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Let A be a discrete subring of C of rank 2. Let X be a compact subset of C with transfinite diameter not less than unity or with transfinite diameter less than unity, void interior, and connected complement. In an earlier paper we characterized the complex valued functions on X which can be uniformly approximated by elements from the ring of polynomials $A[z]$. In this paper the same problem is studied where X is a compact subset of C with transfinite diameter $d(X)$ less than unity and with nonvoid interior. It is also studied for certain compact subsets of C^n where n is any positive integer. These subsets will have the property that every continuous function holomorphic on the interior is uniformly approximable by complex polynomials. A large class of sets of this type is shown to exist.

We endeavor to follow the notation and terminology of Bourbaki [2]. Throughout, X is a compact subset of C^n, $n \geqq 1$. We use the symbol z to stand for the n-tuple of complex numbers $(z_1, \cdots, z_n) \in C^n$. If R is any subring of C, $R[z]$ will denote $R[z_1, \cdots, z_n]$, the ring of polynomials in $z_1, \cdots, z_n$ with coefficients in R. When z is an element of C^n, we define $|z|$ by

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2} \geqq 0 .$$

We write

$$\sum_{k \in N^n} a_k z^k$$

to denote

$$\sum_{k_1, \cdots, k_n = 0}^{\infty} a_{k_1, k_2, \cdots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} .$$

(Here N stands for the nonnegative integers.)

If f is a complex valued function on X and R a subring of C, we say that f is R-approximable on X if for any $\varepsilon > 0$ there exists p in $R[z]$ such that

$$||f - p||_X = \sup_{z \in X} |f(z) - p(z)| < \varepsilon .$$

2. **Polynomial convexity.** If X is a compact subset of C^n, we define its polynomial hull $h(X)$ by

$$h(x) = \{z \in C^n : |p(z)| \leqq \|p\|_X \text{ for all } p \in C[z]\} .$$

It is easy to see that $h(X)$ is also compact. If $X = h(X)$ we say that X is polynomially convex. It suffices to consider only such X by the following (Wermer [7, p. 53]).

PROPOSITION 2.1. Let f be a complex valued function on X and P any subset of $C[z]$. If f is uniformly approximable by elements of P, then there exists exactly one extension f' of f to $h(X)$ such that f' is uniformly approximable by elements of P.

If X is a subset of C and $t_m(z, X)$ is the Chebyshev polynomial of degree m associated with X (Ferguson [3, § 3]) then $t_m(z, X)$ is the same as $t_m(z, h(X))$ which immediately gives $d(X) = d(h(X))$, where d denotes transfinite diameter. In this case ($n = 1$) we also have that $h(X)$ equals X union the bounded components of its complement (Hormander [5, p. 8, Th. 1.3.3]).

By utilizing the maximal principle in each variable separately one can verify the following examples which will be needed later.

EXAMPLE 2.2. If X is defined by

$$X = \{z \in C^n : |z_1| = |z_2| = \cdots = |z_n| = 1\}$$

then we have

$$h(X) = \{z \in C^n : |z_i| \leqq 1, 1 \leqq i \leqq n\} .$$

This example brings out two differences between the cases $n = 1$ and $n > 1$. First, it is clear that X does not contain the topological boundary of $h(X)$ in this example if $n > 1$, although it does if $n = 1$. Secondly, we have seen that for $n = 1$, if $C \sim X$ (complement of X in C) is connected, then $X = h(X)$. This is also false for $n > 1$.

EXAMPLE 2.3. If $n > 1$ and

$$X = \{z \in C^n : |z_1| = 1, z_2 = z_3 = \cdots = z_n = 0\} ,$$

then

$$h(X) = \{z \in C^n : |z_1| \leqq 1, z_2 = z_3 = \cdots = z_n = 0\} .$$

3. **Necessary conditions.** Throughout this section let R be any discrete subring of C.

PROPOSITION 3.1. If f is a complex valued function on a compact subset X of C^n and f is R-approximable on X, then f has a unique extension f' to $h(X)$ which is continuous on $h(X)$, holomorphic on

$h(X)^\circ$ (where superscript $^\circ$ denotes "interior"), and R-approximable on $h(X)$.

Proof. The proof is immediate from 2.1 since uniform limits preserve continuity and holomorphicity.

An example of the use of 3.1 is the following. Let

$$X = \{z \in \boldsymbol{C}: \delta_1 \leqq |z| \leqq \delta_2\},$$

where $0 < \delta_1 < \delta_2 < 1$, and let f be the function $z + 1/2$ restricted to X. By § 2, $h(X) = \delta_2 D$ where D is the closed unit disk in $\boldsymbol{C}$. By the principle of holomorphic continuation, the only continuous extension of f to $h(X)$ which is holomorphic on $\delta_2 D^\circ$ is the function $z + 1/2$ restricted to $\delta_2 D$. But this function is not R-approximable on $\delta_2 D$ for any discrete ring R since any polynomial $p \in R[z]$ has as its value at $z = 0$ an element of R and every nonzero element of R has modulus at least unity. Thus f is not R-approximable on X for any discrete ring R.

In the case $X \subset \boldsymbol{C}$ it is easy to use 3.1 as follows. Given a complex valued function f on X we seek an extension f'' of f to $h(X)$ which is continuous on $h(X)$ and holomorphic on $h(X)^\circ$. If none such exists, then f is not R-approximable on X. If such an f'' exists, then f is R-approximable on X if and only if f'' is R-approximable on $h(X)$. This is true since if f is R-approximable on X, then by 3.1 there exists an R-approximable extension f' of f to $h(X)$ which is continuous on $h(X)$ and holomorphic on $h(X)^\circ$. By § 2 and the maximum modulus principle f' is then uniquely determined by f, and so $f' = f''$ which shows that f'' is R-approximable. The converse is obvious.

This procedure is contingent upon the fact that if $X \subset \boldsymbol{C}$, then there is at most one extension f' of f to $h(X)$ which is continuous on $h(X)$ and holomorphic on $h(X)^\circ$. If $X \subset \boldsymbol{C}^n$, $n > 1$, the extension need not be unique, as the following example shows.

Let

$$X = \{z: |z_1| = 1, z_2 = \cdots = z_n = 0\}.$$

We know from 2.3 that

$$h(X) = \{z: |z_1| \leqq 1, z_2 = \cdots = z_n = 0\}.$$

If we define $f(z) = 1$ for all z in X, then f is certainly $\boldsymbol{Z}$-approximable on X. Also, the function

$$f''(z) = |z_1| \qquad z \in h(X)$$

is an extension of f to $h(X)$ which is continuous on $h(X)$ and holomorphic on $h(X)^\circ$ (indeed, $h(X)^\circ$ is empty). But it is clear that the unique extension f' of f which is $\boldsymbol{Z}$-approximable on $h(X)$ is simply $f'(z) = 1, z \in h(X)$.

PROPOSITION 3.2. Let X be a compact subset of $\boldsymbol{C}^n$ and f a map from X into $\boldsymbol{C}$. Then if f is R-approximable on X, it has a power series expansion about each point in $R^n \cap X^\circ$ with coefficients R.in

Proof. Let z_0 be an element of $R^n \cap X^\circ$. If (p_n) is a sequence of polynomials in $R[z]$ which tends to f uniformly on X then by Bochner and Martin [1, p. 23] the coefficients of the power series expansions of the p_n about z_0 converge to the respective coefficients of the power series expansion of f about z_0. Since R is closed in $\boldsymbol{C}$ we are done.

For completeness we quote Ferguson [3, Prop. 3.1].

PROPOSITION 3.3. Let X be a compact subset of $\boldsymbol{C}$ with $d(X) \geqq 1$. Then a complex valued function f on X is R-approximable on X if and only if it is already an element of $R[z]$.

By restricting a complex valued function on $\boldsymbol{C}^n$ to a function of one complex variable, in certain ways, we can get additional necessary conditions for approximability as follows. For $a \in R^{n-1}$ put

$$X_k = \{z \in \boldsymbol{C}: (a_1, \cdots, a_{k-1}, z, a_k, \cdots, a_{n-1}) \in X\}$$

and $g(z) = f(a_1, \cdots, a_{k-1}, z, a_k, \cdots, a_{n-1})$. X_k is a compact subset of $\boldsymbol{C}$ and if f is R-approximable on X, then clearly g is R-approximable on X_k.

4. Sufficient conditions. Throughout this section we take A to be a discrete subring of $\boldsymbol{C}$ with rank 2. We also always take X to be polynomially convex in view of § 2. The following is obvious.

PROPOSITION 4.1. If z' is an element of A^n, then a complex valued function $f(z)$ is A-approximable on a subset X of $\boldsymbol{C}^n$ if and only if $f(z - z')$ is A-approximable on $z' + X$.

The following definition partially delineates the class of sets to which the main theorem of this section applies.

DEFINITION 4.2. A compact subset X on $\boldsymbol{C}^n$ is said to be Mergelyan if it satisfies the following condition. Any complex valued function which is continuous on X and holomorphic on X° is $\boldsymbol{C}$-approximable on X.

This terminology is motivated by the following theorem, first proved in Mergelyan [6].

THEOREM 4.3. *A compact subset X of C is Mergelyan if and only if its complement $C \sim X$ is connected.*

By § 2 we see that another equivalent condition is that X be polynomially convex, at least in case $X \subset C$. The answer to the corresponding question for compact subsets of C^n, $n > 1$, is not known to the writer, but is certainly more complicated. Indeed, we can see that Lavrent'ev's theorem (Ferguson [3, Prop. 5.1]), which is a consequence of Mergelyan's theorem, fails if we attempt to apply it word for word to compact subsets of C^n, $n > 1$, as follows. Let X be as in 2.3. Then $h(X)$ has void interior, since $n > 1$, and its complement $C^n \sim h(X)$ is connected as can be seen by elementary means. Thus $h(X)$ satisfies the hypotheses of Mergelyan's theorem except that $X \not\subset C$. The continuous function $f''(z) = |z_1|$, $z \in h(X)$, is not C-approximable on $h(X)$ since by 2.1 there exists exactly one such approximable extension of $f(z) = 1$, $z \in X$, to $h(X)$ and this is obviously the function $f'(z) = 1$, $z \in h(X)$. Thus neither Lavrent'ev's nor Mergelyan's theorem extend to higher dimensions.

A theorem which does extend, in a sense, is Runge's theorem. In one of its forms this theorem states that every complex valued function holomorphic in a neighborhood of a compact set X of C is C-approximable on X if and only if $C \sim X$ is connected. From § 2 we know that $C \sim X$ is connected if and only if X is polynomially convex. It is this last criterion which remains in force as we pass to higher dimensions.

PROPOSITION 4.4. Let X be a polynomially convex subset of C^n. Then any function which is holomorphic in a neighborhood of X is C-approximable on X.

Proof. This is a consequence of Hormander [5, p. 91, 4.3.2].

The following exhibits a large class of Mergelyan subsets of C^n, for any n.

PROPOSITION 4.5. Let X be a polynomially convex subset of C^n such that there exists $z' \in X$ and $\delta > 0$ such that whenever $1 < a < 1 + \delta$ we have

$$(aX - (a-1)z')^\circ \supset X .$$

That is, the dilation of X about z' of magnitude a is a neighborhood of X for all $a \in (1, 1+\delta)$. Then X is Mergelyan.

Proof. Choose $\varepsilon > 0$ and without loss of generality assume $z' = 0$. For $1 < a < 1 + \delta$ put $f_a(z) = f(z/a)$; f_a is holomorphic in aX, and by 4.4 there exists $p_a \in C[z]$ such that $\|f_a - p_a\|_X < \varepsilon/2$. It is clear that $\|f_a - f\|_X \to 0$ as $a \to 1$. Hence if a is chosen so that $\|f_a - f\|_X < \varepsilon/2$, then $\|p_a - f\|_X < \varepsilon$.

As an example of this consider

$$X = \{z \in C^n : |z_1| \leqq 1, \cdots, |z_n| \leqq 1\} .$$

We know from § 2 that X is polynomially convex. It is clear that the remaining hypothesis of 4.5 is satisfied with $z' = 0$ and $\delta = \infty$, hence X is Mergelyan. Clearly, the same argument applies for any closed polydisk.

The following result follows from definitions by considering the functions $f(z) = e^{az}$ for all complex numbers a.

PROPOSITION 4.6. If X is a compact convex subset of C^n, then X is polynomially convex.

EXAMPLE 4.7. Let

$$X = \{z \in C^n : |z| \leqq 1\} .$$

Clearly X is compact and convex ($|z + z'| \leqq |z| + |z'|$) and so by 4.6 X is polynominally convex. Clearly, for any $a > 1$

$$(aX)^\circ = \{z \in C^n : |z| < a\} \supset X .$$

Thus, by 4.5, X is Mergelyan.

It is clear that any translation or nonzero dilation of a Mergelyan set is Mergelyan.

The following is the main result of the paper.

THEOREM 4.8. *Let X be a Mergelyan subset of the open unit polycylinder with $0 \in X^\circ$. If f is any complex valued function on X, the following are equivalent:*

(i) *The function f is A-approximable on X;*

(ii) *The function f is continuous on X, holomorphic on X° and the coefficients of its power series expansion about the origin lie in A.*

Proof. By 3.1 and 3.2 we have that (i) implies (ii).

Conversely, assume that (ii) holds and $\varepsilon > 0$. Let $\rho = \max_{1 \leqq i \leqq n} \|z_i\|_X$.

Since X is compact, $\rho < 1$. It is easy to see that

$$\sum_{k \in N^n} \rho^{k_1+\cdots+k_n} < \infty .$$

Let $\delta > 0$ be such that for any $z \in C$ there is an $a \in A$ such that $|z - a| < \delta$. The existence of such a δ is easily demonstrated (Ferguson [3, § 2]). There exists a finite subset F of N^n such that

$$\sum_{k \in N^n \sim F} \rho^{k_1+\cdots+k_n} < \frac{\varepsilon}{3\delta} .$$

Since X is Mergelyan by hypothesis, there exists a sequence of polynomials (p_m) in $C[z]$ such that $(p_m) \rightarrow f$ uniformly on X. Let

$$p_m = \sum_{k \in N^n} a_k^{(m)} z^k .$$

Where, for each m, all but finitely many of the $a_k^{(m)}$ are zero, and

$$f = \sum_{k \in N^n} a_k z^k$$

in a neighborhood of the origin. Then, as noted in the proof of 3.2, for each $k \in N^n$

$$a_k^{(m)} \rightarrow a_k \quad \text{as} \quad m \rightarrow \infty .$$

Thus there exists a positive integer N such that $m > N$ implies

$$\|p_m - f\|_X < \frac{\varepsilon}{3} \tag{1}$$

and

$$|a_k^{(m)} - a_k| < \frac{\varepsilon}{3M(\text{card } F)} \qquad \text{for } k \in F ,$$

where $M = \max_{k \in F} \|z^k\|_X$. Thus if $m > N$ and $[p_m]$ denotes the polynomial p_m with each coefficient replaced by a nearest element of A, we have

$$\begin{aligned} \|p_m - [p_m]\| &\leqq \sum_{k \in F} |a_k^{(m)} - a_k| \, \|z^k\| + \sum_{k \notin F} \delta \|z^k\| \\ &< \sum_{k \in F} \frac{\varepsilon}{3M(\text{card } F)} M + \delta \sum_{k \notin F} \rho^{k_1+\cdots+k_n} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 2\frac{\varepsilon}{3} . \end{aligned}$$

This estimate together with (1) gives

$$\|[p_m] - f\| < \varepsilon .$$

The problem is, in the sense given by 4.1, invariant under translation by elements of A^n; and so the theorem could have been stated for Mergelyan subsets of an open unit polydisk centered at any element of A^n.

The following example is the solution of the problem which motivated this paper.

EXAMPLE 4.9. Let X be a circle of radius $r < 1$ and centered at the origin of the complex plane. There is no loss of generality in the assumption $r < 1$ since it is well known that $d(X) = r$; and if $d(X) \geqq 1$, the problem is trivial by 3.3. Let f be a continuous complex valued function defined on X. By § 2, $h(X) = rD$, where D is the closed unit disk. By 3.1, in order that f be approximable on X it is necessary that f have a continuous extension to $h(X)$ which is holomorphic on $h(X)^\circ$ and A-approximable on $h(X)$.

It is well known that a continuous function g defined on $|z| = 1$ has an extension which is continuous on $|z| \leqq 1$ and holomorphic on $|z| < 1$ if and only if the Fourier coefficients $\hat{g}(n)$ of the function are all zero for $n < 0$ (Hoffman [4, p. 42]). It is easy to see from this that f has a continuous extension f' to $h(X)$ which is holomorphic on $h(X)^\circ$ if and only if

$$\int_{|z|=1} f(rz)z^m dz = 0 \quad \text{for} \quad m = 0, 1, \cdots ,$$

or equivalently,

$$r^{-(m+1)}\int_X f(z)z^m dz = 0 \quad \text{for} \quad m = 0, 1, \cdots .$$

It is clear from the maximum modulus principle that f' is uniquely determined, if it exists. Then by 3.2, if f is to be A-approximable on X the coefficients of the power series expansion of f' about 0 must be in A, that is,

$$\frac{1}{2\pi i}\int_X f(z)z^{-(m+1)}dz \in A \quad \text{for} \quad m = 0, 1, \cdots .$$

In summary then, in order that f be A-approximable on X it is necessary that

$$\frac{1}{2\pi i}\int_X f(z)z^{m-1}dz \begin{cases} = 0 & \text{for} \quad m = 1, 2, \cdots , \\ \in A & \text{for} \quad m = 0, -1, \cdots . \end{cases}$$

These conditions are also sufficient by the preceding discussion and 4.8.

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