ON A PROBLEM OF ILYEFF

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Let \( P(z) \) be a polynomial whose zeros \( z_1, z_2, \cdots, z_n \) \( (n \geq 2) \) lie in \( |z| \leq 1 \). It is shown that \( P'(z) \) always has a zero in \( |z - z_1| \leq 1 \) if \( |z_1| = 1 \) or if \( |z_1| < 1 \) and \( n = 3, 4 \).

In his book *Research Problems in Function Theory* [2] W. K. Hayman mentions the following problem due to L. Ilyeff (Problem 4.5, p. 25): Let \( P(z) \) be a polynomial whose zeros \( z_1, z_2, \cdots, z_n \) \( (n \geq 2) \) lie in \( |z| \leq 1 \). Is it true that \( P'(z) \) always has a zero in \( |z - z_1| \leq 1 \)?

In this note we answer this question in the affirmative if \( |z_1| = 1 \) for arbitrary \( n \) and if \( |z_i| < 1 \) for \( n = 3, 4 \). The case \( n = 2 \) is trivial.

We also show that the disk \( |z - z_1| < 1 \) always contains a zero of \( P'(z) \) regardless of the location of the zeros if \( |P'(z_i)| < n \) and if the polynomial \( P(z) \) is normalized to be a monic polynomial.

2. The boundary case.

**Theorem 1.** Let \( P(z) \) be a polynomial whose zeros \( z_1, z_2, \cdots, z_n \) \( (n \geq 2) \) lie in \( |z| \leq 1 \) such that \( |z_i| = 1 \). Then the disk \( |z - z_1| \leq 1 \) always contains a zero of \( P'(z) \). Furthermore the disk \( |z - z_1| < 1 \) always contains a zero of \( P'(z) \) except when \( P(z) = c(z^n - e^{i\theta}) \).

**Proof.** Without loss of generality we may assume that \( z_1 = 1 \), \( z_k \neq 1 \) for \( k = 2, 3, \cdots, n \) and \( P'(1) = 1 \). We shall show that the polynomial \( P'(z + 1) \) has at least one zero in the closed unit disk. If this is not so then the following representation of \( P'(z + 1) \) is possible [1] for \( |z| < 1 \).

\[
P'(z + 1) = (1 - zf(z))^{n-1}
\]

where \( f(z) \) is analytic in the unit disk and less than one in modulus.

From (1) by differentiation we obtain

\[
P''(1) = (1 - n)f'(0).
\]

The polynomial \( Q(z) \) defined by the relation \( P(z) = (z - 1)Q(z) \) satisfies \( Q(1) = P'(1) = 1 \) and \( 2Q'(1) = P''(1) \). Hence applying (2) we obtain

\[
Q'(1) = \frac{Q'(1)}{Q(1)} = \frac{1}{1 - z_2} + \frac{1}{1 - z_3} + \cdots + \frac{1}{1 - z_n} = \frac{1 - n}{2}f(0)
\]
from which we deduce that $|Q'(1)| < (n - 1)/2$. On the other hand since $|z_k| \leq 1$, $\Re 1/(1 - z_k) \geq 1/2$ and thus $\Re Q'(1) \geq (n - 1)/2$. This contradiction proves the theorem.

To prove the second part of the theorem we observe that $|f(z)| \leq 1$ even if $P'(z + 1) \neq 0$ for $|z| < 1$, so that in this case we also obtain a contradiction unless all the $z_k$ lie on the unit circumference and $f(z)$ is a constant of absolute value one. This implies that $P(z)$ has all its zeros on the unit circumference such that $P'(z)$ has an $(n - 1)$ fold zero on the circle $|z - 1| = 1$.

3. Third and fourth degree polynomials.

**Theorem 2.** Let $P(z)$ be a polynomial of degree three or four whose zeros lie in the closed unit disk. Then any circle of radius one about a zero of $P(z)$ contains a zero of $P'(z)$.

**Proof.** We may assume that $P(z) = (z - x)Q(z)$, where $0 < x < 1$ and the zeros $z_k$, $k = 1, 2, \cdots, n$ of $Q(z)$ lie in $|z| \leq 1$. We shall prove that the polynomial $f(z) = P'(z + x)$ has a zero in $|z| < 1$.

Consider the following polynomials

$$f(z) = \sum_{k=0}^n (k + 1) \frac{Q^{(k)}(x)}{k!} z^k$$

$$g(z) = \sum_{k=0}^n \frac{1}{k - 1} \binom{n}{k} z^k$$

and

$$h(z) = \sum_{k=0}^n \frac{Q^{(k)}(x)}{k!} z^k.$$

By a result due to Szegö [4] every zero $\gamma$ of $h(z)$ has the form $\gamma = -\alpha \beta$, where $\beta$ is a zero of $g(z)$ and $\alpha$ is a point belonging to a circular region containing all the zeros of $f(z)$. The zeros of $g(z)$ have the form $\beta = -1 + \sqrt{1 + p}$ such that $\beta \neq 0$. For $n = 2, 3$ $|\beta| \geq \sqrt{2}$. If $f(z) \neq 0$ in $|z| < 1$ we may choose $\alpha$ such that $|\alpha| \geq 1$. Thus $|\gamma| \geq \sqrt{2}$. Since $h(z) = Q(z + x)$ and $f(z) = P'(z + x)$ it follows that all the zeros of $Q(z)$ satisfy $|z| \leq 1$ and $|z - x| \geq \sqrt{2}$ and no zero of $P'(z)$ lies in $|z - x| < 1$.

Consider now the polynomial $R(z) = P(z - 1 + x) = (z - 1)Q_i(z)$, where $Q_i(z) = Q(z - 1 + x)$. No zero of $R'(z)$ lies in $|z - 1| < 1$. By Theorem 1 we shall obtain a contradiction if we can show that all the zeros of $Q_i(z)$ lie in $|z| < 1$. Indeed the zeros of $Q_i(z)$ satisfy the inequalities $|z - 1 + x| \leq 1$ and $|z - 1| \geq \sqrt{2}$. A straightforward calculation shows that if $z = u + iv$ these inequalities imply
\[ u^2 + v^2 \leq 3 - \left( x + \frac{1}{x} \right) < 1 \]

for \( 0 < x < 1 \). This completes the proof.

4. A particular class of polynomials.

**Theorem 3.** Let \( P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \). If \( P(z_1) = 0 \) and \( |P'(z_1)| < n \), then \( P'(z) \) has a zero in \( |z - z_1| < 1 \).

**Proof.** Write \( P(z) = (z - z_1)Q(z) \) and set \( f(z) = P'(z + z_1) \) and \( f^*(z) = z^{n-1}f(1/z) \). We have \( f(e^{i\theta}) = f^*(e^{i\theta}) \) and

\[
\begin{align*}
  f(z) &= n z^{n-1} + \cdots + Q(z) \\
  f^*(z) &= \overline{Q(z_1)} z^{n-1} + \cdots + n.
\end{align*}
\]

If \( Q(z_1) \neq 0 \) the polynomial \( nf^*(z) - \overline{Q(z_1)} f(z) \) is of degree not exceeding \( n - 2 \) and since \( Q(z_1) = P'(z_1) \) it follows by Rouché's theorem that \( f^*(z) \) has at most \( (n - 2) \) zeros in \( |z| < 1 \). Therefore \( f(z) \) has at least one zero in \( |z| < 1 \). This means that \( P'(z) \) has at least one zero in \( |z - x| < 1 \). If \( Q(z_1) = 0 \) then \( P'(z_1) = 0 \) and the same is true. From Theorem 3 we can deduce that Ilyeff's conjecture is true if all the coefficients of \( Q(z) \) are less than one in modulus. This includes in particular the case where the theorem of Enström-Kakeya [3] is applicable, i.e. when the coefficients of \( Q(z) \) form a monotonically decreasing sequence of positive numbers.

**Bibliography**


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