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EXTENSIONS OF OPIAL'S INEQUALITY

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In this paper certain inequalities involving integrals of powers of a function and of its derivative are proved. The prototype of such inequalities is Opial's Inequality which states that $2 \int_0^x |yy'| dx \leq X \int_0^x y'^2 dx$ whenever y is absolutely continuous on $[0, X]$ with $y(0) = 0$. The extensions dealt with here are all integral inequalities of the form

$$\int_a^b s |y|^p |y'|^q dx \leq K(p, q) \int_a^b r |y'|^{p+q} dx,$$

(or with \leq replaced by \geq), where r, s are nonnegative, measurable functions on $I = [a, b]$, and y is absolutely continuous on I with either $y(a) = 0$, or $y(b) = 0$, or both. In some cases y may be complex-valued, while in other cases y' must not change sign on I . The inequality (as stated) is obtained in case $pq > 0$ and either $p + q \geq 1$ or $p + q < 0$, while the opposite inequality is obtained in case $p < 0, q \geq 1, p + q < 0$, or $p > 0, p + q < 0$. In all cases, necessary and sufficient conditions are obtained for equality to hold.

1. In a recent paper [11], G. S. Yang proved the following generalization of an inequality of Z. Opial [7]:

If y is absolutely continuous on $[a, X]$ with $y(a) = 0$, and if $p, q \geq 1$, then

$$(1) \quad \int_a^x |y|^p |y'|^q dx \leq \frac{q}{p+q} (X-a)^p \int_a^x |y'|^{p+q} dx.$$

Yang's proof is actually valid for $p \geq 0, q \geq 1$. For $p = q = 1, a = 0$, (1) is Opial's result. (See also Olech [6], Beesack [1], Levinson [4], Mallows [5], and Pederson [8] for successively simpler proofs of Opial's inequality; as well as Redheffer [9] for other generalizations of this inequality.) The case $q = 1, p$ a positive integer, was proved by Hua [3], and the result for $q = 1, p \geq 0$ is included in a generalization of Calvert [2]; a short, direct proof of the latter case was also given by Wong [10]. If $q = 1$ the inequality (1) is sharp, but it is not sharp for $q > 1$.

2. The purpose of this paper is to obtain sharp generalizations of (1), and to consider other values of the parameters p, q ; the method of proof is a modification of that of Yang [11]. To this end, we suppose first that y is absolutely continuous on $[a, X]$, where $-\infty \leq a < X \leq \infty$, and that y' does not change sign on (a, X) , so that

$$(2) \quad |y(x)| = \int_a^x |y'(t)| dt, \quad a \leq x \leq X.$$

If r is nonnegative on (a, X) and the integrals exist, then it follows from Hölder's inequality that

$$(3) \quad \int_a^x |y'| dt \leq \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{(p+q-1)/(p+q)} \left(\int_a^x r |y'|^{p+q} dt \right)^{1/(p+q)}$$

if $p + q > 1$, while

$$(4) \quad \int_a^x |y'| dt \geq \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{(p+q-1)/(p+q)} \left(\int_a^x r |y'|^{p+q} dt \right)^{1/(p+q)}$$

if either $p + q < 0$ or $0 < p + q < 1$. Taking the case $p + q > 1$, we suppose first that $p > 0, q > 0$. Then,

$$(5) \quad |y|^p \leq \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p(p+q-1)/(p+q)} \left(\int_a^x r |y'|^{p+q} dt \right)^{p/(p+q)} \\ a \leq x \leq X.$$

Now, set $z(x) = \int_a^x r |y'|^{p+q} dt$. So $z' = r |y'|^{p+q}$, and

$$|y'|^q = r^{-\{q/(p+q)\}} (z')^{q/(p+q)}.$$

Thus, if s is nonnegative on (a, X) ,

$$s |y|^p |y'|^q \leq s r^{-\{q/(p+q)\}} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p(p+q-1)/(p+q)} z^{p/(p+q)} (z')^{q/(p+q)}.$$

If we assume the existence of the following integrals, then applying Hölder's inequality again, with indices $(p+q)/p$ and $(p+q)/q$, we obtain

$$(6) \quad \int_a^X s |y|^p |y'|^q dx \leq K_1(X, p, q) \left(\int_a^X z^{p/q} z' dx \right)^{q/(p+q)} \\ = K_1(X, p, q) \int_a^X r |y'|^{p+q} dx,$$

since $z(a) = 0$ and $(p+q)/q > 0$. Here,

$$(7) \quad K_1(X, p, q) \\ = \left(\frac{q}{p+q} \right)^{q/(p+q)} \left\{ \int_a^X s^{(p+q)/p} r^{-\{q/p\}} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p+q-1} dx \right\}^{p/(p+q)}.$$

Similarly, if $p < 0$ and $q < 0$, then (5) again follows from (2) and (4). As above, since $(p+q)/p > 1$ and $(p+q)/q > 1$ again, we obtain inequality (6). This proves the main part of

THEOREM 1. *Let p, q be real numbers such that $pq > 0$, and*

either $p + q > 1$, or $p + q < 0$, and let r, s be nonnegative, measurable functions on (a, X) such that $\int_a^X r^{-1/(p+q-1)} dx < \infty$, and the constant $K_1(X, p, q)$ defined by (7) is finite, where $-\infty \leq a < X \leq \infty$. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X) , then

$$(8) \quad \int_a^X s |y|^p |y'|^q dx \leq K_1(X, p, q) \int_a^X r |y'|^{p+q} dx.$$

Equality holds in (8) if and only if either $q > 0$ and $y \equiv 0$, or

$$(9) \quad s = k_1 r^{(q-1)/(p+q-1)} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p(1-q)/q},$$

and

$$y = k_2 \int_a^x r^{-\{1/(p+q-1)\}} dt,$$

for some constants $k_1 (\geq 0)$, k_2 real.

It only remains to prove the assertion concerning (9). Now, equality holds in (8) only if it holds in (3)—or (4)—and in Hölder's inequality leading to (6); that is, only if both

$$r |y'|^{p+q} = A r^{-\{1/(p+q-1)\}} \quad \text{or} \quad y' = k_2 r^{-\{1/(p+q-1)\}},$$

and

$$z^{p/q} z' = B s^{(p+q)/p} r^{-(q/p)} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p+q-1}.$$

The first of these conditions is equivalent to the second of equations (9) since $y(a) = 0$. Using this condition and the definition of z , the second reduces to

$$R^{(p+q)(1-q)/q} = C s^{(p+q)/p} (R')^{(p+q)(q-1)/q}, \quad \left(R \equiv \int_a^x r^{-\{1/(p+q-1)\}} dt \right),$$

which is equivalent to the first of equations (9). Finally, if s is given by (9), it is easy to verify that the corresponding value of K_1 in (7) is

$$k_1 \frac{q}{p+q} \left(\int_a^X r^{-\{1/(p+q-1)\}} dt \right)^{p/q},$$

and hence is finite. Similarly, choosing y as in (9),

$$\int_a^X r |y'|^{p+q} dx = |k_2|^{p+q} \int_a^X r^{-\{1/(p+q-1)\}} dx < \infty,$$

completing the proof of the theorem.

COROLLARY 1. *If $pq > 0$, $p + q > 1$, (8) holds even if y is complex-valued. Equality holds if and only if s and y are given by (9) with $k_1 \geq 0$, k_2 complex.*

Proof. The inequality (8) follows as above but in place of (2) we have

$$|y(x)| \leq \int_a^x |y'(t)| dt, \quad a \leq x \leq X.$$

Equality holds in (8) only if, in addition to

$$|y'| = Ar^{-\{1/(p+q-1)\}}, \quad z^{p/q} z' = Bs^{(p+q)/p} r^{-(q/p)} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p+q-1},$$

we also have

$$|y(x)| = \int_a^x |y'(t)| dt;$$

thus only if

$$y(x) = \left(A \int_a^x r^{-\{1/(p+q-1)\}} dt \right) e^{i\theta(x)},$$

which, in view of the condition on $|y'|$, leads to $\theta'(x) \equiv 0$ and, therefore, only if

$$y = Ae^{i\alpha} \int_a^x r^{-\{1/(p+q-1)\}} dt = k_2 \int_a^x r^{-\{1/(p+q-1)\}} dt.$$

The rest follows as before.

REMARK 1. If $pq > 0$ and $p + q = 1$, then in place of (5) we have

$$|y|^p \leq M^p \left(\int_a^x r |y'| dt \right)^p,$$

where $M(x) = \text{ess. sup}_{t \in [a, x]} r^{-1}(t)$ and r is a positive, measurable function on (a, X) . Therefore, if

$$\tilde{K}_1(X, p, q) = q^q \left\{ \int_a^X Ms^{1/p} r^{-(q/p)} dx \right\}^p < \infty,$$

then

$$(10) \quad \int_a^X s |y|^p |y'|^q dx \leq \tilde{K}_1(X, p, q) \int_a^X r |y'| dx.$$

As in the corollary above, equality holds in (10) if and only if $y \equiv 0$, or

$$r = \text{const.} > 0 \quad \text{and} \quad y = k \left(\int_a^x s^{1/p} dt \right)^q,$$

k complex.

We only state the next theorem, since its proof is the same as that of Theorem 1, with $[a, x]$ replaced by $[x, b]$ throughout.

THEOREM 2. *Let p, q be real numbers satisfying the same conditions as in Theorem 1, and let r, s be nonnegative measurable functions on (X, b) , where $-\infty \leq X < b \leq \infty$, such that $\int_X^b r^{-1/(p+q-1)} dx < \infty$, and*

$$(11) \quad K_2(X, p, q) = \left(\frac{q}{p+q} \right)^{q/(p+q)} \left\{ \int_X^b s^{(p+q)/p} r^{-(q/p)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt \right)^{p+q-1} dx \right\}^{p/(p+q)}$$

is finite. If y is absolutely continuous on $[X, b]$, $y(b) = 0$, (and y' does not change sign on (X, b) in case $q < 0$), then

$$(12) \quad \int_X^b s |y|^p |y'|^q dx \leq K_2(X, p, q) \int_X^b r |y'|^{p+q} dx.$$

Equality holds in (12) if and only if either $q > 0$ and $y \equiv 0$, or

$$s = k_3 r^{(q-1)/(p+q-1)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt \right)^{p(1-q)/q},$$

and

$$y = k_4 \int_x^b r^{-\{1/(p+q-1)\}} dt,$$

for some constants $k_3 (\geq 0), k_4$ real.

REMARK 2. As above, if $pq > 0$ and $p+q > 1$, then (12) holds even if y is complex-valued. Also, if $p+q = 1$, r is a positive, measurable function on (X, b) , $\hat{M}(x) = \text{ess. sup}_{t \in [x, b]} r^{-1}(t)$ and

$$\tilde{K}_2(X, p, q) = q^q \left\{ \int_X^b \hat{M} s^{1/p} r^{-(q/p)} dx \right\}^p < \infty,$$

then

$$(13) \quad \int_X^b s |y|^p |y'|^q dx \leq \tilde{K}_2(X, p, q) \int_X^b r |y'| dx,$$

where y is again complex-valued. Equality holds if and only if $r = \text{const.} > 0$ and $y = \hat{k} \left(\int_x^b s^{1/p} dt \right)^q$.

COROLLARY 2. *Let $pq > 0$ with $p + q > 1$, and let r, s be non-negative, measurable functions on (a, b) , where $-\infty \leq a < b \leq \infty$, such that $\int_a^b r^{-\{1/(p+q-1)\}} dx < \infty$, and*

$$(14) \quad (K(p, q) \equiv) K_1(X_1, p, q) = K_2(X, p, q) < \infty ,$$

where K_1, K_2 are defined by (7), (11) respectively, and $X(a < X < b)$ is the (unique) solution of equation (14). If y is complex-valued, absolutely continuous on $[a, b]$, with $y(a) = y(b) = 0$, then

$$(15) \quad \int_a^b s |y|^p |y'|^q dx \leq K(p, q) \int_a^b r |y'|^{p+q} dx .$$

Moreover, equality holds if and only if either $y \equiv 0$, or

$$s = \begin{cases} \alpha_1 r^{(q-1)/(p+q-1)} \left(\int_a^x r^{-\{1/(p+q-1)\}} dt \right)^{p(1-q)/q}, & a \leq x < X, \\ \alpha_2 r^{(q-1)/(p+q-1)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt \right)^{p(1-q)/q}, & X < x \leq b, \end{cases}$$

and

$$y = \begin{cases} \beta_1 \int_a^x r^{-\{1/(p+q-1)\}} dt, & a \leq x \leq X, \\ \beta_2 \int_x^b r^{-\{1/(p+q-1)\}} dt, & X \leq x \leq b, \end{cases}$$

where α_1, α_2 are nonnegative constants, and β_1, β_2 are complex constants such that

$$\beta_1 \int_a^X r^{-\{1/(p+q-1)\}} dt = \beta_2 \int_X^b r^{-\{1/(p+q-1)\}} dt .$$

Proof. The conclusion follows from Corollary 1 and Theorem 2 since, on choosing X to be the unique solution of equation (14), we have

$$\begin{aligned} \int_a^b s |y|^p |y'|^q dx &= \int_a^X s |y|^p |y'|^q dx + \int_X^b s |y|^p |y'|^q dx \\ &\leq K_1(X, p, q) \int_a^X r |y'|^{p+q} dx + K_2(X, p, q) \int_X^b r |y'|^{p+q} dx \\ &= K(p, q) \int_a^b r |y'|^{p+q} dx . \end{aligned}$$

Moreover, equality holds in (15) if and only if it holds in both (8) and (12).

REMARK 3. As before, if $pq > 0$ and $p + q = 1$, then for r a positive, measurable function on (a, b) ,

$$(16) \quad \int_a^b |s| |y|^p |y'|^q dx \leq \tilde{K}(p, q) \int_a^b r |y'| dx,$$

where

$$(\tilde{K}(p, q) \equiv) \tilde{K}_1(X, p, q) = \tilde{K}_2(X, p, q).$$

Equality holds in (16) if and only if either $y \equiv 0$, or

$$r(x) = \begin{cases} c_1(>0), & a \leq x < X, \\ c_2(>0), & X < x \leq b, \end{cases} \quad \text{and} \quad y = \begin{cases} \gamma_1 \left(\int_a^x s^{1/p} dt \right)^q, & a \leq x \leq X, \\ \gamma_2 \left(\int_x^b s^{1/p} dt \right)^q, & X \leq x \leq b, \end{cases}$$

where

$$\gamma_1 \left(\int_a^x s^{1/p} dt \right)^q = \gamma_2 \left(\int_x^b s^{1/p} dt \right)^q.$$

EXAMPLES

1. Setting $r = s \equiv 1$ in (8) or (10), we obtain as an improvement of (1),

$$(17) \quad \int_a^x |y|^p |y'|^q dx \leq \frac{q^{q/(p+q)}}{p+q} (X-a)^p \int_a^x |y'|^{p+q} dx$$

if $pq > 0$, $p+q \geq 1$. It may be remarked that (17) is also true if $p = 0$. Equality holds in (17) in case $p+q > 1$ if and only if either $p = 0$, or else $y \equiv 0$, or else $q = 1$ and $y = A(x-a)$; if $p+q = 1$, equality holds if and only if $y = A(x-a)$. In case $q = 1$, (17) reduces to the results of Hua, Yang, Calvert and Wong, while Opial's original inequality is obtained for $p = q = 1$. (Note that if $p < 0$ and $q < 0$, $K_1(X, p, q) = \infty$.)

2. Taking $q = 1$, $s \equiv 1$ in (15), we obtain

$$(18) \quad \int_a^b |y^p y'| dx \leq \frac{1}{p+1} \left(\int_a^X r^{-(1/p)} dx \right)^p \int_a^b r |y'|^{p+1} dx,$$

if $p \geq 0$, and y is complex-valued, absolutely continuous on $[a, b]$ with $y(a) = y(b) = 0$. Here, X is the unique solution of

$$\int_a^X r^{-(1/p)} dx = \int_X^b r^{-(1/p)} dx, \quad \int_a^b r^{-(1/p)} dx < \infty.$$

Equality holds in (18) if and only if $y = A \int_a^x r^{-(1/p)} dt$ for $a \leq x \leq X$ and $y = B \int_x^b r^{-(1/p)} dt$ for $X \leq x \leq b$. In case $p = 1$, (18) reduces to a result of Beesack [2].

3. Taking $r \equiv 1$, $s \equiv (x-a)^{p(1-q)/q}$ in Theorem 1,

$$(19) \quad \int_a^X (x-a)^{p(1-q)/q} |y|^p |y'|^q dx \leq \frac{q}{p+q} (X-a)^{p/q} \int_a^X |y'|^{p+q} dx.$$

Equality holds if and only if either $q > 0$ and $y \equiv 0$, or $y = A(x-a)$. As a special case of (19), let $y = u^{1/2}$, $p = q = -1$, $a = 0$. Then

$$\int_0^X \frac{x^2}{|u'|} dx < X \int_0^X \frac{|u|}{|u'|^2} dx \quad \text{unless } u = Ax^2.$$

4. Taking $r \equiv (x-a)^{p(p+q-1)/(p+q)}$, $s \equiv 1$ in Theorem 1,

$$(20) \quad \int_a^X |y|^p |y'|^q dx \leq \left(\frac{q}{p+q} \right)^{1-p} (X-a)^{p/(p+q)} \int_a^X (x-a)^{p(p+q-1)/(p+q)} |y'|^{p+q} dx.$$

Equality holds if and only if either $q > 0$ and $y \equiv 0$, or $y = A(x-a)^{q/(p+q)}$. As a special case of (20), let $y = u^{1/2}$, $p = q = -1$, $a = 0$. Then

$$\int_0^X \frac{dx}{|u'|} < \frac{1}{2} X^{1/2} \int_0^X \frac{x^{-3/2} |u|}{|u'|^2} dx \quad \text{unless } u = Ax.$$

3. To obtain lower bounds for $\int_a^X s |y|^p |y'|^q dx$ (or $\int_a^b s |y|^p |y'|^q dx$) consider first the case when $p+q > 1$. If, in addition, $p < 0$, (3) yields

$$(21) \quad |y|^p \geq \left(\int_a^x r^{-(1/(p+q-1))} dt \right)^{p(p+q-1)/(p+q)} \left(\int_a^x r |y'|^{p+q} dt \right)^{p/(p+q)}.$$

If s is non-negative on (a, X) , then

$$s |y|^p |y'|^q \geq s r^{-(q/(p+q))} \left(\int_a^x r^{-(1/(p+q-1))} dt \right)^{p(p+q-1)/(p+q)} z^{p/(p+q)} (z')^{q/(p+q)},$$

where $z(x) = \int_a^x r |y'|^{p+q} dt$.

Thus, Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$ —note that the latter lies between 0 and 1—gives

$$(22) \quad \int_a^X s |y|^p |y'|^q dx \geq K_1(X, p, q) \int_a^X r |y'|^{p+q} dx,$$

where $K_1(X, p, q)$ is defined by (7).

Similarly, if $p > 0$ and $p+q < 0$, then (4) yields (21). Again, if s is non-negative on (a, X) , Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$ —note that $0 < (p+q)/q < 1$ still holds—leads to (22). Equality holds in (22) if and only if it holds in (3)—or (4)—and in Hölder's inequality leading to (22); that is, if and only if s, y are given by (9). This proves

THEOREM 3. *Let p, q be real numbers such that either $p < 0$ and*

$p + q > 1$, or $p > 0$ and $p + q < 0$. Let r, s be nonnegative measurable functions on (a, X) such that $\int_a^X r^{-1/(p+q-1)} dx < \infty$, and the constant $K_1(X, p, q)$ defined by (7) is finite, where $-\infty \leq a < X \leq \infty$. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X) , then (22) holds. There is equality in (22) if and only if s and y are as defined in (9).

COROLLARY 3. If $p < 0$ and $p + q > 1$, (22) holds even if y is complex-valued. Equality holds if and only if s and y are given by (9) with $k_1 \geq 0, k_2$ complex.

The proof of this is essentially the same as that of Corollary 1.

REMARK 4. If $p < 0$ and $p + q = 1$, then in place of (21) we have

$$|y|^p \geq M^p \left(\int_a^x r |y'| dt \right)^p,$$

where $M(x) = \text{ess sup}_{t \in [a, x]} r^{-1}(t)$ and r is a positive, measurable function on (a, X) .

Thus, if

$$\tilde{K}_1(X, p, q) = q^q \left\{ \int_a^X M s^{1/p} r^{-(q/p)} dx \right\}^p < \infty,$$

then

$$(23) \quad \int_a^X s |y|^p |y'|^q dx \geq \tilde{K}_1(X, p, q) \int_a^X r |y'| dx.$$

As in the corollary above, equality holds in (23) if and only if

$$r = \text{const.} > 0 \quad \text{and} \quad y = k \left(\int_a^x s^{1/p} dt \right)^q,$$

k complex.

Replacing $[a, x]$ by $[x, b]$ throughout Theorem 3, we obtain

THEOREM 4. Let p, q be real numbers satisfying the same conditions as in Theorem 3, and let r, s be non-negative measurable functions on (X, b) , where $-\infty \leq X < b \leq \infty$, such that $\int_X^b r^{-1/(p+q-1)} dx < \infty$, and $K_2(X, p, q)$ defined by (11) is finite. If y is absolutely continuous on $[X, b]$, $y(b) = 0$, (and y' does not change sign on (X, b) in case $p > 0$), then

$$(24) \quad \int_X^b s |y|^p |y'|^q dx \geq K_2(X, p, q) \int_X^b r |y'|^{p+q} dx.$$

Equality holds in (24) if and only if

$$(25) \quad \begin{aligned} s &= k_3 r^{(q-1)/(p+q-1)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt \right)^{p(1-q)/q}, \text{ and} \\ y &= k_4 \int_x^b r^{-\{1/(p+q-1)\}} dt, \end{aligned}$$

for some constants $k_3 (\geq 0)$, k_4 real.

REMARK 5. If $p < 0$ and $p + q > 1$, then (24) holds even if y is complex-valued. Also, if $p < 0$, $p + q = 1$ and r is a positive, measurable function on (X, b) , and

$$\hat{M}(x) = \operatorname{ess\,sup}_{t \in [x, b]} r^{-1}(t), \quad \tilde{K}_2(X, p, q) = q^q \int_X \hat{M} s^{1/p} r^{-(q/p)} dx \Big\} < \infty,$$

then

$$(26) \quad \int_X s |y|^p |y'|^q dx \geq \tilde{K}_2(X, p, q) \int_X r |y'| dx,$$

where y is again complex-valued. Equality holds if and only if $r = \operatorname{const.} > 0$ and $y = \hat{k} \left(\int_x^b s^{1/p} dt \right)^q$.

COROLLARY 4. Let $p < 0$ and $p + q > 1$. Let r, s be nonnegative, measurable functions on (a, b) , $-\infty \leq a < b \leq \infty$, such that $\int_a^b r^{-\{1/(p+q-1)\}} dx$ is finite. Let y be complex-valued, absolutely continuous on $[a, b]$ with $y(a) = y(b) = 0$. Then,

$$(27) \quad \int_a^b s |y|^p |y'|^q \geq K(p, q) \int_a^b r |y'|^{p+q} dx,$$

where $K(p, q)$ is defined by (14). Moreover, equality holds if and only if s and y are defined as in theorem 2.

The proof is immediate in view of Theorems 3 and 4, Corollary 3 and Remark 5.

REMARK 6. Again if $p < 0$ and $p + q = 1$, then for $r(x)$ positive, measurable on (a, b) ,

$$(28) \quad \int_a^b s |y|^p |y'|^q dx \geq \tilde{K}(p, q) \int_a^b r |y'| dx,$$

where $\tilde{K}(p, q)$ is defined as in Remark 3. Further, equality holds in (28) if and only if r and y are defined as in Remark 3.

Our next result is an extension of Theorem 3 to the case when $0 < p + q < 1$ and $q > 1$. (Note that in Theorem 3 the restriction $q > 1$ is implicit since $p + q > 1$ and $p < 0$ imply $q > 1$.)

THEOREM 5. *Let $p < 0, q > 1$ and $0 < p + q < 1$. Let r, s be non-negative, measurable functions on (a, X) such that $\int_a^X r^{-\{1/(p+q-1)\}} dx$ and $\int_a^X s^{-\{1/(q-1)\}} dx$ are finite. If y is complex-valued, absolutely continuous on $[a, X]$, $y(a) = 0$, then*

$$(29) \quad \int_a^X s |y|^p |y'|^q dx \geq \hat{K}_1(X, p, q) \int_a^X r |y'|^{p+q} dx,$$

where

$$(30) \quad \hat{K}_1(X, p, q) = \left(\frac{q}{p+q} \right)^q \left(\int_a^X s^{-\{1/(q-1)\}} dx \right)^{1-q} \left(\int_a^X r^{-\{1/(p+q-1)\}} dx \right)^{p+q-1}.$$

Equality holds in (29) if and only if s and y are as defined by (9) with k_2 complex.

Proof. Since $p/q < 0$,

$$|y|^{p/q} \geq \left(\int_a^x |y'| dt \right)^{p/q}, \quad a \leq x \leq X.$$

Therefore,

$$(31) \quad \int_a^X |y|^{p/q} |y'| dx \geq \frac{q}{p+q} \left(\int_a^X |y'| dx \right)^{(p+q)/q}.$$

From Hölder's inequality with indices q and its conjugate, it follows that

$$\int_a^X |y|^{p/q} |y'| dx \leq \left(\int_a^X s^{-\{1/(q-1)\}} dx \right)^{(q-1)/q} \left(\int_a^X s |y|^p |y'|^q dx \right)^{1/q};$$

and also with indices $p+q$ and its conjugate, that

$$\int_a^X |y'| dx \geq \left(\int_a^X r^{-\{1/(p+q-1)\}} dx \right)^{(p+q-1)/(p+q)} \left(\int_a^X r |y'|^{p+q} dx \right)^{1/(p+q)}$$

In view of the above inequalities, (29) follows from (31).

Again, equality holds in (29) if and only if

$$|y| = \int_a^x |y'| dt, \quad A_1 s^{-\{1/(q-1)\}} = s |y|^p |y'|^q,$$

and

$$A_2 r^{-\{1/(p+q-1)\}} = r |y'|^{p+q};$$

that is, if and only if

$$|y'| = a_2 r^{-\{1/(p+q-1)\}}, \quad |y| = a_2 \int_a^x r^{-\{1/(p+q-1)\}} dt,$$

and

$$s = k_3 r^{(q-1)/(p+q-1)} \left(\int_a^x r^{-1/(p+q-1)} dt \right)^{p(1-q)/q};$$

thus, as in Corollary 1, if and only if s and y are as defined by (9) with k_4 complex.

REMARK 7. If $p < 0$, $0 < p + q < 1$ and $q = 1$, $s(x)$ positive and measurable on (a, X) , then in place of (29) the following holds:

$$(32) \quad \int_a^X s |y|^p |y'| dx \geq \frac{M^{*-1}}{p+1} \left(\int_a^X r^{-(1/p)} dx \right)^p \int_a^X r |y'|^{p+1} dx,$$

where $M^* = M^*(X) = \text{ess sup}_{x \in [a, X]} s^{-1}(x)$. Equality holds in (32) if and only if $s = \text{const.} > 0$ and $y = k^* \int_a^X r^{-(1/p)} dt$, k^* complex.

Replacing $[a, x]$ by $[x, b]$ throughout Theorem 5, we obtain

THEOREM 6. Let p, q be real numbers satisfying the same conditions as in Theorem 5. Let r, s nonnegative, measurable functions on (X, b) such that $\int_X^b r^{-(1/(p+q-1))} dx$ and $\int_X^b s^{-(1/(q-1))} dx$ are finite. If y is complex-valued, absolutely continuous on $[X, b]$, $y(b) = 0$, then

$$(33) \quad \int_X^b s |y|^p |y'|^q dx \geq \hat{K}_2(X, p, q) \int_X^b r |y'|^{p+q} dx,$$

where

$$\hat{K}_2(X, p, q) = \left(\frac{q}{p+q} \right)^q \left(\int_X^b s^{-(1/(q-1))} dx \right)^{1-q} \left(\int_X^b r^{-(1/(p+q-1))} dx \right)^{p+q-1}.$$

Equality holds in (33) if and only if s and y are defined by (25) with k_4 complex.

As a direct consequence of Theorem 5 and 6 we have

COROLLARY 5. Let p, q be real numbers satisfying the same conditions as in Theorem 5. Let r, s be nonnegative measurable functions on (a, b) such that $\int_a^b r^{-1/(p+q-1)} dx$ and $\int_a^b s^{-1/(q-1)} dx$ are finite. If y is complex-valued, absolutely continuous on $[a, b]$ with $y(a) = y(b) = 0$, then,

$$(34) \quad \int_a^b s |y|^p |y'|^q dx \geq \hat{K}(p, q) \int_a^b r |y'|^{p+q} dx,$$

where $\hat{K}(p, q) = \hat{K}_1(X, p, q) = \hat{K}_2(X, p, q)$, with X the unique solution ($a < X < b$) of the latter equation. Moreover equality holds in (34) if and only if s and y are defined as in corollary 1.

REMARK 8. Let $p < 0$, $0 < p + q < 1$ and $q = 1$; $s(x)$ positive and measurable on (X, b) . Then, for complex-valued, absolutely continuous y on $[X, b]$ such that $y(b) = 0$,

$$(35) \quad \int_X^b s |y|^p |y'| dx \geq \frac{\hat{M}^{*-1}}{p+1} \left(\int_X^b r^{-1/p} dx \right)^p \int_X^b r |y'|^{p+1} dx,$$

where $\hat{M}^* = \hat{M}^*(X) = \text{ess sup}_{x \in [X, b]} s^{-1}(x)$.

Finally, if y is complex-valued, absolutely continuous on $[a, b]$ such that $y(a) = y(b) = 0$, and if s is positive and continuous on (a, b) , then (32) and (35) yield

$$(36) \quad \int_a^b s |y|^p |y'| dx \geq \frac{\bar{M}^{-1}}{p+1} \left(\int_a^x r^{-1/p} dx \right)^p \int_a^b r |y'|^{p+1} dx,$$

where $\bar{M} = M^*(X)$ and X is the unique solution ($a < X < b$) of the equation $\hat{M}^*(X) \left(\int_a^X r^{-(1/p)} dx \right)^p = M^*(X) \left(\int_a^X r^{-(1/p)} dx \right)^p$. Equality holds in (36) if and only if $s = \text{const.} > 0$ and

$$y = k_1^* \left(\int_a^x r^{-(1/p)} dt \right)^p \left(k_2^* \left(\int_x^b r^{-(1/p)} dt \right)^p \right)$$

according as $a \leq x \leq X$ ($X \leq x \leq b$).

Examples can be constructed for special cases of r and s as before. However, we content ourselves with noting that if $s(x) \equiv 1$, (32) reduces to the following inequality of Calvert's paper [2, p. 75],

$$\int_a^X |u^{p-1} u'| \geq \frac{1}{p} \left(\int_a^X r^{1-q} \right)^{p-1} \int_a^X r |u'|^p, \quad 0 < p < 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{p} = 1.$$

4. Let u be a given function and let

$$y = u^{q/(p+q)} \quad (p+q \neq 0).$$

If p and q are such that $q/(p+q) > 0$, then it is obvious that y is absolutely continuous on an interval if and only if u is, and that y vanishes at a point if and only if u does. A simple computation gives

$$|y|^p |y'|^q = \left(\frac{q}{p+q} \right)^q |u'|^q \quad \text{and} \quad |y'|^{p+q} = \left(\frac{q}{p+q} \right)^{p+q} |u|^{-p} |u'|^{p+q},$$

that is,

(37)

$$|y|^p |y'|^q = \left(\frac{P+Q}{Q} \right)^{P+Q} |u'|^{P+Q} \quad \text{and} \quad |y'|^{p+q} = \left(\frac{P+Q}{Q} \right)^Q |u|^{-P} |u'|^{P+Q},$$

where $p = -P$, $p+q = Q$.

In view of (37) and Theorem 1 we have

THEOREM 7. *Let P, Q be real numbers such that either $P < 0$, $Q > 1$ and $P + Q > 0$ or $P > 0$ and $P + Q < 0$. Let r, s be nonnegative, measurable functions on (a, X) such that $\int_a^X s^{-1/(Q-1)} dx < \infty$. Let the constant*

(38)

$$K_1^*(X, P, Q) = \left(\frac{Q}{P+Q} \right)^{\{(P+Q)/Q\}-P} \left\{ \int_a^X r^{-(Q/P)} s^{(P+Q)/P} \left(\int_a^x s^{-\{1/(Q-1)\}} dt \right)^{Q-1} dx \right\}^{P/Q}$$

be finite. If u is absolutely continuous on $[a, X]$, $u(a) = 0$, and u' does not change sign on (a, X) , then

$$(39) \quad \int_a^X s |u|^P |u'|^Q dx \geq K_1^*(X, P, Q) \int_a^X r |u'|^{P+Q} dx.$$

Equality holds in (38) if and only if

$$r = k_1^* s^{(P+Q-1)/(Q-1)} \left(\int_a^x s^{-\{1/(Q-1)\}} dt \right)^{P-\{P/(P+Q)\}}, \quad \text{and} \\ u = k_2^* \left(\int_a^x s^{-\{1/(Q-1)\}} dt \right)^{Q/(P+Q)},$$

for some constants $k_1^ (\geq 0)$, k_2^* real.*

Theorems 3 and 7 lead to

COROLLARY 6. *Let p, q be real numbers as in Theorem 3. Let r, s be nonnegative measurable functions on (a, X) such that $K_1(X, p, q)$, $K_1^*(X, p, q)$ defined by (7), (38) respectively are finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X) , then*

$$\int_a^X s |y|^p |y'|^q dx \geq \max(K_1, K_1^*) \int_a^X r |y'|^{p+q} dx.$$

Moreover, equality holds if and only if s and y are defined by (9) or

$$(40) \quad r = k_1^* s^{(p+q-1)/(q-1)} \left(\int_a^x s^{-\{1/(q-1)\}} dt \right)^{p-\{p/(p+q)\}}, \quad \text{and} \\ y = k_2^* \left(\int_a^x s^{-\{1/(q-1)\}} dt \right)^{q/(p+q)},$$

for some constants $k_1^ (\geq 0)$, k_2^* real.*

Proof. The inequality is immediate in view of (22) and (39) and the fact that $q > 1$ is implicit if $p < 0$. Again, a straight-forward computation shows that (9) holds if and only if (40) holds. Thus, equality holds in (22) if and only if it holds in (39). Also, then $K_1 = K_1^*$. This completes the proof.

REMARK 9. If $r = s \equiv 1$, K_1^* are meaningful constants when $p + q > 0$ and $q > 0$ respectively. Therefore, in Corollary 6 if $r = s \equiv 1$ and $p < 0$, $p + q > 1$,

$$K_1 = \frac{q^{q/(p+q)}}{p+q} (X-a)^p, \quad K_1^* = \frac{q^{1-p}}{(p+q)^{(p/q)+1-p}} (X-a)^p.$$

It is easy to verify that $\ln x/(1-x^{-1})$ is an increasing function of x for $x > 1$. Thus,

$$\frac{1}{1-\frac{1}{q}} \ln q > \frac{1}{1-\frac{1}{p+q}} \ln(p+q),$$

whence

$$q^{p-(p/(p+q))} < (p+q)^{p-(p/q)}.$$

Consequently, in this case $K_1^* > K_1$.

Another example where $K_1^* \geq K_1$ is when $r = (x-a)^{p(p+q-1)/(p+q)}$, $s = (x-a)^{p(1-q)/q}$, $p < 0$ and $p+q > 1$. Then,

$$K_1 = \left(\frac{q}{p+q}\right)^{1-p} \left(\frac{q}{q+(p+q)(1-q)}\right)^{p/(p+q)} (X-a)^{p/(p+q) + \{(1-q)p/q\}},$$

and

$$K_1^* = \frac{q}{p+q} \left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p/q} (X-a)^{p/(p+q) + (1-q)p/q}.$$

If $q \leq 2$, $q + (p+q)(1-q) > 0$ and therefore, in view of

$$0 < -p/(p+q)(q-1) < 1$$

and $-\ln x$ convex if $x > 0$, we have

$$(q + (p+q)(1-q))^{-\{p/(p+q)(q-1)\}} \cdot q^{1+\{p/(p+q)(q-1)\}} \leq p+q,$$

whence

$$\left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p+q} \leq \left(\frac{q}{p+q}\right)^{-q(p+q)} \left(\frac{q}{q+(p+q)(1-q)}\right)^q,$$

that is,

$$\left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p/q} \geq \left(\frac{q}{p+q}\right)^{-p} \left(\frac{q}{q+(p+q)(1-q)}\right)^{p/(p+q)} \\ \text{if } 2 \geq q > p+q > 1,$$

proving that $K_1^* \geq K_1$ in this case.

As above, in view of (37) and Theorem 3 we have

THEOREM 8. *Let P, Q be real numbers such that $PQ > 0$, and either $Q > 1$ or $Q < 0$. Let r, s be nonnegative, measurable functions on (a, X) such that $\int_a^X s^{-1/(Q-1)} dx < \infty$, and the constant K_1^* defined by (38) is finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X) , then*

$$(40) \quad \int_a^X s |u|^P |u'|^Q dx \leq K_1^* \int_a^X r |u'|^{P+Q} dx.$$

Equality holds in (40) if and only if r and u are as defined in Theorem 7.

REMARK 10. If P and Q above satisfy

$$P > 0, P + Q > 1 \quad \text{and} \quad 0 < Q < 1,$$

then (37) and Theorem 5 yield

$$(41) \quad \int_a^X s |u|^P |u'|^Q dx \leq \hat{K}_1(X, P, Q) \int_a^X r |u'|^{P+Q} dx,$$

where \hat{K}_1 is defined by (30). Here u can be taken as complex-valued. Equality holds if and only if it holds in (29), that is if and only if s and $u(=y)$ are as defined by (9) with k_s complex.

If $P > 0$ and $Q = 1$, then (37) and (23) yield

$$(42) \quad \int_a^X s |u|^P |u'| dx \leq \hat{K} \int_a^X r |u'|^{P+1} dx$$

where s is a positive, measurable function on (a, X) and

$$(43) \quad \hat{K}(P) = \frac{1}{P+1} \left(\int_a^X M^* s^{(P+1)/P} r^{-(1/P)} dx \right)^P, \quad M^*(x) = \operatorname{ess\,sup}_{t \in [a, x]} s^{-1}(t).$$

Equality holds in (42) if and only if $s = \operatorname{const.} > 0$ and $u = k \left(\int_a^x r^{-(1/P)} dt \right)$, k complex.

Combining Theorems 1 and 8 and Remark 10 we have

COROLLARY 7. *Let p, q be real numbers such that $pq > 0$. Let r, s be nonnegative, measurable functions on (a, X) such that*

$$\int_a^X r^{-(1/(p+q-1))} dx, \int_a^X s^{-(1/(q-1))} dx$$

(or $M^*(x)$ if $p > 0, q = 1$) exist, and the constants K_1, K_1^*, \hat{K}_1 and $\hat{K}(p)$ are finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$,

and y' does not change sign on (a, X) , then

$$\int_a^X s |y|^p |y'|^q dx \leq K \int_a^X r |y'|^{p+q} dx ,$$

where $K = \min(K_1, K_1^*)$ if $\alpha) q > 1$ or $q < 0$, $= \min(K_1, \hat{K}_1)$ if $\beta) 0 < q < 1$ and $p + q > 1$, $= \min(K_1, \hat{K})$ if $\gamma) q = 1$. Moreover, equality holds if and only if it holds in both (8) and (40), (8) and (41), (8) and (42) according as $\alpha), \beta), \gamma)$ is the case.

REMARK 11. If $r = s \equiv 1$ and $q > 1$ (so $p > 0$) in Corollary 7, the fact that $\ln x/(1 - x^{-1})$ is an increasing function of x for $x > 1$ leads to $K_1^* > K_1$ and thus $K = K_1$. Again, if $r = s \equiv 1$ and $q = 1$ above, $K = K_1 = \hat{K}$. Also, if $r = s \equiv 1$ and $0 < q < 1 < p + q$ then

$$K_1 = \frac{q^{q/(p+q)}}{p+q} (X-a)^p , \quad \hat{K}_1 = \left(\frac{q}{p+q} \right)^q (X-a)^p .$$

That $\hat{K}_1 > K_1$ follows from the fact that for $0 < q < 1 < p + q$,

$$\frac{q}{q-1} \ln q < 1 < \frac{p+q}{p+q-1} \ln(p+q) ,$$

whence

$$\left(1 - \frac{1}{q}\right) \ln(p+q) < \left(1 - \frac{1}{p+q}\right) \ln q .$$

Similar results could be stated on $[X, b]$ and $[a, b]$.

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