

# Pacific Journal of Mathematics

**DUAL GROUPS OF VECTOR SPACES**

WILLIAM CHARLES WATERHOUSE

## DUAL GROUPS OF VECTOR SPACES

WILLIAM C. WATERHOUSE

Let  $E$  be a topological vector space over a field  $K$  having a nontrivial absolute value. Let  $E'$  be the dual space of continuous linear maps  $E \rightarrow K$ , and  $\hat{E}$  the dual group of continuous characters  $E \rightarrow R/Z$ .  $\hat{E}$  is a vector space over  $K$  by  $(\alpha\varphi)(x) = \varphi(\alpha x)$ , and composition with a nonzero character of  $K$  is a linear map of  $E'$  into  $\hat{E}$ . This map is always an isomorphism if  $K$  is locally compact, while if  $K$  is not locally compact it is never an isomorphism unless  $\hat{E} = 0$ . When  $K$  is locally compact,  $E'$  is in addition topologically isomorphic to  $\hat{E}$  if each is given its topology of uniform convergence on compact sets. This leads to conditions on  $E$  which imply that  $E$  is topologically isomorphic to  $(\hat{E})^\wedge$ .

**THEOREM 1.** *Let  $K$  be a field with absolute value. Then  $\hat{K}$  is one-dimensional over  $K$  if and only if  $K$  is locally compact.*

*Proof.* The sufficiency of local compactness is fairly well known (cf. [4, p. 92-3] for the characteristic zero case). To prove it, one takes a nonzero character  $\pi$  of  $K$  and considers the subspace  $K\pi$  of  $\hat{K}$ . It is easy to check that  $a \mapsto a\pi$  is a bicontinuous linear map, so  $K\pi$  is complete and hence closed in  $\hat{K}$ . On the other hand,  $K\pi$  separates the points of  $K$ , so by Pontrjagin duality it is dense in  $\hat{K}$ . Thus  $\hat{K} = K\pi$ .

Suppose conversely that  $\hat{K}$  is one-dimensional, and choose a nonzero  $\pi$  in  $\hat{K}$ . The completion of  $K$  will again be a field, say  $L$ , and  $\pi$  extends to a character of  $L$ . Then every  $a \in L$  gives a character  $a\pi$  of  $L$ . If  $a \neq b$ , then  $a - b$  is invertible, and so  $\pi((a - b)c)$  cannot be zero for all  $c$ . Thus no two of the characters  $a\pi$  are equal, and hence no two can agree on the dense set  $K$ . This contradicts one-dimensionality of  $\hat{K}$  unless  $K = L$ , and we conclude that  $K$  must be complete. Hence if  $K$  is archimedean, it is locally compact.

We now assume that  $K$  is nonarchimedean. Let  $A = \{x: |x| \leq 1\}$ ,  $M = \{x: |x| < 1\}$ . Let  $\pi$  be a character of the discrete group  $A/M$  with  $\pi(1) \neq 0$ ; we extend  $\pi$  to a character of the discrete group  $K/M$  and interpret it as an element of  $\hat{K}$ . Let  $c > 1$  be an element of the value group, and consider the group  $G_c/M$ , where  $G_c = \{x: |x| \leq c\}$ . All characters of this discrete group extend to characters of  $K$  vanishing on  $M$ , and by one-dimensionality they all come from multiples of  $\pi$ .

Now if  $a \in A$ , then  $aM \subset M$ , so  $a\pi$  vanishes on  $M$ ; conversely, if  $a\pi$  vanishes on  $M$ , then  $1/a \notin M$  and  $a \in A$ . Similarly,  $a\pi$  vanishes

on  $G_c$  if and only if  $|a| < 1/c$ . Thus the dual group of the discrete abelian group  $G_c/M$  is (algebraically) isomorphic to  $A/\{a: |a| < 1/c\}$ , which is isomorphic to  $G_c/M$  itself under multiplication by an element of absolute value  $c$ . A theorem of Kakutani [3, p. 396-7] shows that an infinite discrete abelian group has a dual group of strictly larger cardinality; hence  $G_c/M$  must be finite. This implies both that  $A/M$  is finite and that the value group is discrete; since  $K$  has these two properties and is complete, it is locally compact [1, p. 119].

**COROLLARY.** *Suppose  $K$  is not locally compact. Let  $E$  be a topological vector space over  $K$  with  $\hat{E} \neq 0$ . For any  $\pi \in \hat{K}$ , the map  $E' \rightarrow \hat{E}$  given by composition with  $\pi$  fails to be surjective.*

*Proof.* If  $E' = 0$  or  $\pi = 0$ , the statement is obvious. Suppose then there is a  $0 \neq f \in E'$ , and choose an  $x \in E$  with  $f(x) \neq 0$ . The subspace  $Kx$  is topologically isomorphic to  $K$ , so its dual space is one-dimensional and is generated by the restriction of  $f$ . Hence all elements in  $\hat{E}$  coming from  $E'$  restrict to multiples of  $\pi \circ f$  on  $Kx$ . If  $\tau$  is a character of  $K$  not a multiple of  $\pi$ , then  $\tau \circ f \in \hat{E}$  is not in the image of  $E'$ .

**REMARK.** A topological field  $K$  is called *locally retrobounded* if for every pair of neighborhoods  $U, V$  of zero there is an  $a \neq 0$  in  $K$  such that  $a\{x^{-1}: x \in V\} \subset U$ ; for example, an ordered field is locally retrobounded in its order topology. Every such field admits either an absolute value or a valuation which defines its topology [1, §5, Exerc. 2]. The proof of Theorem 1 works equally well for a valuation into any ordered abelian group, and hence Theorem 1 and its corollary hold for all locally retrobounded fields.

**THEOREM 2.** *Suppose  $K$  is locally compact,  $0 \neq \pi \in \hat{K}$ . Let  $E$  be a topological vector space over  $K$ . Then the map  $E' \rightarrow \hat{E}$  given by  $f \mapsto \pi \circ f$  is a vector space isomorphism. It is a homeomorphism if  $E'$  and  $\hat{E}$  have their topologies of uniform convergence on compact sets.*

*Proof.* If  $0 \neq f$ , then  $f(E) = K$ , so  $\pi \circ f \neq 0$ ; thus the map is injective. Now let  $\varphi \in \hat{E}$ . For each  $x \in E$  there is a unique linear functional on  $Kx$  inducing  $\varphi|Kx$ , since  $Kx \cong K$  and  $\hat{K}$  is one-dimensional. We define  $f(x)$  to be this functional evaluated at  $x$ ; this gives us a homogeneous function  $f: E \rightarrow K$ . For any  $x, y \in E$  and  $a \in K$ , we have

$$\begin{aligned} 0 &= \varphi(ax) + \varphi(ay) - \varphi(ax + ay) = \pi f(ax) + \pi f(ay) - \pi f(ax + ay) \\ &= \pi[f(ax) + f(ay) - f(ax + ay)] = \pi(a[f(x) + f(y) - f(x + y)]); \end{aligned}$$

hence  $f(x) + f(y) - f(x + y) = 0$ , and  $f$  is linear. If finally  $f$  were not continuous, then  $f^{-1}(a)$  would be dense in  $E$  for every  $a \in K$ . Hence  $f(U) = K$  for any neighborhood  $U$  of zero, so  $\varphi(U) = \pi \circ f(U) = \pi(K)$  for all such  $U$  and  $\varphi$  would not be continuous.

Now the map  $E' \rightarrow \hat{E}$  is an isomorphism, and it is obviously continuous; we need only prove it is open. The map  $K' \rightarrow \hat{K}$  is a homeomorphism, since (as we noted in the proof of Theorem 1)  $\hat{K} \cong K$ . Hence, given any neighborhood  $U$  of zero in  $K$ , we can find an open  $V$  and a compact set  $B$  such that, for  $g$  in  $K'$ ,  $\pi \circ g(B) \subset V$  implies  $g(a) \in U$  for  $|a| \leq 1$ . But if  $C$  is any compact set in  $E$ ,  $BC$  will again be compact. It is easy to see then that if  $f \in E'$  and  $\pi \circ f(BC) \subset V$ , then  $f(C) \subset U$ ; this means that  $E' \rightarrow \hat{E}$  is open.

Let  $K$  again be locally compact, and let  $E$  be a locally convex topological vector space over  $K$ . (In the archimedean case, the requisite theory is standard, cf. [2]; van Tiel has shown that exactly the same theory holds in the nonarchimedean case [6].) In view of Theorem 2, we identify  $E'$  and  $\hat{E}$  furnished with the topology of uniform convergence on compact sets.

**THEOREM 3.** *If  $E$  is quasi-complete and barrelled, then  $E$  is topologically isomorphic to  $(\hat{E})^\wedge$ .*

*Proof.* Since  $E$  is locally convex, the map  $E \rightarrow (\hat{E})^\wedge$  is injective. Since  $E$  is quasi-complete, the closed convex hull of a compact set is compact; thus the topology on  $\hat{E}$  is that of uniform convergence on convex compact sets. This is weaker than the Mackey topology, and hence the map  $E \rightarrow (\hat{E})^\wedge$  is bijective.

If  $S$  is a compact balanced set in  $\hat{E}$ , then its polar  $S^\circ$  is a barrel in  $E$ , and hence is a neighborhood of 0 in  $E$ . These polars are a neighborhood basis at 0 in  $(\hat{E})^\wedge$ , so the map  $E \rightarrow (\hat{E})^\wedge$  is continuous.

Finally, if  $U$  is a neighborhood of 0 in  $E$ ,  $U^\circ$  is equicontinuous and therefore compact in  $\hat{E}$ ; hence  $U^{\circ\circ}$  is a neighborhood of 0 in  $(\hat{E})^\wedge$ . But  $E$  has a neighborhood basis at 0 consisting of closed absolutely convex sets  $U$ , and for them  $U = U^{\circ\circ}$ . Thus the map is open.

As particular cases of Theorem 3, we get

**COROLLARY.** *If  $E$  is either complete and metrizable or reflexive,  $E$  is topologically isomorphic to  $(\hat{E})^\wedge$ .*

For the real and complex fields, Theorem 2 and these two cases of Theorem 3 were proved by M. F. Smith [5].

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