

Pacific Journal of Mathematics

NÖRLUND SUMMABILITY OF FOURIER SERIES

MASAKO IZUMI AND SHIN-ICHI IZUMI

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The Nörlund summability was first applied to the theory of Fourier series by E. Hille and J. D. Tamarkin. Many other mathematicians have since worked in this field. Recently T. Singh has proved a nice theorem concerning the Nörlund summability of Fourier series. In Part I, we shall give a generalization.

Absolute Nörlund summability was defined by L. McFadden and he proved a theorem concerning the absolute Nörlund summability of the Fourier series of functions of the Lipschitz class which was generalized by S. N. Lal. We shall give another generalization of McFadden's theorem in Part II.

PART I.

1. Let $\sum_{n=0}^{\infty} a_n$ be a given series and (s_n) be the sequence of its partial sums. Let (p_n) be a sequence of real numbers and $P_n = p_0 + p_1 + \cdots + p_n$. We suppose that $P_n \neq 0$ for all n . The series $\sum a_n$ is called to be summable (N, p_n) to s when $\lim_{n \rightarrow \infty} t_n$ exists and is equal to s , where

$$t_n = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k = \frac{1}{P_n} \sum_{k=0}^n p_k s_{n-k}.$$

t_n is called the n th (N, p_n) mean or n th Nörlund mean.

In the special case in which $p_n = \binom{n + \alpha - 1}{n} = A_n^{\alpha-1} (\alpha > 0)$, the Nörlund mean reduces to the (C, α) mean. Another special case that $p_n = 1/(n + 1)$, is called the Harmonic mean.

The condition for the regularity of summability (N, p_n) , is

$$p_n/P_n \rightarrow 0 \quad \text{and} \quad \sum_{k=0}^n |p_k| = O(|P_n|) \quad \text{as } n \rightarrow \infty.$$

If (p_n) is a positive sequence, then the second condition is satisfied. It is also easy to see that, if (p_n) is an increasing sequence, then a $(C, 1)$ summable sequence is summable (N, p_n) .

We shall define $p(t)$ on the interval $(0, \infty)$ such that $p(n) = p_n$ for $n = 0, 1, 2, \dots$ and that $p(t)$ is continuous on $(0, \infty)$ and is linear in each interval $(k, k + 1) (k = 0, 1, 2, \dots)$. We put $P(t) = \int_0^t p(u) du$, then, $P(n) = (1/2)p_0 + p_1 + \cdots + p_{n-1} + (1/2)p_n \cong P_n$ as $n \rightarrow \infty$ when $P_n \rightarrow \infty$ and $p_n/P_n \rightarrow 0$.

T. Singh [4] has proved the following theorems:

THEOREM S1. *If (p_n) is a positive sequence such that $p_n \downarrow$ and $P_n \rightarrow \infty$ and further if*

$$(1) \quad \Phi(t) = \int_0^t |\varphi_x(u)| du = o(p(1/t)/P(1/t)) \quad \text{as } t \rightarrow 0,$$

then the Fourier series of f is summable (N, p_n) to $f(x)$ at the point x , where $\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)$.

THEOREM S2. *If (p_n) satisfies the conditions of Theorem S1 and*

$$(2) \quad \Psi(t) = \int_0^t |\psi_x(u)| du = o(p(1/t)/P(1/t)) \quad \text{as } t \rightarrow 0,$$

then the conjugate Fourier series of f is summable (N, p_n) to

$$-\frac{1}{\pi} \int_0^\pi \frac{\psi_x(u)}{2 \tan u/2} du$$

at the point x when the last integral exists, where

$$\psi_x(u) = f(x+u) - f(x-u).$$

We shall prove that we can replace the condition $p_n \downarrow$ by the more general one

$$(3) \quad \int_1^n u |p'(u)| du = O(P_n) \quad \text{as } n \rightarrow \infty.$$

If (p_n) is monotone, then the condition (3) is equivalent to

$$(4) \quad np_n = O(P_n),$$

since

$$\int_1^n up'(u)du = [up(u)]_1^n - \int_1^n p(u)du = np(n) - P_n + O(1).$$

If (p_n) is decreasing, then (4) is satisfied automatically. In general, condition (3) implies (4).

If the condition (4) is satisfied, then (1) implies

$$(5) \quad \Phi(t) = o(t) \quad \text{as } t \rightarrow 0.$$

2. Our first theorem is as follows.

THEOREM 1. *If (p_n) is a positive sequence such that $P_n \rightarrow \infty$, (3) holds, and condition (1) is satisfied, then the Fourier series of f is summable (N, p_n) to $f(x)$ at the point x .*

Proof. We write $\varphi_x(u) = \varphi(u)$ and by $t_n(x)$ we denote the n th Nörlund mean of the Fourier series of f at the point x , then

$$\begin{aligned} t_n(x) - f(x) &= \frac{1}{2\pi P_n} \int_0^\pi \frac{\varphi(t)}{\sin t/2} \left(\sum_{k=0}^n p_{n-k} \sin(k + 1/2)t \right) dt \\ &= \frac{1}{2\pi P_n} \int_0^\pi \frac{\varphi(t)}{2 \sin t/2} L_n(t) dt \\ &= \frac{1}{2\pi P_n} \left(\int_0^{1/n} + \int_{1/n}^\pi \right) = \frac{1}{2\pi} (I + J) . \end{aligned}$$

By (5),

$$\begin{aligned} |I| &\leq \frac{A}{P_n} \int_0^{1/n} \frac{|\varphi(t)|}{t} \left(\sum_{k=0}^n p_{n-k} \cdot kt \right) dt \\ &\leq An \int_0^{1/n} |\varphi(t)| dt = o(1) . \end{aligned}$$

We write

$$\begin{aligned} |L_n(t)| &= \left| \sum_{k=0}^n p_k \sin(n - k + 1/2)t \right| \\ &\leq \sum_{k=0}^{[1/t]} p_k + \left| \sum_{k=[1/t]}^n p_k \sin(n - k + 1/2)t \right| \\ &= L'_n(t) + L''_n(t) , \end{aligned}$$

then we have, by Abel's lemma,

$$|L''_n(t)| \leq A \left\{ \frac{p([1/t])}{t} + \frac{p(n)}{t} + \frac{1}{t} \int_{1/t}^n |p'(u)| du \right\} .$$

Therefore

$$\begin{aligned} |J| &= \left| \frac{1}{P_n} \int_{1/n}^\pi \frac{\varphi(t)}{\sin t/2} L_n(t) dt \right| \\ &\leq \frac{A}{P_n} \left\{ \int_{1/n}^\pi \frac{|\varphi(t)|}{t} L'_n(t) dt + \int_{1/n}^\pi \frac{|\varphi(t)|}{t} L''_n(t) dt \right\} \\ &= \frac{A}{P_n} (J_1 + J_2) , \end{aligned}$$

where

$$\begin{aligned} |J_1| &\leq \int_{1/n}^\pi \frac{|\varphi(t)|}{t} P(1/t) dt \leq \left[\frac{\Phi(t)}{t} P(1/t) \right]_{1/n}^\pi \\ &\quad + \int_{1/n}^\pi \frac{\Phi(t)}{t^2} P(1/t) dt + \int_{1/n}^\pi \frac{\Phi(t)}{t} \frac{p(1/t)}{t^2} dt \\ &= o(P_n) + o(1) \int_{1/n}^\pi \frac{p(1/t)}{t^2} dt + o(1) \int_{1/n}^\pi \frac{p(1/t)}{t^2} dt \\ &= o(P_n) \end{aligned}$$

and

$$\begin{aligned}
 J_2 &\leq A \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \frac{p(1/t)}{t} dt + \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \frac{p(n)}{t} dt \right. \\
 &\quad \left. + \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt \int_{1/t}^n |p'(u)| du \right\} \\
 &\leq A \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} P(1/t) dt + p_n \left[\frac{\Phi(t)}{t^2} \right]_{1/n}^{\pi} \right\} \\
 &\quad + p_n \int_{1/n}^{\pi} \frac{\Phi(t)}{t^3} dt + \left[\frac{\Phi(t)}{t^2} \int_{1/t}^n |p'(u)| du \right]_{1/n}^{\pi} \\
 &\quad + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^3} dt \int_{1/t}^n |p'(u)| du + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} \frac{|p'(1/t)|}{t^2} dt \\
 &\leq AJ_1 + o(np_n) + o(np_n) + O\left(\int_1^n |p'(u)| du\right) \\
 &\quad + o\left(\int_1^n u |p'(u)| du\right) + o\left(\int_1^n u |p'(u)| du\right) \\
 &= o(P_n).
 \end{aligned}$$

Thus we get $J = o(1)$ and then we have proved the theorem.

3. THEOREM 2. *If (p_n) is a positive sequence satisfying the conditions in Theorem 1 and*

$$\Psi(t) = \int_0^t |\psi_x(u)| du = o(p(1/t)/P(1/t)) \quad \text{as } t \rightarrow 0,$$

then the conjugate Fourier series of f is summable (N, p_n) to

$$-\frac{1}{\pi} \lim_{n \rightarrow \infty} \int_{1/n}^{\pi} \frac{\psi_x(t)}{2 \tan t/2} dt$$

when the last limit exists.

Proof. Let $\psi_x(t) = \psi(t)$ and $\tilde{t}_n(x)$ be the n th (N, p_n) mean of the conjugate Fourier series of f , then

$$\begin{aligned}
 \tilde{t}_n(x) &= \left(-\frac{1}{\pi} \int_{1/n}^{\pi} \frac{\psi_x(t)}{2 \tan t/2} dt \right) \\
 &= \frac{-1}{2\pi P_n} \int_0^{1/n} \frac{\psi_x(t)}{\sin t/2} \left(\sum_{k=0}^n \left(\cos \frac{1}{2}t - \cos \left(k + \frac{1}{2} \right)t \right) \right) dt \\
 &\quad + \frac{1}{2\pi P_n} \int_{1/n}^{\pi} \frac{\psi(t)}{\sin t/2} \left(\sum_{k=0}^n p_{n-k} \cos \left(k + \frac{1}{2} \right)t \right) dt.
 \end{aligned}$$

Applying the method of proof of Theorem 1 to above integrals, we obtain the theorem.

PART II.

4. Let (t_n) be the sequence of Nörlund means of the series $\sum a_n$. If $\sum |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is called to be summable $|N, p_n|$ or absolutely summable (N, p_n) .

L. McFadden [6] proved the following theorem.

THEOREM M. *Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If $f \in \text{Lip } \alpha (0 < \alpha < 1)$ and*

$$(6) \quad \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1/2} P_n} < \infty,$$

then the Fourier series of f is $|N, p_n|$ summable.

This was generalized in the following form by S. L. Lal [3]:

THEOREM L. *Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If the continuity modulus $\omega(t) = \omega(t; f)$ of f satisfies the conditions*

$$(7) \quad \sum_{n=1}^{\infty} \frac{\omega(1/n^\delta)}{n} < \infty \quad (0 < \delta < 1)$$

and

$$(8) \quad \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/2} P_n} < \infty,$$

then the Fourier series of f is $|N, p_n|$ summable.

We shall prove the following theorem:

THEOREM 3. *Let $2 \geq p > 1$, $1/p + 1/q = 1$ and let (p_n) be a positive, decreasing and convex sequence tending to zero. If*

$$(9) \quad \sum_{n=1}^{\infty} p_n^2 n^{p-2} < \infty$$

and

$$(10) \quad \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty$$

where $\omega(\delta)$ is the continuity modulus of f , and further if

$$\sum_{m=n}^{\infty} \frac{1}{m^p (\omega(1/m))^{p-1}} \leq \frac{A}{(n\omega(1/n))^{p-1}}$$

(or more specially, if $u^{-\delta}\omega(u) \downarrow$ as $u \uparrow$ for a positive $\delta < 1$), then the Fourier series of f is $|N, p_n|$ summable.

If (p_n) decreases monotonically then (9) is equivalent to

$$\hat{\xi}_1(x) = \sum_{n=1}^{\infty} p_n \cos nx \quad \text{and} \quad \hat{\xi}_2(x) = \sum_{n=1}^{\infty} p_n \sin nx$$

belong to L^p .

From (9) we have

$$\begin{aligned} P_n &= \sum_{k=1}^n p_k = \sum_{k=1}^n p_k k^{1-2/p} \cdot k^{2/p-1} \\ &\leq \left(\sum_{k=1}^n p_k^q k^{p-2} \right)^{1/p} \left(\sum_{k=1}^n k^{(2-p)q/p} \right)^{1/q} \leq A n^{1/p} \end{aligned}$$

and then

$$(11) \quad \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} \cdot n^{1/p}} \leq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty .$$

Therefore, under the condition (9), the condition (10) is stronger than (11). If p decreases from 2, then (9) becomes weaker but (10) becomes stronger.

In the case $p = 1$, we have the following.

THEOREM 4. *Let (p_n) be a decreasing and convex sequence tending to zero, such that*

$$(12) \quad \sum_{n=1}^{\infty} \frac{p_n}{n} < \infty$$

and

$$(13) \quad \sum_{n=k}^{\infty} \frac{1}{nP_n} \leq \frac{A}{P_k} .$$

If f has the continuity modulus $\omega(\delta)$ such that

$$(14) \quad \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_n} < \infty ,$$

then the Fourier series of f is summable $|N, p_n|$.

It is known ([7], Chap. V, § 1) that if (p_n) is a decreasing and convex sequence tending to zero, then $\hat{\xi}_1(x)$ is integrable and that if (p_n) is decreasing and satisfies the condition (12), then $\hat{\xi}_2(x)$ is integrable. We have also

$$P_n = \sum_{k=1}^n p_k = \sum_{k=1}^n (p_k/k)k \leq An ,$$

and then we have $\sum \omega(1/n)/n < \infty$, i.e., (11) holds also in this case.

5. For the proof of Theorem 3, we use the following lemmas.

LEMMA 1. ([5]) *If (p_n) is a positive and decreasing sequence, then*

$$\left| \sum_{k=a}^b p_k e^{i(n-k)t} \right| \leq AP(1/t)$$

for any a and $b > a$ and for any integer n .

LEMMA 2. ([6]) *If (p_n) is a positive and decreasing sequence and $\xi(t) = \sum_{k=0}^{\infty} p_k e^{ikt}$, then*

$$|\xi(x + 2t) - \xi(x)| \leq A \frac{t}{x} P\left(\frac{1}{t}\right) \quad \text{for all } x \text{ in } (t, \pi) .$$

LEMMA 3. ([2] and [1]) *If $\lambda(t)$ is a positive increasing function on $(1, \infty)$ and $f \in L^p$ ($1 < p \leq 2$), then*

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{\lambda(2n)} \left(\sum_{m=n}^{\infty} \rho_m^q \right)^{1/q} \\ \leq A \int_0^1 \frac{dt}{t^2 \lambda(1/t)} \left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p dx \right)^{1/p} \end{aligned}$$

where $\rho_m^2 = a_m^2 + b_m^2$, a_m and b_m being the m th Fourier coefficients of f and $1/p + 1/q = 1$.

6. We shall now prove Theorem 3. By the definition, we have

$$\begin{aligned} t_n - t_{n-1} &= \frac{1}{\pi} \int_0^\pi \varphi(t) \left\{ \sum_{k=0}^n \left(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \right) D_k(t) \right\} dt \\ &= \frac{1}{\pi} \int_0^\pi \varphi(t) \left\{ \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \cos kt \right\} dt \\ &= \frac{1}{\pi} \int_0^\pi \varphi(t) \left\{ \sum_{k=0}^{n-1} \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) \cos (n-k)t \right\} dt \\ &= \frac{1}{\pi} \frac{1}{P_n P_{n-1}} \int_0^\pi \varphi(t) \left\{ \sum_{k=0}^{n-1} (p_k P_n - p_n P_k) \cos (n-k)t \right\} dt \end{aligned}$$

where $p_{-1} = P_{-1} = 0$, and then

$$\begin{aligned}
(15) \quad \pi |t_n - t_{n-1}| &\leq \frac{1}{P_{n-1}} \left| \int_0^\pi \varphi(t) \left(\sum_{k=0}^{\infty} p_k \cos(n-k)t \right) dt \right| \\
&\quad + \frac{1}{P_{n-1}} \left| \int_0^{1/n} \varphi(t) \left(\sum_{k=n}^{\infty} p_k \cos(n-k)t \right) dt \right| \\
&\quad + \frac{p_n}{P_n P_{n-1}} \left| \int_0^{1/n} \varphi(t) \left(\sum_{k=0}^{n-1} P_k \cos(n-k)t \right) dt \right| \\
&\quad + \frac{1}{P_{n-1}} \left| \int_{1/n}^\pi \varphi(t) \left(\sum_{k=n}^{\infty} p_k \cos(n-k)t \right) dt \right| \\
&\quad + \frac{p_n}{P_n} \int_{1/n}^\pi \varphi(t) \left(\sum_{k=0}^{n-1} P_k \cos(n-k)t \right) dt \Big| \\
&= I_n + J_n + K_n + L_n,
\end{aligned}$$

following McFadden [6]. We shall begin to estimate I_n .

$$\begin{aligned}
I_n &\leq \frac{1}{P_{n-1}} \left\{ \left| \int_0^\pi \varphi(t) \left(\sum_{k=0}^{\infty} p_k \cos kt \right) \cos nt dt \right| \right. \\
&\quad \left. + \left| \int_0^\pi \varphi(t) \left(\sum_{k=0}^{\infty} p_k \sin kt \right) \sin nt dt \right| \right\} \\
&= \frac{1}{P_{n-1}} \left\{ \left| \int_0^\pi \varphi(t) \xi_1(t) \cos nt dt \right| \right. \\
&\quad \left. + \left| \int_0^\pi \varphi(t) \xi_2(t) \sin nt dt \right| \right\} \\
&= I_{n,1} + I_{n,2}.
\end{aligned}$$

Let

$$A_n = \int_0^\pi \varphi(t) \xi_1(t) \cos nt dt \quad \text{and} \quad E_n = \left(\sum_{m=n}^{\infty} |A_m|^q \right)^{1/q},$$

then

$$\begin{aligned}
\mathcal{I} &= \sum_{n=2}^{\infty} |I_{n,1}| = \sum_{n=2}^{\infty} \frac{|A_n|}{P_{n-1}} = \sum_{j=1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \frac{|A_n|}{P_{n-1}} \\
&\leq \sum_{j=1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} |A_n|^q \right)^{1/q} \left(\sum_{n=2^j}^{2^{j+1}-1} \frac{1}{P_{n-1}^p} \right)^{1/p} \\
&\leq \sum_{j=1}^{\infty} E_{2^j} \left(\sum_{n=2^j}^{2^{j+1}-1} \frac{1}{P_{n-1}^p} \right)^{1/p} \\
&\leq 4 \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} \frac{E_n}{n} \left(\sum_{m=2n}^{4n-1} \frac{1}{P_{m-1}^p} \right)^{1/p} \\
&= 4 \sum_{n=1}^{\infty} \frac{E_n}{n} \left(\sum_{m=2n}^{4n-1} \frac{1}{P_{m-1}^p} \right)^{1/p} \leq A \sum_{n=1}^{\infty} \frac{E_n}{n^{1/q} P_n}.
\end{aligned}$$

If we put $\lambda(n) = n^{1/q} P_n$, then $\lambda(n) \uparrow \infty$ as n increases. By Lemma 3, we get

$$\begin{aligned} \mathcal{F} &\leq A \int_0^1 \frac{dt}{t^{2\lambda}(1/t)} \left(\int_{-\pi}^{\pi} |\Delta_t(\varphi \cdot \xi_1)|^p dx \right)^{1/p} \\ &= A \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)} \left(\int_{-\pi}^{\pi} |\Delta_t(\varphi \cdot \xi_1)|^p dx \right)^{1/p} \end{aligned}$$

where

$$\begin{aligned} \Delta_t(\varphi \cdot \xi_1) &= \varphi(x+t)\xi_1(x+t) - \varphi(x-t)\xi_1(x-t) \\ &= \xi_1(x+t)[\varphi(x+t) - \varphi(x-t)] + \varphi(x-t)[\xi_1(x+t) - \xi_1(x-t)] \\ &= \xi_1(x+t)\Delta_t\varphi + \varphi(x-t)\Delta_t\xi_1, \end{aligned}$$

and then

$$\begin{aligned} \mathcal{F} &\leq A \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)} \left(\int_{-\pi}^{\pi} |\xi_1(x+t)\Delta_t\varphi|^p dx \right)^{1/p} \\ &\quad + A \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)} \left(\int_{-\pi}^{\pi} |\varphi(x-t)\Delta_t\xi_1|^p dx \right)^{1/p} \\ &= \mathcal{F}' + \mathcal{F}'' . \end{aligned}$$

By (9) and (10), we have

$$\begin{aligned} \mathcal{F}' &\leq A \int_0^1 \frac{\omega(t)dt}{t^{1+1/p}P(1/t)} \leq A \int_1^{\infty} \frac{\omega(1/u)}{u^{1/q}P(u)} du \\ &\leq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q}P_n} \leq A \end{aligned}$$

and, by Lemma 2 and (11),

$$\begin{aligned} \mathcal{F}'' &\leq A + \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)} \left(\int_t^{\pi} \omega(x)^p |\xi_1(x+2t) - \xi_1(x)|^p dx \right)^{1/p} \\ &\leq A + \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)} \left(\int_t^{\pi} \omega(x)^p \left(\frac{t}{x} P\left(\frac{1}{t}\right) \right)^p dx \right)^{1/p} \\ &\leq A + \int_0^1 \frac{dt}{t^{1/p}} \left(\int_t^{\pi} \frac{\omega(x)^p}{x^p} dx \right)^{1/p} \\ &\leq A + \int_1^{\infty} \frac{\omega(1/u)^{1/q}}{u^{1/q}} \cdot \frac{1}{u\omega(1/u)^{1/q}} \left(\int_1^u \frac{\omega(1/v)^p}{v^{2-p}} dv \right)^{1/p} du \\ &\leq A + \left(\int_1^{\infty} \frac{\omega(1/u)}{u} du \right)^{1/q} \left(\int_1^{\infty} \frac{du}{u^p \omega(1/u)^{p/q}} \int_1^u \frac{\omega(1/v)^p}{v^{2-p}} dv \right)^{1/p} \\ &\leq A + A \left(\int_1^{\infty} \frac{\omega(1/v)^p}{v^{2-p}} dv \int_v^{\infty} \frac{du}{u^p \omega(1/u)^{p-1}} \right)^{1/p} \\ &\leq A + A \left(\int_1^{\infty} \frac{\omega(1/v)}{v} dv \right)^{1/p} \leq A . \end{aligned}$$

Thus we have proved that $\mathcal{F} = \sum_{n=2}^{\infty} |I_{n,1}| < \infty$. Similarly we can

prove that $\sum_{n=1}^{\infty} |I_{n,2}| < \infty$, and therefore $\sum_{n=1}^{\infty} I_n < \infty$.

Secondly, we shall estimate the sum $\sum_{n=1}^{\infty} J_n$. Since $\sum_{k=n}^{\infty} p_k \cos(k-n)x$ is a nonnegative function, we have

$$\begin{aligned} \sum_{n=1}^{\infty} J_n &\leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} \int_0^{1/n} \left(\sum_{k=n}^{\infty} p_k \cos(k-n)t \right) dt \\ &\leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} \left(\frac{p_n}{n} + \sum_{k=n+1}^{\infty} \frac{p_n}{k-n} \sin \frac{k-n}{n} \right) \\ &\leq A + \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} p_n \leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} + A \leq A, \end{aligned}$$

since $\sum_{k=1}^n (\sin kx/k)$ is uniformly bounded.

Thirdly,

$$\begin{aligned} \sum_{n=1}^{\infty} K_n &\leq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{\omega(1/n)}{n} \sum_{k=0}^{n-1} P_k \\ &\leq A \sum_{k=1}^{\infty} P_k \sum_{n=k+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{\omega(1/n)}{n} \\ &\leq A \sum_{k=1}^{\infty} P_k \frac{\omega(1/k)}{k} \sum_{n=k+1}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \\ &\leq A \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k} \leq A. \end{aligned}$$

Finally,

$$\begin{aligned} &\sum_{k=n}^{\infty} p_k \cos(k-n)t + \frac{p_n}{P_n} \sum_{k=0}^{n-1} P_k \cos(n-k)t \\ &= \left\{ \frac{p_n}{2} + \sum_{k=n}^{\infty} (p_k - p_{k+1}) \frac{\sin(n-k+1/2)t}{2 \sin t/2} \right\} \\ &\quad + \left\{ \sum_{k=0}^{n-1} p_k \frac{\sin(n-k+1/2)t}{2 \sin t/2} - \frac{1}{2} P_{n-1} \right\} \end{aligned}$$

and then

$$\begin{aligned} L_n &\leq \frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \frac{\varphi(t)}{2 \sin t/2} \left(\sum_{k=n}^{\infty} (p_k - p_{k+1}) \sin(n-k+1/2)t \right) dt \right| \\ &\quad + \frac{p_n}{P_n P_{n-1}} \left| \int_{1/n}^{\pi} \frac{\varphi(t)}{2 \sin t/2} \left(\sum_{k=0}^{n-1} p_k \sin(n-k+1/2)t \right) dt \right| \\ &\quad + \frac{1}{2} \frac{p_n}{P_{n-1}} \left(1 - \frac{P_{n-1}}{P_n} \right) \int_0^{\pi} |\varphi(t)| dt \\ &= L'_n + L''_n + L'''_n, \end{aligned}$$

as in McFadden [6]. Now

$$\begin{aligned}
 \sum_{n=1}^{\infty} L'_n &\leq \sum_{n=1}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t^2} dt \\
 &\leq \sum_{n=1}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} \sum_{k=1}^n \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} dt + A \\
 &\leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} dt \sum_{n=k}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} + A \\
 &\leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} \frac{p(1/t)}{P(1/t)} dt + A \leq A \sum_{k=1}^{\infty} \frac{\omega(1/n)p_n}{P_n} + A \\
 &\leq A \sum_{k=1}^{\infty} \frac{\omega(1/n)}{n} + A \leq A, \\
 \sum_{n=1}^{\infty} L''_n &\leq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt \\
 &= \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^n \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt + A \\
 &\leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}} + A \\
 &\leq \int_0^1 \frac{\omega(t)}{t} dt + A \leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} + A \leq A
 \end{aligned}$$

and

$$\sum_{n=1}^{\infty} L''_n \leq A \sum_{n=1}^{\infty} \frac{p_n^2}{P_n P_{n-1}} \leq A \sum_{n=1}^{\infty} \frac{1}{n^2} \leq A.$$

Collecting above estimations, we get $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$.

7. We shall now prove Theorem 4. We start from (15). First,

$$\begin{aligned}
 I_n &= \frac{1}{P_{n-1}} \left| \int_0^{\pi} \varphi(t) \left(\sum_{k=0}^{\infty} p_k \cos(n-k)t \right) dt \right| \\
 &\leq \frac{1}{P_{n-1}} \left\{ \left| \int_0^{\pi} \varphi(t) \xi_1(t) \cos nt dt \right| + \left| \int_0^{\pi} \varphi(t) \xi_2(t) \cos nt dt \right| \right\} \\
 &= I_{n,1} + I_{n,2}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \sum_{n=1}^{\infty} I_{n,1} &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\varphi(t + \pi/2n) \xi_1(t + \pi/2n) - \varphi(t - \pi/2n) \xi_1(t - \pi/2n)| dt \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\xi_1(t + \pi/2n)| \cdot |\varphi(t + \pi/2n) - \varphi(t - \pi/2n)| dt \\
 &\quad + \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\varphi(t - \pi/2n)| \cdot |\xi_1(t + \pi/2n) - \xi_1(t - \pi/2n)| dt \\
 &= \mathcal{J}_1 + \mathcal{J}_2,
 \end{aligned}$$

where, since ξ_1 is integrable and $|\varphi(t + \pi/2n) - \varphi(t - \pi/2n)| \leq A\omega(1/n)$,

$$\mathcal{F}_1 \leq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} < \infty$$

by the condition (14) and, since¹⁾

$$\xi_1(x) = \sum_{\nu=0}^{\infty} (\nu + 1) \Delta^2 p_{\nu} \cdot K_{\nu}(x),$$

we get

$$\begin{aligned} \mathcal{F}_2 &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{\pi/n}^{\pi} \omega(x) |\xi_1(x + \pi/n) - \xi_1(x)| dx + A \\ &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^{\infty} (\nu + 1) \Delta^2 p_{\nu} \int_{\pi/n}^{\pi} \omega(x) |K_{\nu}(x + \pi/n) - K_{\nu}(x)| dx + A \\ &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \left(\sum_{\nu=0}^n (\nu + 1) \Delta^2 p_{\nu} + \sum_{\nu=n+1}^{\infty} (\nu + 1) \Delta^2 p_{\nu} \right) \\ &\quad \times \int_{\pi/n}^{\pi} \omega(x) |K_{\nu}(x + \pi/n) - K_{\nu}(x)| dx + A \\ &= \mathcal{F}'_2 + \mathcal{F}''_2 + A. \end{aligned}$$

It is well known that

$$|K'_{\nu}(x)| \leq A\nu^2 \quad \text{and} \quad |K'_{\nu}(x)| \leq A/x^2$$

and then

$$\begin{aligned} \int_0^{\pi/\nu} |K_{\nu}(x + \pi/n) - K_{\nu}(x)| dx &\leq \frac{\pi}{n} A\nu^2 \cdot \frac{\pi}{\nu} \leq A\nu/n, \\ |K_{\nu}(x + \pi/n) - K_{\nu}(x)| &\leq A/nx^2 \quad \text{in } (\pi/\nu, \pi). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{F}'_2 &\leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^n (\nu + 1) \Delta^2 p_{\nu} \left(\int_{\pi/n}^{\pi/\nu} + \int_{\pi/\nu}^{\pi} \right) \omega(x) |K_{\nu}(x + \pi/n) - K_{\nu}(x)| dx \\ &\leq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \left\{ \sum_{\nu=0}^n (\nu + 1)^2 \Delta^2 p_{\nu} \sum_{k=\nu}^n \frac{1}{k^2} \omega\left(\frac{1}{k}\right) + \sum_{\nu=0}^n (\nu + 1) \Delta^2 p_{\nu} \sum_{k=1}^{\nu} \omega\left(\frac{1}{k}\right) \right\} \\ &\leq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \left\{ \sum_{k=1}^n \frac{1}{k^2} \omega\left(\frac{1}{k}\right) \cdot kP_k + \sum_{k=1}^n \omega\left(\frac{1}{k}\right) (k(p_k - p_{k+1}) + p_{k+1}) \right\} \\ &\leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) P_k \sum_{n=k}^{\infty} \frac{1}{nP_{n-1}} \\ &\quad + A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \left\{ k(p_k - p_{k+1}) + p_{k+1} \right\} \sum_{n=k}^{\infty} \frac{1}{nP_{n-1}} \\ &\leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) + A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \left\{ \frac{k(p_k - p_{k+1})}{p_k} + \frac{p_{k+1}}{P_k} \right\} \\ &\leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) \end{aligned}$$

¹⁾ $\Delta^2 p_{\nu} = p_{\nu} - 2p_{\nu+1} + p_{\nu+2}$ and K_{ν} is the ν th Fejér kernel.

by the condition (11) and the relation ([6], p. 183)

$$k(p_k - p_{k+1})/P_k \leq A/k.$$

Further,

$$\begin{aligned} \mathcal{F}_2'' &\leq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \sum_{\nu=n+1}^{\infty} (\nu+1) \Delta^2 p_\nu \sum_{k=1}^n \omega\left(\frac{1}{k}\right) \\ &\leq A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \sum_{n=k}^{\infty} \frac{(n+2)(p_{n+1} - p_{n+2}) + p_{n+2}}{nP_{n-1}} \\ &\leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) < \infty. \end{aligned}$$

Thus we have proved that $\sum I_{n,1} < \infty$. The estimation of $I_{n,2}$ is similar to that of $I_{n,1}$ and thus $\sum I_n < \infty$.

Now, $\sum_{k=n}^{\infty} p_k \cos(k-n)x$ is a positive function and then

$$\begin{aligned} \sum_{n=1}^{\infty} J_n &\leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} \int_0^{1/n} \left(\sum_{k=n}^{\infty} p_k \cos(k-n)t \right) dt \\ &\leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} p_n \leq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} < \infty. \end{aligned}$$

Convergence of $\sum_{n=1}^{\infty} K_n$ and $\sum_{n=1}^{\infty} L_n$ is proved as in the proof of Theorem 3. Thus we have established Theorem 4.

REFERENCES

1. M. and S. Izumi, *On the Leindler's theorem*, Proc. Japan Acad. **42** (1966), 533-534.
2. L. Leindler, *Über Strukturbedingungen für Fourierschereihe*, Math. Zeitschr. **88** (1965), 418-431.
3. S. N. Lal, *On the absolute Nörlund summability of Fourier series*, Archiv der Mathematik **15** (1964), 214-222.
4. T. Singh, *Nörlund summability of Fourier series and its conjugate series*, Annali di Matematica Pura ed Applicato (Bologna) (4) **64** (1964), 123-132.
5. E. Hille and J. D. Tamarkin, *On the summability of Fourier series, I*, Trans. Amer. Math. Soc. **34** (1932), 757-783; II, Ann. of Math. (2) **34** (1933), 329-348, 602-605; III, Math. Ann. **108** (1933), 525-577; IV, Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 376-380; V, Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 369-372.
6. L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168-207.
7. A. Zygmund, *Trigonometric series*, Cambridge, 1959.

Received September 12, 1966.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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