UNIFORM APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS. II

Le Baron O. Ferguson
Let $A$ be a discrete subring of $C$ of rank 2. Let $X$ be a compact subset of $C$ with transfinite diameter not less than unity or with transfinite diameter less than unity, void interior, and connected complement. In an earlier paper we characterized the complex valued functions on $X$ which can be uniformly approximated by elements from the ring of polynomials $A[z]$. In this paper the same problem is studied where $X$ is a compact subset of $C$ with transfinite diameter $d(X)$ less than unity and with nonvoid interior. It is also studied for certain compact subsets of $C^n$ where $n$ is any positive integer. These subsets will have the property that every continuous function holomorphic on the interior is uniformly approximable by complex polynomials. A large class of sets of this type is shown to exist.

We endeavor to follow the notation and terminology of Bourbaki [2]. Throughout, $X$ is a compact subset of $C^n$, $n \geq 1$. We use the symbol $z$ to stand for the $n$-tuple of complex numbers $(z_1, \cdots, z_n) \in C^n$. If $R$ is any subring of $C$, $R[z]$ will denote $R[z_1, \cdots, z_n]$, the ring of polynomials in $z_1, \cdots, z_n$ with coefficients in $R$. When $z$ is an element of $C^n$, we define $|z|$ by

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2} \geq 0.$$

We write

$$\sum_{k \in N^n} a_k z^k$$

to denote

$$\sum_{k_1, \cdots, k_n=0}^\infty a_{k_1, k_2, \cdots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}.$$

(Here $N$ stands for the nonnegative integers.)

If $f$ is a complex valued function on $X$ and $R$ a subring of $C$, we say that $f$ is $R$-approximable on $X$ if for any $\varepsilon > 0$ there exists $p$ in $R[z]$ such that

$$\|f - p\|_X = \sup_{z \in X} |f(z) - p(z)| < \varepsilon.$$

2. Polynomial convexity. If $X$ is a compact subset of $C^n$, we define its polynomial hull $h(X)$ by
\[
\begin{align*}
\mathcal{h}(x) &= \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_x \text{ for all } p \in \mathbb{C}[z] \}.
\end{align*}
\]
It is easy to see that \( \mathcal{h}(X) \) is also compact. If \( X = \mathcal{h}(X) \) we say that \( X \) is polynomially convex. It suffices to consider only such \( X \) by the following (Wermer [7, p. 53]).

**Proposition 2.1.** Let \( f \) be a complex valued function on \( X \) and \( P \) any subset of \( \mathbb{C}[z] \). If \( f \) is uniformly approximable by elements of \( P \), then there exists exactly one extension \( f' \) of \( f \) to \( \mathcal{h}(X) \) such that \( f' \) is uniformly approximable by elements of \( P \).

If \( X \) is a subset of \( \mathbb{C} \) and \( t_m(z, X) \) is the Chebyshev polynomial of degree \( m \) associated with \( X \) (Ferguson [3, §3]) then \( t_m(z, h(X)) \) immediately gives \( d(X) = d(h(X)) \), where \( d \) denotes transfinite diameter. In this case (\( n = 1 \)) we also have that \( h(X) \) equals \( X \) union the bounded components of its complement (Hormander [5, p. 8, Th. 1.3.3]).

By utilizing the maximal principle in each variable separately one can verify the following examples which will be needed later.

**Example 2.2.** If \( X \) is defined by
\[
X = \{z \in \mathbb{C}^*: |z_1| = |z_2| = \cdots = |z_n| = 1\}
\]
then we have
\[
\mathcal{h}(X) = \{z \in \mathbb{C}^*: |z_i| \leq 1, 1 \leq i \leq n\}.
\]
This example brings out two differences between the cases \( n = 1 \) and \( n > 1 \). First, it is clear that \( X \) does not contain the topological boundary of \( \mathcal{h}(X) \) in this example if \( n > 1 \), although it does if \( n = 1 \). Secondly, we have seen that for \( n = 1 \), if \( C \sim X \) (complement of \( X \) in \( C \)) is connected, then \( X = \mathcal{h}(X) \). This is also false for \( n > 1 \).

**Example 2.3.** If \( n > 1 \) and
\[
X = \{z \in \mathbb{C}^*: |z_1| = 1, z_2 = z_3 = \cdots = z_n = 0\},
\]
then
\[
\mathcal{h}(X) = \{z \in \mathbb{C}^*: |z_1| \leq 1, z_2 = z_3 = \cdots = z_n = 0\}.
\]

3. **Necessary conditions.** Throughout this section let \( R \) be any discrete subring of \( \mathbb{C} \).

**Proposition 3.1.** If \( f \) is a complex valued function on a compact subset \( X \) of \( \mathbb{C}^n \) and \( f \) is \( R \)-approximable on \( X \), then \( f \) has a unique extension \( f' \) to \( \mathcal{h}(X) \) which is continuous on \( \mathcal{h}(X) \), holomorphic on
Proof. The proof is immediate from 2.1 since uniform limits preserve continuity and holomorphicity.

An example of the use of 3.1 is the following. Let

\[ X = \{ z \in \mathbb{C} : \delta_1 \leq |z| \leq \delta_2 \} , \]

where \( 0 < \delta_1 < \delta_2 < 1 \), and let \( f \) be the function \( z + 1/2 \) restricted to \( X \). By § 2, \( h(X) = \delta_2 D \) where \( D \) is the closed unit disk in \( \mathbb{C} \). By the principle of holomorphic continuation, the only continuous extension of \( f \) to \( h(X) \) which is holomorphic on \( \delta_2 D^\circ \) is the function \( z + 1/2 \) restricted to \( \delta_2 D \). But this function is not \( R \)-approximable on \( \delta_2 D \) for any discrete ring \( R \) since any polynomial \( p \in R[z] \) has as its value at \( z = 0 \) an element of \( R \) and every nonzero element of \( R \) has modulus at least unity. Thus \( f \) is not \( R \)-approximable on \( X \) for any discrete ring \( R \).

In the case \( X \subset \mathbb{C} \) it is easy to use 3.1 as follows. Given a complex valued function \( f \) on \( X \) we seek an extension \( f'' \) of \( f \) to \( h(X) \) which is continuous on \( h(X) \) and holomorphic on \( h(X)^\circ \). If none such exists, then \( f \) is not \( R \)-approximable on \( X \). If such an \( f'' \) exists, then \( f \) is \( R \)-approximable on \( X \) if and only if \( f'' \) is \( R \)-approximable on \( h(X) \). This is true since if \( f \) is \( R \)-approximable on \( X \), then by 3.1 there exists an \( R \)-approximable extension \( f' \) of \( f \) to \( h(X) \) which is continuous on \( h(X) \) and holomorphic on \( h(X)^\circ \). By § 2 and the maximum modulus principle \( f' \) is then uniquely determined by \( f \), and so \( f' = f'' \) which shows that \( f'' \) is \( R \)-approximable. The converse is obvious.

This procedure is contingent upon the fact that if \( X \subset \mathbb{C} \), then there is at most one extension \( f' \) of \( f \) to \( h(X) \) which is continuous on \( h(X) \) and holomorphic on \( h(X)^\circ \). If \( X \subset \mathbb{C}^*, n > 1 \), the extension need not be unique, as the following example shows.

Let

\[ X = \{ z : |z_1| = 1, z_2 = \cdots = z_n = 0 \} . \]

We know from 2.3 that

\[ h(X) = \{ z : |z_1| \leq 1, z_2 = \cdots = z_n = 0 \} . \]

If we define \( f(z) = 1 \) for all \( z \) in \( X \), then \( f \) is certainly \( Z \)-approximable on \( X \). Also, the function

\[ f''(z) = |z_1| \quad z \in h(X) \]
is an extension of $f$ to $h(X)$ which is continuous on $h(X)$ and holomorphic on $h(X)^o$ (indeed, $h(X)^o$ is empty). But it is clear that the unique extension $f'$ of $f$ which is $Z$-approximable on $h(X)$ is simply $f'(z) = 1, z \in h(X)$.

**Proposition 3.2.** Let $X$ be a compact subset of $C^n$ and $f$ a map from $X$ into $C$. Then if $f$ is $R$-approximable on $X$, it has a power series expansion about each point in $R^n \cap X^o$ with coefficients $R$ in

*Proof.* Let $z_0$ be an element of $R^n \cap X^o$. If $(p_n)$ is a sequence of polynomials in $R[z]$ which tends to $f$ uniformly on $X$ then by Bochner and Martin [1, p. 23] the coefficients of the power series expansions of the $p_n$ about $z_0$ converge to the respective coefficients of the power series expansion of $f$ about $z_0$. Since $R$ is closed in $C$ we are done.

For completeness we quote Ferguson [3, Prop. 3.1].

**Proposition 3.3.** Let $X$ be a compact subset of $C$ with $d(X) \geq 1$. Then a complex valued function $f$ on $X$ is $R$-approximable on $X$ if and only if it is already an element of $R[z]$.

By restricting a complex valued function on $C^n$ to a function of one complex variable, in certain ways, we can get additional necessary conditions for approximability as follows. For $a \in R^{n-1}$ put

$$X_k = \{z \in C: (a_1, \cdots, a_{k-1}, z, a_k, \cdots, a_{n-1}) \in X\}$$

and $g(z) = f(a_1, \cdots, a_{k-1}, z, a_k, \cdots, a_{n-1})$. $X_k$ is a compact subset of $C$ and if $f$ is $R$-approximable on $X$, then clearly $g$ is $R$-approximable on $X_k$.

4. Sufficient conditions. Throughout this section we take $A$ to be a discrete subring of $C$ with rank 2. We also always take $X$ to be polynomially convex in view of § 2. The following is obvious.

**Proposition 4.1.** If $z'$ is an element of $A^n$, then a complex valued function $f(z)$ is $A$-approximable on a subset $X$ of $C^n$ if and only if $f(z - z')$ is $A$-approximable on $z' + X$.

The following definition partially delineates the class of sets to which the main theorem of this section applies.

**Definition 4.2.** A compact subset $X$ on $C^n$ is said to be Mergelyan if it satisfies the following condition. Any complex valued function which is continuous on $X$ and holomorphic on $X^o$ is $C$-approximable on $X$. 


This terminology is motivated by the following theorem, first proved in Mergelyan [6].

**Theorem 4.3.** A compact subset $X$ of $C$ is Mergelyan if and only if its complement $C \sim X$ is connected.

By § 2 we see that another equivalent condition is that $X$ be polynomially convex, at least in case $X \subset C$. The answer to the corresponding question for compact subsets of $C^*$, $n > 1$, is not known to the writer, but is certainly more complicated. Indeed, we can see that Lavrent’ev’s theorem (Ferguson [3, Prop. 5.1]), which is a consequence of Mergelyan’s theorem, fails if we attempt to apply it word for word to compact subsets of $C^*$, $n > 1$, as follows. Let $X$ be as in 2.3. Then $h(X)$ has void interior, since $n > 1$, and its complement $C^* \sim h(X)$ is connected as can be seen by elementary means. Thus $h(X)$ satisfies the hypotheses of Mergelyan’s theorem except that $X \not\subset C$. The continuous function $f''(z) = |z|, z \in h(X)$, is not $C$-approximable on $h(X)$ since by 2.1 there exists exactly one such approximable extension of $f(z) = 1, z \in X$, to $h(X)$ and this is obviously the function $f'(z) = 1, z \in h(X)$. Thus neither Lavrent’ev’s nor Mergelyan’s theorem extend to higher dimensions.

A theorem which does extend, in a sense, is Runge’s theorem. In one of its forms this theorem states that every complex valued function holomorphic in a neighborhood of a compact set $X$ of $C$ is $C$-approximable on $X$ if and only if $C \sim X$ is connected. From § 2 we know that $C \sim X$ is connected if and only if $X$ is polynomially convex. It is this last criterion which remains in force as we pass to higher dimensions.

**Proposition 4.4.** Let $X$ be a polynomially convex subset of $C^*$. Then any function which is holomorphic in a neighborhood of $X$ is $C$-approximable on $X$.

**Proof.** This is a consequence of Hormander [5, p. 91, 4.3.2].

The following exhibits a large class of Mergelyan subsets of $C^*$, for any $n$.

**Proposition 4.5.** Let $X$ be a polynomially convex subset of $C^*$ such that there exists $z' \in X$ and $\delta > 0$ such that whenever $1 < a < 1 + \delta$ we have

$$(aX - (a - 1)z')^\circ \supset X.$$  

That is, the dilation of $X$ about $z'$ of magnitude $a$ is a neighborhood
of $X$ for all $a \in (1, 1 + \delta)$. Then $X$ is Mergelyan.

Proof. Choose $\varepsilon > 0$ and without loss of generality assume $z' = 0$. For $1 < a < 1 + \delta$ put $f_a(z) = f(az)$; $f_a$ is holomorphic in $aX$, and by 4.4 there exists $p_a \in C[z]$ such that $\|f_a - p_a\|_x < \varepsilon/2$. It is clear that $\|f_a - f\|_x \to 0$ as $a \to 1$. Hence if $a$ is chosen so that $\|f_a - f\|_x < \varepsilon/2$, then $\|p_a - f\|_x < \varepsilon$.

As an example of this consider

$$X = \{z \in C^n: |z_1| \leq 1, \ldots, |z_n| \leq 1\}.$$  

We know from § 2 that $X$ is polynomially convex. It is clear that the remaining hypothesis of 4.5 is satisfied with $z' = 0$ and $\delta = \infty$, hence $X$ is Mergelyan. Clearly, the same argument applies for any closed polydisk.

The following result follows from definitions by considering the functions $f(z) = e^{az}$ for all complex numbers $a$.

**Proposition 4.6.** If $X$ is a compact convex subset of $C^n$, then $X$ is polynomially convex.

**Example 4.7.** Let

$$X = \{z \in C^n: |z| \leq 1\}.$$  

Clearly $X$ is compact and convex ($|z + z'| \leq |z| + |z'|$) and so by 4.6 $X$ is polynomially convex. Clearly, for any $a > 1$

$$(aX)^c = \{z \in C^n: |z| < a\} \supset X.$$  

Thus, by 4.5, $X$ is Mergelyan.

It is clear that any translation or nonzero dilation of a Mergelyan set is Mergelyan.

The following is the main result of the paper.

**Theorem 4.8.** Let $X$ be a Mergelyan subset of the open unit polycylinder with $0 \in X^c$. If $f$ is any complex valued function on $X$, the following are equivalent:

(i) The function $f$ is $A$-approximable on $X$;

(ii) The function $f$ is continuous on $X$, holomorphic on $X^c$ and the coefficients of its power series expansion about the origin lie in $A$.

Proof. By 3.1 and 3.2 we have that (i) implies (ii). Conversely, assume that (ii) holds and $\varepsilon > 0$. Let $\rho = \max_{i \leq n} \|z_i\|_x$. 

Since $X$ is compact, $\rho < 1$. It is easy to see that
\[ \sum_{k \in N^n} \rho^{k_1 + \cdots + k_n} < \infty . \]

Let $\delta > 0$ be such that for any $z \in C$ there is an $a \in A$ such that $|z - a| < \delta$. The existence of such a $\delta$ is easily demonstrated (Ferguson [3, § 2]). There exists a finite subset $F$ of $N^n$ such that
\[ \sum_{k \in N^n - F} \rho^{k_1 + \cdots + k_n} < \frac{\varepsilon}{3\delta} . \]

Since $X$ is Mergelyan by hypothesis, there exists a sequence of polynomials $(p_m)$ in $C[z]$ such that $(p_m) \to f$ uniformly on $X$. Let
\[ p_m = \sum_{k \in N^n} a^{(m)}_k z^k . \]

Where, for each $m$, all but finitely many of the $a^{(m)}_k$ are zero, and
\[ f = \sum_{k \in N^n} a_k z^k \]
in a neighborhood of the origin. Then, as noted in the proof of 3.2, for each $k \in N^n$
\[ a^{(m)}_k \to a_k \quad \text{as} \quad m \to \infty . \]

Thus there exists a positive integer $N$ such that $m > N$ implies
\[ ||p_m - f||_X < \frac{\varepsilon}{3} \]
and
\[ |a^{(m)}_k - a_k| < \frac{\varepsilon}{3M(\text{card } F)} \quad \text{for} \quad k \in F , \]
where $M = \max_{k \in F} ||z^k||_X$. Thus if $m > N$ and $[p_m]$ denotes the polynomial $p_m$ with each coefficient replaced by a nearest element of $A$, we have
\[ ||p_m - [p_m]|| \leq \sum_{k \in F} |a^{(m)}_k - a_k| ||z^k|| + \sum_{k \in F} \delta ||z^k|| \]
\[ < \sum_{k \in F} \frac{\varepsilon}{3M(\text{card } F)} M + \delta \sum_{k \in F} \rho^{k_1 + \cdots + k_n} \]
\[ < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 2\frac{\varepsilon}{3} . \]

This estimate together with (1) gives
\[ ||[p_m] - f|| < \varepsilon . \]
The problem is, in the sense given by 4.1, invariant under translation by elements of $A^n$; and so the theorem could have been stated for Mergelyan subsets of an open unit polydisk centered at any element of $A^n$.

The following example is the solution of the problem which motivated this paper.

**Example 4.9.** Let $X$ be a circle of radius $r < 1$ and centered at the origin of the complex plane. There is no loss of generality in the assumption $r < 1$ since it is well known that $d(X) = r$; and if $d(X) \geq 1$, the problem is trivial by 3.3. Let $f$ be a continuous complex valued function defined on $X$. By § 2, $h(X) = rD$, where $D$ is the closed unit disk. By 3.1, in order that $f$ be approximable on $X$ it is necessary that $f$ have a continuous extension to $h(X)$ which is holomorphic on $h(X)^\circ$ and $A$-approximable on $h(X)$.

It is well known that a continuous function $g$ defined on $|z| = 1$ has an extension which is continuous on $|z| \leq 1$ and holomorphic on $|z| < 1$ if and only if the Fourier coefficients $\hat{g}(n)$ of the function are all zero for $n < 0$ (Hoffman [4, p. 42]). It is easy to see from this that $f$ has a continuous extension $f'$ to $h(X)$ which is holomorphic on $h(X)^\circ$ if and only if

$$\int_{|z|=1} f(rz)z^m dz = 0 \quad \text{for } m = 0, 1, \ldots,$$

or equivalently,

$$r^{-(m+1)} \int_X f(z)z^m dz = 0 \quad \text{for } m = 0, 1, \ldots.$$

It is clear from the maximum modulus principle that $f'$ is uniquely determined, if it exists. Then by 3.2, if $f$ is to be $A$-approximable on $X$ the coefficients of the power series expansion of $f'$ about 0 must be in $A$, that is,

$$\frac{1}{2\pi i} \int_X f(z)z^{-(m+1)} dz \in A \quad \text{for } m = 0, 1, \ldots.$$

In summary then, in order that $f$ be $A$-approximable on $X$ it is necessary that

$$\frac{1}{2\pi i} \int_X f(z)z^{m-1} dz = 0 \quad \text{for } m = 1, 2, \ldots,$$

$$\in A \quad \text{for } m = 0, -1, \ldots.$$

These conditions are also sufficient by the preceding discussion and 4.8.
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