DUAL GROUPS OF VECTOR SPACES

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Let $E$ be a topological vector space over a field $K$ having a nontrivial absolute value. Let $E'$ be the dual space of continuous linear maps $E \rightarrow K$, and $\hat{E}$ the dual group of continuous characters $E \rightarrow R/Z$. $\hat{E}$ is a vector space over $K$ by $(a \varphi)(x) = \varphi(ax)$, and composition with a nonzero character of $K$ is a linear map of $E'$ into $\hat{E}$. This map is always an isomorphism if $K$ is locally compact, while if $K$ is not locally compact it is never an isomorphism unless $\hat{E} = 0$. When $K$ is locally compact, $E'$ is in addition topologically isomorphic to $\hat{E}$ if each is given its topology of uniform convergence on compact sets. This leads to conditions on $E$ which imply that $E$ is topologically isomorphic to $\langle \hat{E} \rangle^\sim$.

**Theorem 1.** Let $K$ be a field with absolute value. Then $K$ is one-dimensional over $K$ if and only if $K$ is locally compact.

**Proof.** The sufficiency of local compactness is fairly well known (cf. [4, p. 92-3] for the characteristic zero case). To prove it, one takes a nonzero character $\pi$ of $K$ and considers the subspace $K\pi$ of $\hat{K}$. It is easy to check that $a \mapsto a\pi$ is a bicontinuous linear map, so $K\pi$ is complete and hence closed in $\hat{K}$. On the other hand, $K\pi$ separates the points of $K$, so by Pontrjagin duality it is dense in $\hat{K}$. Thus $\hat{K} = K\pi$.

Suppose conversely that $\hat{K}$ is one-dimensional, and choose a nonzero $\pi$ in $\hat{K}$. The completion of $K$ will again be a field, say $L$, and $\pi$ extends to a character of $L$. Then every $a \in L$ gives a character $a\pi$ of $L$. If $a \neq b$, then $a - b$ is invertible, and so $\pi((a - b)c)$ cannot be zero for all $c$. Thus no two of the characters $a\pi$ are equal, and hence no two can agree on the dense set $K$. This contradicts one-dimensionality of $\hat{K}$ unless $K = L$, and we conclude that $K$ must be complete. Hence if $K$ is archimedean, it is locally compact.

We now assume that $K$ is nonarchimedean. Let $A = \{x: |x| \leq 1\}$, $M = \{x: |x| < 1\}$. Let $\pi$ be a character of the discrete group $A/M$ with $\pi(1) \neq 0$; we extend $\pi$ to a character of the discrete group $K/M$ and interpret it as an element of $\hat{K}$. Let $c > 1$ be an element of the value group, and consider the group $G_c/M$, where $G_c = \{x: |x| \leq c\}$. All characters of this discrete group extend to characters of $K$ vanishing on $M$, and by one-dimensionality they all come from multiples of $\pi$.

Now if $a \in A$, then $aM \subset M$, so $a\pi$ vanishes on $M$; conversely, if $a\pi$ vanishes on $M$, then $1/a \notin M$ and $a \in A$. Similarly, $a\pi$ vanishes
on $G$, if and only if $|a| < 1/c$. Thus the dual group of the discrete abelian group $G/M$ is (algebraically) isomorphic to $A/\{a: |a| < 1/c\}$, which is isomorphic to $G_c/M$ itself under multiplication by an element of absolute value $c$. A theorem of Kakutani [3, p. 396-7] shows that an infinite discrete abelian group has a dual group of strictly larger cardinality; hence $G_c/M$ must be finite. This implies both that $A/M$ is finite and that the value group is discrete; since $K$ has these two properties and is complete, it is locally compact [1, p. 119].

**COROLLARY.** Suppose $K$ is not locally compact. Let $E$ be a topological vector space over $K$ with $E \neq 0$. For any $\pi \epsilon K$, the map $E' \rightarrow \hat{E}$ given by composition with $\pi$ fails to be surjective.

**Proof.** If $E' = 0$ or $\pi = 0$, the statement is obvious. Suppose then there is a $0 \neq f \epsilon E'$, and choose an $x \epsilon E$ with $f(x) \neq 0$. The subspace $Kx$ is topologically isomorphic to $K$, so its dual space is one-dimensional and is generated by the restriction of $f$. Hence all elements in $\hat{E}$ coming from $E'$ restrict to multiples of $\pi \circ f$ on $Kx$. If $\tau$ is a character of $K$ not a multiple of $\pi$, then $\tau \circ f \epsilon \hat{E}$ is not in the image of $E'$.

**REMARK.** A topological field $K$ is called **locally retrobounded** if for every pair of neighborhoods $U, V$ of zero there is an $a \neq 0$ in $K$ such that $a\{x^{-1}: x \epsilon V\} \subset U$; for example, an ordered field is locally retrobounded in its order topology. Every such field admits either an absolute value or a valuation which defines its topology [1, §5, Exerc. 2]. The proof of Theorem 1 works equally well for a valuation into any ordered abelian group, and hence Theorem 1 and its corollary hold for all locally retrobounded fields.

**THEOREM 2.** Suppose $K$ is locally compact, $0 \neq \pi \epsilon \hat{K}$. Let $E$ be a topological vector space over $K$. Then the map $E' \rightarrow \hat{E}$ given by $f \mapsto \pi \circ f$ is a vector space isomorphism. It is a homeomorphism if $E'$ and $\hat{E}$ have their topologies of uniform convergence on compact sets.

**Proof.** If $0 \neq f$, then $f(E) = K$, so $\pi \circ f \neq 0$; thus the map is injective. Now let $\varphi \epsilon \hat{E}$. For each $x \epsilon E$ there is a unique linear functional on $Kx$ inducing $\varphi | Kx$, since $Kx \cong K$ and $\hat{K}$ is one-dimensional. We define $f(x)$ to be this functional evaluated at $x$; this gives us a homogeneous function $f: E \rightarrow K$. For any $x, y \epsilon E$ and $a \epsilon K$, we have

$$0 = \varphi(ax) + \varphi(ay) - \varphi(ax + ay) = \pi f(ax) + \pi f(ay) - \pi f(ax + ay) = \pi [f(ax) + f(ay) - f(ax + ay)] = \pi (a[f(x) + f(y) - f(x + y)]).$$
hence $f(x) + f(y) - f(x + y) = 0$, and $f$ is linear. If finally $f$ were not continuous, then $f^{-1}(a)$ would be dense in $E$ for every $a \in K$. Hence $f(U) = K$ for any neighborhood $U$ of zero, so $\varphi(U) = \pi \circ f(U) = \pi(K)$ for all such $U$ and $\varphi$ would not be continuous.

Now the map $E' \to \hat{E}$ is an isomorphism, and it is obviously continuous; we need only prove it is open. The map $K' \to \hat{K}$ is a homeomorphism, since (as we noted in the proof of Theorem 1) $\hat{K} \cong K$. Hence, given any neighborhood $U$ of zero in $K$, we can find an open $V$ and a compact set $B$ such that, for $g$ in $K'$, $\pi \circ g(B) \subset V$ implies $g(a) \in U$ for $|a| \leq 1$. But if $C$ is any compact set in $E$, $BC$ will again be compact. It is easy to see then that if $f \in E'$ and $\pi \circ f(BC) \subset V$, then $f(C) \subset U$; this means that $E' \to \hat{E}$ is open.

Let $K$ again be locally compact, and let $E$ be a locally convex topological vector space over $K$. (In the archimedean case, the requisite theory is standard, cf. [2]; van Tiel has shown that exactly the same theory holds in the nonarchimedean case [6].) In view of Theorem 2, we identify $E'$ and $\hat{E}$ furnished with the topology of uniform convergence on compact sets.

**Theorem 3.** If $E$ is quasi-complete and barrelled, then $E$ is topologically isomorphic to $(\hat{E})^\circ$.

**Proof.** Since $E$ is locally convex, the map $E \to (\hat{E})^\circ$ is injective. Since $E$ is quasi-complete, the closed convex hull of a compact set is compact; thus the topology on $\hat{E}$ is that of uniform convergence on convex compact sets. This is weaker than the Mackey topology, and hence the map $E \to (\hat{E})^\circ$ is bijective.

If $S$ is a compact balanced set in $\hat{E}$, then its polar $S^\circ$ is a barrel in $E$, and hence is a neighborhood of 0 in $E$. These polars are a neighborhood basis at 0 in $(\hat{E})^\circ$, so the map $E \to (\hat{E})^\circ$ is continuous.

Finally, if $U$ is a neighborhood of 0 in $E$, $U^\circ$ is equicontinuous and therefore compact in $\hat{E}$; hence $U^{\circ \circ}$ is a neighborhood of 0 in $(\hat{E})^\circ$. But $E$ has a neighborhood basis at 0 consisting of closed absolutely convex sets $U$, and for them $U = U^{\circ \circ}$. Thus the map is open.

As particular cases of Theorem 3, we get

**Corollary.** If $E$ is either complete and metrizable or reflexive, $E$ is topologically isomorphic to $(\hat{E})^\circ$.

For the real and complex fields, Theorem 2 and these two cases of Theorem 3 were proved by M. F. Smith [5].
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