the covariance matrices of the random variables $X(t_j) - X(t_{j-1})$, $j = 1, \cdots, i$ and the functions D_{i-1}/D_i for the covariance matrices of the random variables $X(t_j) - X(t_0)$, $j = 1, \cdots, i$ are equal. This follows from row and column operations on the matrices. Referring to Lemma 1 and using Eq. (8), the following lemma is obtained.

LEMMA 2. *Let $X(t)$ be a real valued stationary Gaussian process with covariance $\gamma(h)$ which is convex for $h \in [0, \delta]$ for some $\delta > 0$. Let $t_0 < t_1 < \cdots < t_n$, $(t_n - t_0) \le \delta$. Then*

$$\text{Prob}\left\{ \max_{1 \le i \le n} |\xi_i| \le a \right\} \le \prod_{i=1}^{n} \sqrt{2/\pi} \int_0^{\sqrt{2}\,a/\alpha_{ii}} e^{-x^2/2} dx ,$$

where ξ_i can be either $X(t_i) - X(t_{i-1})$ or $X(t_i) - X(t_0)$ and $\alpha_{ii} = \sigma(t_i - t_{i-1})$, $i = 1, \cdots, n$.

Using Lemma 2, the following theorem is easily obtained.

THEOREM 1. *Let $X(t)$ be a real valued stationary Gaussian process with covariance function $\gamma(h)$ that is convex for $h \in [0, \delta]$ for some $\delta > 0$. Let $\sigma(h) = 2(1 - \gamma(h))$. Then if*

$$(9) \qquad \sigma(h) \ge \frac{C}{|\log |h||}, \qquad C > 0 ,$$

$h \in [0, a]$, for some $a > 0$, it follows that almost all sample functions of $X(t)$ are unbounded in every interval.

The result in Theorem 1 was first obtained by Belyaev, who

employed a lemma due to Ventcel which is similar to Lemma 2. An alternate way of proving Theorem 1 is to use the Chung-Erdos version of the Borel-Cantelli lemma [2]. Theorem 1 is contained in Theorem 2. It has been displayed separately because its proof is simple and because it is used to conclude the proof of Theorem 2.

In preparation for Theorem 2, we shall prove some lemmas on determinants associated with Toeplitz matrices. A matrix $\{c_{j,k}\}$ $j, k = 0$, $1, \cdots, n$ is a Toeplitz matrix if it is positive definite and if $c_{j,k} = c_{j-k}$. We shall normalize these numbers by taking $c_0 = 1$. The elements of a Toeplitz matrix can be represented as the Fourier-Stieltjes transforms of a distribution function $\alpha(x)$ on the unit circle,

$$(10) \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} d\alpha(x) \ .$$

Conversely, distribution functions on the unit circle give rise to positive definite sequences $\{c_k\}$ such that finite subsets of these sequences determine Toeplitz matrices. If the distribution function has at least $n + 1$ points of increase, the determinants of the matrices $\{c_{j-k}\}$, j, $k = 0, 1, \cdots, i, 0 < i \leq n$ are strictly positive. A distribution function will be said to be of order k if it has at least k points of increase.

Let $\{c_k\}$ $k = 0, \pm 1, \cdots$, be a positive definite sequence and $\alpha(x)$ the corresponding distribution function. We shall denote the determinant of the matrix given by $c_0, c_{-1}, c_1, \cdots, c_{-n}$ by $D_{n+1}(\alpha)$. Also, the function $D_n(\alpha)/D_{n+1}(\alpha)$ will be referred to as $D_n/D_{n+1}(\alpha)$.

The following lemma is a minor modification of Theorem (a) on page 38 in Ref. 4.

LEMMA 3. *Let $\alpha(x)$ be a distribution function on $[-\pi, \pi]$ of order $n + 1$ and $g(z)$ be any polynomial of the nth degree in e^{ix}, $a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $z = e^{ix}$, subject to the condition that $|a_0| = 1$. Then*

$$(11) \qquad \frac{D_{n+1}}{D_n}(\alpha) = \min_{g(z)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x) \ ,$$

where the minimum is taken over all the polynomials $g(z)$.

Since $\alpha(x)$ defines a probability measure on the unit circle, so does its translate $\alpha(x - \xi)$. Define this new distribution function by $\alpha_\xi(x)$. The proof of the following lemma follows immediately from Lemma 3.

LEMMA 4. *Let $\alpha(x)$ be a distribution function of order $n + 1$ on $[-\pi, \pi]$. Then*

$$\frac{D_{n+1}}{D_n}(\alpha(x)) = \frac{D_{n+1}}{D_n}(\alpha_\xi(x)) \, .$$

Let Θ be the set of all probability measures on the unit circle. We shall now consider the set of probability measures $\{\alpha * \theta, \theta \in \Theta\}$, where $*$ denotes convolution.

LEMMA 5. *Let $\alpha(x)$ be a distribution of order $n + 1$ on $[-\pi, \pi]$, and let θ be any distribution function on $[-\pi, \pi]$. Then*

$$\frac{D_{n+1}}{D_n}(\alpha) \leqq \frac{D_{n+1}}{D_n}(\alpha * \theta) \, .$$

Proof. Let θ_ξ, $(-\pi \leqq \xi \leqq \pi)$ be the extreme points of Θ, i.e., the measures that assign probability one at the point ξ. Clearly the extreme points of $\{\alpha * \theta, \theta \in \Theta\}$ are contained in the extreme points of $\{\alpha * \theta_\xi\}$. Consider the convolution of α with an element in the convex hull of $\{\theta_\xi\}$, i.e., $\alpha * \sum_{j=1}^n t_j \theta_{\xi j}$, $\sum_{j=1}^n t_j = 1$. Then

$$(12) \qquad \frac{D_{n+1}}{D_n}(\alpha) = \sum_{j=1}^n t_j \frac{D_{n+1}}{D_n}(\alpha * \theta_{\xi j}) \leqq \frac{D_{n+1}}{D_n}\left(\alpha * \sum_{j=1}^n t_j \theta_{\xi j}\right) \, .$$

The equality in Eq. (12) follows by Lemma 4, the inequality follows from Eq. (11).

Since D_{n+1}/D_n is a uniformly continuous function of the appropriate $2n + 1$ Fourier-Stieltjes coefficients, Eq. (12) can be extended to the weak closure of the convex hull of $\{\alpha * \theta_\xi\}$. This is $\{\alpha * \theta, \theta \in \Theta\}$ itself. Thus, the lemma is proved.

Let $X(t)$ be a normalized stationary Gaussian process with covariance φ. Consider the stationary sequence $\{\varphi(jh)\}, j = 0, \pm 1, \cdots$, and let $\alpha(x)$ be the probability distribution on the unit circle associated with $\{\varphi(jh)\}$. For this same process $X(t)$, consider the stationary sequence of Gaussian random variables $\{X(t + jh) - X(t + (j-1)h)\} j = 0, \pm 1, \cdots$. Denote the elements of the covariance matrix of these random variables by d_n, i.e.,

$$d_n = E\{(X(t + jh) - X(t + (j-1)h)(X(t + kh) - X(t + (k-1)h)\}$$
$$= 2\varphi((j - k)h) - \varphi((j - k - 1)h) - \varphi((j - k + 1)h) \, ,$$

where $n = j - k$. Thus, we have the following equation for d_n,

$$(13) \qquad d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \, |\, e^{ix} - 1\,|^2 \, d\alpha(x) \, .$$

Also, d_n is a positive definite sequence; it is defined by the distribution function $\alpha_2(x) = \int_{-\pi}^{x} |e^{iy} - 1|^2 d\alpha(y)$. We shall normalize the sequence d_n by dividing it by d_0; thus an alternate representation for Eq. (13) is

$$d_n/d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} d(\alpha_2(x)/d_0) \ .$$

LEMMA 6. *Let $\alpha(x)$ be a distribution function of order $n + 2$ and let $\alpha_2(x)$ be the distribution function corresponding to the second difference of the coefficients of $\alpha(x)$ (i.e., c_n and d_n as given above). Then*

$$\frac{D_{n+2}}{D_{n+1}}(\alpha(x)) \leqq \frac{D_{n+1}}{D_n}(\alpha_2(x)) \ .$$

Proof.

(14) $$\frac{D_{n+1}}{D_n}(\alpha_2(x)/d_0) = \min_{g(z)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 |e^{ix} - 1|^2 d(\alpha(x)/d_0) \ .$$

Note that the $1/d_0$ factor can be cancelled from each side. The polynomial $g(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ is subject to the restriction that $|a_0| = 1$. Equation (14) can also be written as

(15) $$\frac{D_{n+1}}{D_n}(\alpha_2(x)) = \min_{f(z)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z)|^2 d\alpha(x) \ ,$$

where the polynomial $f(z) = \alpha_0 z^{n+1} + \alpha_1 z^n + \cdots + \alpha_{n+1}$ is subject to the restrictions that $|\alpha_0| = 1$ and that $\alpha_{n+1} = \alpha_n + \alpha_{n-1} + \cdots + \alpha_0$. The lemma follows directly from Eq. (15), since the right-hand side of Eq. (15) is greater than $D_{n+2}/D_{n+1}(\alpha(x))$. We can now prove Theorem 2.

THEOREM 2. *Let $X(t)$ be a normalized stationary Gaussian process with covariance $\varphi(h)$ that is convex for $h \in [0, \delta]$, for some $\delta > 0$. Suppose that $E\{(X(t + h) - X(t))^2\} = 2(1 - \varphi(h)) = \sigma^2(h)$ satisfies the following inequality:*

(16) $$\sigma^2(h) \geqq \frac{C}{|\log|h||} \ , \qquad C > 0 \ ,$$

for $h \in [0, a]$, for some $a > 0$. Then with probability one the sample paths of $X(t)$ are unbounded in every interval. Moreover, let Γ be the set of all normalized covariance functions. If $\gamma(h) \in \Gamma$, then the stationary Gaussian process with covariance $\gamma(h) \cdot \varphi(h)$ has sample

paths that are unbounded in every interval.

Proof. Without loss of generality we can assume $\delta \leqq a$ and that $\varphi(\delta - \varepsilon) > \varphi(\delta) > 0$ for sufficiently small $\varepsilon > 0$. Consider the sequence $\varphi\{[(j/2n)\delta]\}$ $j = 0, 1, \cdots$. Because the Gaussian process is real, $\{\varphi(j/2n)\delta\} = \{\varphi(-j/2n)\delta\}$. Let $\alpha(x)$ be the distribution function defined by this sequence. By Eq. (6) we see that $\alpha(x)$ is of order $2n + 1$. Let $\mu(x)$ be the distribution function given by $\{\gamma(j/2n)\delta\}$ $j = 0, 1, \cdots$. Then $(\alpha * \mu)$ and $(\alpha * \mu)_2$ are of order $2n$.

Let $X(t)$ be the stationary Gaussian process with covariance $\gamma(h)\cdot\varphi(h)$. Consider the random variables

$$\left\{X\!\left(\frac{j}{2n}\delta\right) - X\!\left(\frac{j-1}{2n}\delta\right)\right\}j = 1, \cdots, n + 1 \; .$$

They are linearly independent because $(\alpha * \mu)_2$ is of order $2n$. From Lemmas 6 and 5 we obtain

$$(17) \qquad \frac{D_n}{D_{n+1}}((\alpha * \mu)_2) \leqq \frac{D_{n+1}}{D_{n+2}}(\alpha * \mu) \leqq \frac{D_{n+1}}{D_{n+2}}(\alpha) \leqq \frac{2}{\sigma^2(\delta/2n)} \; ,$$

where the last inequality is obtained using Ostrowski's results as follows: The term $D_{n+1}/D_{n2}(\alpha)$ is a function on the Toeplitz matrix which has as its elements $\varphi\{[(j/2n)\delta]\}$ $j = 0, 1, \cdots, n + 1$. Perform the following row and column operations on this matrix: Subtract the jth row from the $j + 1^{\text{st}}$ row, $j = 1, 2, \cdots, n$, leaving the first row fixed, then subtract the jth column from the $j + 1^{\text{st}}$ column $j = 1, 2, \cdots, n$, leaving the first column fixed. These operations leave $D_{n+1}(\alpha)$ and $D_{n+2}(\alpha)$ unchanged. The convexity of $\varphi(h)$ is used to show that all of diagonal terms of the resulting matrix are negative, and that the matrix has a dominant principle diagonal. The function $D_{n+1}/D_{n+2}(\alpha)$ is evaluated using Eq. (7). This accounts for the final inequality in Eq. (17).

The remainder of the proof follows from Lemma 2 and Theorem 1.

A result similar to Theorem 2 can be obtained for Gaussian processes with stationary increments. Theorem 2 extends the class of discontinuous stationary Gaussian processes from those for which

$$E\{(X(t + h) - X(t))^2\} = 2(1 - \varphi(h))$$

to those for which $E\{(X(t + h) - X(t))^2\} = 2(1 - \gamma(h)\cdot\varphi(h))$. The class of Gaussian processes with stationary increments that have discontinuous sample paths can also be extended from those processes for which $E\{(X(t + h) - X(t))^2\} = 2(1 - \varphi(h))$ for $h \in [0, \delta]$ for some $\delta > 0$ and $\varphi \in \varPhi$ to those processes for which

$$E\{(X(t + h) - X(t))^2\} = 2(1 - \gamma(h) \cdot \varphi(h))$$

for $h \in [0, \delta]$ for some $\delta > 0$, where $\gamma(h)$ is any normalized covariance function. The proof of this result for Gaussian processes with stationary increments is contained in the proof of Theorem 2, since in Theorem 2 we are concerned with the covariance matrix of the increments of a stationary Gaussian process. This matrix is the same as the covariance matrix of the increments of a Gaussian process with stationary increments as long as the function $\sigma^2(h)$(i.e., as in Eq. (1)) is the same in both cases.

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