A FAMILY OF FUNCTORS DEFINED ON GENERALIZED PRIMARY GROUPS

RAY MINES, III
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Let \( G \) denote an abelian group; \( G \) is called a generalized \( p \)-primary group if \( qG = G \) for all primes \( q \neq p \). Let \( \alpha \) be an ordinal, and let \( \delta: G \to E_\alpha \) satisfy the following four conditions:

1. \( E_\alpha \) is \( p^\alpha \)-Ext-injective,
2. \( p^\alpha E_\alpha = 0 \),
3. \( \delta \) is \( p^\alpha \)-pure in \( E_\alpha \),
4. \( \ker \delta = p^\alpha G \).

Define \( p^\alpha * G \) to be that subgroup of \( E_\alpha \) such that \( p^\alpha (E_\alpha/\delta(G)) = p^\alpha * G/\delta(G) \). If \( \alpha \) is a limit ordinal, let

\[
L_\alpha(G) = \lim_{\beta < \alpha} G/p^\beta G.
\]

Let \( U(G) = \text{Ext}(Z(p^\infty), G) \) and \( U_\alpha(G) = U(G)/p^\alpha U(G) \).

Then we have the following \( p^\alpha \)-pure containments: \( G/p^\alpha G \cong \delta(G) \subseteq U_\alpha(G) \subseteq p^\alpha *(G) \subseteq L_\alpha U_\alpha(G) \), whenever \( \alpha \) is a countable limit of lesser hereditary ordinals. We have \( p^\alpha *(G) = U_\alpha(G) \) for all groups \( G \) if and only if \( p^\alpha \) Ext is hereditary. From this we obtain a new proof of the fact that \( p^\alpha \) Ext is hereditary when \( \alpha \) is a countable limit of lesser hereditary ordinals. We also obtain an example of a cotorsion group \( G \) such that \( G/p^\alpha G \) is not equal to \( L_\alpha(G) \), thus refuting a conjecture of Harrison. A group \( G \) is called generally complete if \( L_\alpha(G)/\delta(G) \) is reduced for all limit ordinals \( \alpha \). A generalized \( p \)-primary group \( G \) is generally complete if and only if it is cotorsion.

A result of Kulikov [7] will be studied and generalized, and an application to the study of cotorsion groups will be given.

Throughout this paper the word “group” will mean “abelian group”. The notation of [2] will be followed. The letter \( p \) will indicate a prime.

The elements of the group \( \text{Ext}(A, B) \) are equivalence classes of extensions \( E: 0 \to B \to E \to A \to 0 \). However, no distinction will be made between equivalence classes and an element of the equivalence class. Thus, it will be said that \( E \) is an element of \( \text{Ext}(A, B) \). Also, \( B \) will be considered as a subgroup of \( E \). The arrow \( \to \) will denote a monomorphism, and the arrow \( \Rightarrow \) will denote an epimorphism. The element \( \text{Ext}(f, g)E \), for \( E \in \text{Ext}(A, B) \), \( f: B \to B' \), and \( g: A' \to A \), will be denoted by \( gEf \). All other notation will be that used in Chapter III of [8].

Recall that a subgroup \( H \) of a group \( G \) is said to be \( p^\alpha \)-pure in \( G \) if the extension \( H \to G \to G/H \) is an element of \( p^\alpha \text{Ext}(G/H, H) \); \( G/H \) is said to be a \( p^\alpha \)-pure quotient of the group \( G \). A group \( G \) is said to be \( p^\alpha \)-projective if \( p^\alpha \text{Ext}(G, A) = 0 \) for all groups \( A \); \( G \) is called \( p^\alpha \)-injective if \( p^\alpha \text{Ext}(A, G) = 0 \) for all groups \( G \).
The functor $p^\alpha \text{Ext}(\cdot, \cdot)$ is said to be hereditary (or shorter, $\alpha$ is called a hereditary ordinal) if every $p^\alpha$-pure subgroup of a $p^\alpha$-projective is $p^\alpha$-projective, or, equivalently, if every $p^\alpha$-pure quotient of a $p^\alpha$-injective is $p^\alpha$-injective. In § 3 a new proof will be given to show that $p^\alpha \text{Ext}$ is hereditary if $\alpha$, is a countable limit of lesser hereditary ordinals.

We shall use the notation $\lambda(G)$ to denote the length of $G$; i.e., the least ordinal $\alpha$ satisfying $p^{\alpha+1}G = p^\alpha G$.

1. The functor $p^\alpha$. In [9] it is shown that for all ordinals $\alpha$ there exists an exact sequence

$$Z \to G_\alpha \to H_\alpha,$$

such that for all group $G$ the following hold.

(1) $p^\alpha G \to G \xrightarrow{\delta} \text{Ext}(H_\alpha, G) \xrightarrow{\epsilon} \text{Ext}(G_\alpha, G)$

is exact, and $\text{Im}(\delta)$ is $p^\alpha$-pure in $\text{Ext}(H_\alpha, G)$. Here we have identified $G$ with $\text{Hom}(Z, G)$ in the usual way;

(2) $H_\alpha$ is a $p^\alpha$-projective $p$-group, so $p^\alpha \text{Ext}(H_\alpha, G) = 0$, and $\text{Ext}(H_\alpha, G)$ is $p^\alpha$-injective;

(3) The sequences for $\alpha$ and $\alpha + n$ are connected by

$$
\begin{array}{c}
Z \xrightarrow{p^\alpha} G_{\alpha+n} \xrightarrow{\delta} H_{\alpha+n} \\
\downarrow \quad \downarrow \\
Z \xrightarrow{p^\alpha} G_\alpha \xrightarrow{\delta} H_\alpha
\end{array}
$$

(4) If $\alpha$ is a limit ordinal, then

$$H_\alpha = \bigoplus_{\beta < \alpha} H_\beta;$$

(5) $p^\alpha H_{\alpha+1}$ is cyclic of order $p$ and $H_\alpha = H_{\alpha+1}/p^\alpha H_{\alpha+1};$

(6) $p^\alpha H_\alpha = 0.$

Let $p^\alpha G$ denote $\varepsilon^{-1}(p^\alpha \text{Ext}(G_\alpha, G))$; then $G/p^\alpha G = \text{Im}\delta$ is a $p^\alpha$-pure subgroup of $p^\alpha G$.

**Theorem 1.1.** Let $E$ be $p^\alpha$-injective such that $p^\alpha E = 0$, that there exists a homomorphism $\gamma: G \to E$ with kernel $p^\alpha G$, and that $\text{Im}\gamma$ a $p^\alpha$-pure subgroup of $E$. Let $G^*$ denote the subgroup of $E$ satisfying $G^*/\gamma(G) = p^\alpha(E/\gamma(G))$. Then there exists an isomorphism $g: p^\alpha G \to G^*$, such that $g\delta = \gamma$.

**Proof.** For convenience in the remainder of this paper we will denote $\text{Ext}(H_\alpha, G)$ and $\text{Ext}(G_\alpha, G)$ by $E_\alpha(G)$ and $F_\alpha(G)$, respectively, or simply by $E_\alpha$ and $F_\alpha$ if no confusion can result. For this proof
let $E / \gamma(G) = F$. Replace Im $\gamma$ and Im $\delta$ by $G / p^\alpha G$. Then the following sequences are exact:

$$G / p^\alpha G \rightarrow E_{\alpha} \rightarrow F_{\alpha}$$

$$G / p^\alpha G \rightarrow E \rightarrow F.$$ 

Before continuing with the proof we prove the following:

**Lemma 1.2.** If $f$, $g$ are homomorphisms from $E_{\alpha}$ to $E$ (or $E$ to $E_{\alpha}$) such that $f | G / p^\alpha G = g | G / p^\alpha G$, then $f | p^\alpha G = g | p^\alpha G (f | G^* = g | G^*)$.

**Proof.** Assume $f, g: E_{\alpha} \rightarrow E$, the proof for $f, g: E \rightarrow E_{\alpha}$ being the same. Let $h = f - g$; then $h(G / p^\alpha G) = 0$. Therefore, $h$ can be lifted to a homomorphism $h^*$ of $F_{\alpha}$ into $E$. Since $p^\alpha E = 0$, we have $h^* | p^\alpha F = 0$. Thus, $h | p^\alpha G = 0$; so $f | p^\alpha G = g | p^\alpha G$.

We now continue the proof of Theorem 1.1. Since $E$ is $p^\alpha$-injective, there exists a homomorphism $g': E_{\alpha} \rightarrow E$ such that the following diagram commutes.

$$G / p^\alpha G \rightarrow E_{\alpha} \rightarrow F_{\alpha}$$

$$\downarrow g' \downarrow \bar{g}$$

$$G / p^\alpha G \rightarrow E \rightarrow F;$$

$\bar{g}$ arises in the usual way. Let $g = g' | p^\alpha G$. Since $\bar{g}(p^\alpha F_{\alpha}) \subseteq p^\alpha F$, it follows that $g(p^\alpha G) \subseteq G^*$. Similarly, there exists a homomorphism $f': E \rightarrow E_{\alpha}$ such that

$$G / p^\alpha G \rightarrow E \rightarrow F$$

$$\downarrow f' \downarrow \bar{f}$$

$$G / p^\alpha G \rightarrow E_{\alpha} \rightarrow F_{\alpha}$$

commutes. Let $f = f' | G^*$; then clearly $f(G^*) \subseteq p^\alpha G$. Consider $f' \circ g': E_{\alpha} \rightarrow E_{\alpha}$. By Lemma 1.2

$$f \circ g = f' \circ g' | p^\alpha G = 1_{E_{\alpha}} | p^\alpha G = 1p^\alpha G.$$ 

Similarly, $g \circ f = g' \circ f' | G^* = 1_{G^*}$. Thus, $g$ is an isomorphism of $p^\alpha G \rightarrow G^*$, and clearly $g\delta = \gamma$.

It follows that, if $E$ is a $p^\alpha$-injective having the following properties:

1. There exists a homomorphism $\gamma: G \rightarrow E$ with $\ker \gamma = p^\alpha G$ and $\text{Im } \gamma$ $p^\alpha$-pure in $E$;
2. $p^\alpha E = 0$,

then $p^\alpha G$ can be taken as the subgroup of $E$ with the property that
$p^\alpha G/\gamma(G) = p^\alpha(E/\gamma(G))$.

Let $U(G) = \text{Ext}(Z(p^\alpha), G)$ and $U_\alpha(G) = U(G)/p^\alpha U(G)$. In [11] it is shown that for all ordinals $\alpha$, $U_\alpha(G)$ is contained in $p^\alpha G$ and $\delta(G) \subseteq U_\alpha(G)$. In [11] Nunke has shown that $\alpha$ is a hereditary ordinal if and only if $U_\alpha(G) = p^\alpha(G)$ for all groups $G$.

The remaining part of this section will be spent in proving the following theorem.

**Theorem 1.3.** Let $\alpha$ be an ordinal such that for all $\gamma < \alpha$ there exists a hereditary $\beta$ with $\gamma < \beta < \alpha$. Then $p^\alpha G \subseteq \lim_{\beta < \alpha} U_\beta(G)$.

The proof of this theorem follows from a series of lemmas. We first observe that $\{U_\beta(G), \pi_\beta\}$ is an inverse system, where for $\beta > \gamma$ $\pi_\beta: U_\beta(G) \rightarrow U_\gamma(G)$ is the natural projection with kernel $p^\beta U_\gamma(G)$.

**Lemma 1.4.** Let $\beta$ and $\gamma$ be ordinals with $\gamma < \beta$. Then there exists a homomorphism $\pi_\beta: p^\beta G \rightarrow p^\gamma G$ agreeing with the natural projection of $G/p^\beta G$ onto $G/p^\gamma G$ when restricted to $G/p^\beta G$. Moreover if $\alpha < \beta < \gamma$, then $\pi_\beta^\alpha \pi_\gamma^\beta = \pi_\gamma^\alpha$.

**Proof.** The extensions

$$G/p^\beta G \rightarrow E_\beta \rightarrow F_\beta$$

and

$$G/p^\gamma G \rightarrow E_\gamma \rightarrow F_\gamma$$

are $p^\beta$-pure and $p^\gamma$-pure, respectively. Since $\beta > \gamma$, the top extension is also $p^\gamma$-pure. As $E_\gamma$ is $p^\gamma$-injective, there exists a map $\mu_\gamma^\beta$ of $E_\beta$ into $E_\gamma$ such that the following diagram commutes:

\[
\begin{array}{ccc}
p^\gamma G/p^\beta G & \rightarrow & E_\beta \\
\downarrow & & \downarrow \mu_\gamma^\beta \\
G/p^\gamma G & \rightarrow & E_\gamma \\
\pi & \downarrow & \lambda_\gamma^\beta \\
& F_\beta & \rightarrow & F_\gamma
\end{array}
\]

where $\pi$ is the canonical projection. The homomorphism $\lambda_\gamma^\beta$ arises in the usual way. Define $\pi_\beta^\gamma$ by $\pi_\beta^\gamma = \mu_\gamma^\beta | p^\beta G$.

As in the proof of Theorem 1.1, $\text{Im} \pi_\beta^\gamma$ is contained in $p^\gamma G$, and, as in Lemma 1.2, the homomorphism is unique. If $\alpha < \beta < \gamma$, then let $\mu_\gamma^\alpha = \mu_\gamma^\beta \mu_\beta^\alpha$. 

**Lemma 1.5.** Let $\beta$ and $\gamma$ be ordinals with $\beta < \gamma$. Let $\pi$ denote the canonical projection of $G/p_\beta G$ onto $G/p_\gamma G$. If $\pi_\gamma$ is a homomorphism of $U_\beta(G)$ into $p_\gamma^*(G)$ agreeing with $\pi$ on $G/p_\beta G$, then $\pi_\gamma$ is the canonical projection of $U_\beta(G)$ onto $U_\gamma(G)$.

**Proof.** Let $\mu$ denote the natural projection of $U_\gamma(G)$ onto $U_\beta(G)$. Consider the homomorphism $\pi_\gamma - \mu : p_\beta^*(G)$ into $p_\gamma^*(G)$ defined in Lemma 1.4 is the natural projection of $U_\beta(G)$ onto $U_\gamma(G)$.

The homomorphism $\delta : G \rightarrow E_\beta$ can be defined in terms of $\delta_\beta : G \rightarrow E_\beta$ by $\delta(x)_\beta = \delta_\beta(x)$. Then the homomorphism $\mu^\alpha_\beta$ used in the proof of Lemma 1.4 can be taken as the natural coordinate projection. So the intersection over all $\beta < \alpha$ of Ker $\pi_\beta$ is zero.

**Theorem 1.7.** If $\alpha$ is a limit ordinal, then the set $\{p^\alpha G, \pi_\beta^\alpha\}$ is an inverse system, and there is an isomorphic copy of $p^\alpha G$ in $\lim_{\beta < \alpha} p^\beta G$.

**Proof.** Lemma 1.4 shows that $\{p^\alpha G, \pi_\beta^\alpha\}$ is an inverse system. The homomorphisms $\pi_\beta^\alpha : p^\alpha G \rightarrow p^\beta G$ gives a family of maps of the group $p^\alpha G$ into this inverse system satisfying $\pi_\gamma^\beta \pi_\alpha^\gamma = \pi_\alpha^\gamma$. Thus, there is a homomorphism $\mu : p^\alpha G \rightarrow \lim_{\beta < \alpha} p^\beta G$. The ker $\mu = \bigcap_{\beta < \alpha} \ker \pi_\beta = 0$. Thus, $\mu$ is a monomorphism.

We are now in a position to prove Theorem 1.3.
Proof of Theorem 1.3. We will show that for all $\gamma < \alpha$ the image of $\pi_\gamma$ is contained in $U_\gamma(G)$. Let $\gamma < \alpha$; then there exists a hereditary ordinal $\beta$ such that $\gamma <\beta < \alpha$. Since $p^\beta G = U_\beta(G)$, it follows that the image of $\pi_\beta$ is contained in $U_\beta(G)$. Lemma 1.4 and 1.5 show that $\pi_\beta$ maps $p^\alpha G$ into $U_\gamma(G)$. Since $\{U_\beta(G),\pi_\beta\}$ is an inverse family and $\pi_\beta$, it follows that there exists a homomorphism $\mu: p^\alpha G \rightarrow \lim_{\beta < \alpha} U_\beta(G)$.

As in the proof of Theorem 1.7, $\ker \mu = 0$. Thus $\mu$ is a monomorphism.

**Corollary 1.8.** The group $G/p^\alpha G$ is a $p^\alpha$-pure subgroup of the group $\lim_{\beta < \alpha} U_\beta(G)$.

**Proof.** Since $\Pi_{\beta < \alpha} U_\beta(G) \subseteq E_\alpha$, it follows that $\lim_{\beta < \alpha} U_\beta(G) \subseteq E_\alpha$. The group $G/p^\alpha G$ is a $p^\alpha$-pure subgroup of $E_\alpha$, and

$$G/p^\alpha G \subseteq p^\alpha G \subseteq \lim_{\beta < \alpha} U_\beta(G).$$

2. The functor $L_\alpha$. Let $G$ be a group and $\alpha$ a limit ordinal. Then the family $\{p^\beta G\}_{\beta < \alpha}$ forms a neighborhood system at zero for the group $G$. This topology will be called the natural topology. If the length of $G = \lambda(G) = \alpha$, then the topology is a Hausdorff topology. If $\alpha \neq \lambda(G)$, then $\{p^\beta G\}_{\beta < \alpha}$ leads to a topology on $G/p^\alpha G$, given by $\{p^\beta G/p^\alpha G\}_{\beta < \alpha}$. This topology is a Hausdorff topology on $G/p^\alpha G$. The family, $\{p^\beta G\}_{\beta < \alpha}$, leads to a uniformity on $G$, respectively $G/p^\alpha G$. Therefore, we can consider the completion of $G, (G/p^\alpha G)$ with respect to this uniformity. Let $L_\alpha(G)$ denote the completion of $G$ if $\lambda(G) = \alpha$, or completion of $G/p^\alpha G$ if $\lambda(G) > \alpha$.

In [12], Zelinsky showed that $L_\alpha(G) = \lim_{\beta < \alpha} G/p^\beta G$. We remark that notation $L_\alpha(G)$ is consistent with the notation used by Harrison in [4]. Let $\pi_\beta: L_\alpha(G) \rightarrow G/p^\beta G$ be the natural projection of $\lim G/p^\beta G$ onto $G/p^\beta G$. A base for the topology on $L_\alpha G$ is given by $\{\ker \pi_\beta\}_{\beta < \alpha}$. We shall call this topology the induced topology. We shall now study the functor $L_\alpha$ on the following class of groups introduced by Kulikov in [6] and [7].

**Definition 2.1.** A group $G$ is a generalized $p$-primary group (g.p. group), if $G$ is divisible by all primes other than $p$.

The following theorem is due to Kulikov [7].

**Theorem 2.2.** Let $G$ be a g.p. group. Let $\alpha$ be an ordinal less than or equal to the length of $G$, satisfying the following condition:
There exists a countable increasing sequence of ordinals whose limit is $\alpha$.

Then if $\delta$ is the natural map of $G$ into $\lim_{\beta<\alpha} G/p^\beta G$, with kernel equal to $p^\alpha G$:

1. $\delta(G) + p^\beta L_\alpha(G) = L_\alpha(G)$, for all $\beta < \alpha$;
2. $L_\alpha(G)/\delta(G)$ is divisible;
3. $\delta(G) \cap p^\beta L_\alpha(G) = p^\beta \delta(G)$ for all $\beta < \alpha$;
4. $G/p^\beta G = L_\alpha(G)/p^\beta L_\alpha(G)$, for all $\beta < \alpha$.

Notice that condition (1) states that $\delta(G)$ is dense in $L_\alpha(G)$ in the natural topology; and condition (4) shows that $L_\alpha(G)$ is complete in the natural topology, since

$$L_\alpha(L_\alpha(G)) = \lim_{\beta<\alpha} L_\alpha(G)/p^\beta L_\alpha(G) = \lim G/p^\beta G = L_\alpha(G).$$

We will show that conditions (1), (2), and (4) are equivalent and that when they happen, the natural topology and the induced topology on $L_\alpha(G)$ are the same. However, we first shall prove the following.

**Theorem 2.3.** If $G$ is a g.p. group and $\alpha$ is a limit ordinal, then $G/p^\alpha G$ is $p^\alpha$-pure in $L_\alpha(G)$.

**Proof.** Since $G/p^\beta G$ is contained in $E_\alpha$, it follows that

$$L_\alpha(G) \subseteq \Pi_{\beta<\alpha} G/p^\beta G \subseteq \Pi E_\beta = E_\alpha.$$

The embedding $\delta: G \to L_\alpha(G)$ is the map, $\delta: G \to E_\alpha$, with its range cut down to $L_\alpha(G)$. Since $G/p^\alpha G$ is a $p^\alpha$-pure in $E_\alpha$, the theorem follows.

Notice that this theorem generalized condition (3) of Kulikov's theorem.

**Theorem 2.4.** If $G$ is a g.p. group and $\alpha$ is a limit ordinal less than or equal to the length of $G$, then the following are equivalent:

1. $\delta(G)$ is dense in $L_\alpha(G)$ in the natural topology; i.e., $\delta(G) + p^\beta L_\alpha(G) = L_\alpha(G)$ for all $\beta < \alpha$.
2. $L_\alpha(G)/\delta(G)$ is divisible.
3. $p^\beta L_\alpha(G) = \ker \pi_\beta$ for $\beta < \alpha$, where $\pi_\beta$ is the natural projection, $L_\alpha(G)$, onto $G/p^\beta G$; i.e., the natural topology and the induced topology are the same.

**Proof.** First we shall show that (1) implies (3). Note that $\pi_\beta L_\alpha(G) \subseteq G/p^\beta G$; it follows that $p^\beta L_\alpha G \subseteq \ker \pi_\beta$. If $x \in \ker \pi_\beta$, then $x = y + z$, with $y \in \delta(G)$ and $z \in p^\beta L_\alpha G$. Then $z \in \ker \pi_\beta$. Thus, $y \in \delta(G) \cap \ker \pi_\beta = p^\beta G$. It follows that $x \in p^\beta G + p^\beta L_\alpha G = p^\beta L_\alpha G$. Thus, $\ker \pi_\beta = p^\beta L_\alpha(G)$.  


We will now show (3) implies (1). A neighborhood system for $L_a(G)$ in the product topology is given by $\{\ker\pi_\alpha | \beta < \alpha\}$. If condition (3) holds, then $\{p^\beta L_aG | \beta < \alpha\}$ is a neighborhood system for $L_aG$. The group $\delta(G)$ is dense in $L_a(G)$ in the product topology. If condition (3) holds, then $\delta(G)$ is dense in $L_a(G)$ in the natural topology.

In order to show (1) is equivalent to (2), we first observe that, since $G$ is generalized primary, all groups in question are divisible by all primes other than $p$. Thus, it only has to be shown that $\delta(G)$ is dense in $L_a(G)$ if and only if $L_a(G)/\delta(G)$ is $p^\alpha$-divisible. The proof of this fact follows from a series of lemmas.

**Lemma 2.5.** If $\beta < \alpha$ and $\pi_\beta$ is the map defined in (3) of Theorem 2.4, then $L_aG = \delta(G) + \ker\pi_\beta$.

**Proof.** If $x \in L_aG$, then there exists $y \in G$ such that $y + p^\beta G = \pi_\beta(x)$. Then $\delta(y) - x \in \ker\pi_\beta$.

**Lemma 2.6.** Let $G, L_aG, \pi_\beta$ be as above. If $x \in \ker\pi_\beta$ and the image of $x$ in $L_a(G)/\delta(G)$ is in $p^\beta(L_a(G)/\delta(G))$, then $x \in p^\beta L_a(G)$.

**Proof.** The proof is by induction on $\beta$. If $\beta = 1$, then $\pi_1(x) = 0$, and $x$ maps into $p(L_aG/\delta(G))$. Thus, there exists a $y \in L_a(G)$ such that $x + \delta(G) = py + \delta(G)$, and so $x - py \in \delta(G)$. Since $\pi_1(x - py) = 0$, $x - py \in \ker\pi_1 \cap \delta(G) = p\delta(G)$. Thus, there exists a $z \in G$ such that $x - py = p\delta(z)$, or $x = p(y + \delta(z)) \in pL_aG$.

If $\beta > 1$, then let $\pi^*_\beta$ be the natural projection of $G/p^\beta G \to G/p^\gamma G$. If $\beta = \gamma + 1$, then $0 = \pi^*_\beta \pi_\beta(x) - \pi^*_\gamma(x)$. So $x \in \ker\pi_\tau$, and $x$ maps into $p^\tau(L_a(G)/\delta(G))$. Hence, $x \in p^\tau L_a(G)$. We must show $x \in p^\tau L_a(G)$.

Since $x \in p^\tau[L_a(G)/\delta(G)]$, there exists a $y' \in L_a(G)$ such that $y' + \delta(G) \in p^\tau(L_a(G)/\delta(G))$ and $x + \delta(G) = py' + \delta(G)$; thus, $x - py' \in \delta(G)$. Since $x \in p^\beta L_a(G)$, we see that $x - py' \in pL_aG \cap \delta(G) = p\delta(G)$; so $x = p(y' + z)$ for some $z \in \delta(G)$. Let $y = y' + z$. Then $x = py$ and $y + \delta(G) = y' + \delta(G) \in p^\tau(L_a(G)/\delta(G))$. By Lemma 2.5, $L_a(G) = \delta(G) + \ker\pi_\tau$. So there exists $y'' \in \ker\pi_\tau, g \in \delta(G)$, such that $y = y'' + g$. Then $y'' + \delta(G) = y + \delta(G) \in p^\tau(L_a(G)/\delta(G))$. Thus, $y'' \in p^\tau L_a(G)$ by the induction hypothesis. It follows that $pg = x - py'' \in \ker\pi_\beta$, so $pg \in \delta(G) \ker\pi_\beta = p^\beta \delta(G)$, and we see that $x \in p^\beta L_a(G)$.

Let $\beta$ be a limit ordinal. Then $\pi_\gamma(x) - \pi^*_\beta \pi_\beta(x) = 0$, and $x + \delta(G) \in p^\beta(L_a(G)/\delta(G)) \subseteq p^\tau(L_a(G)/\delta(G))$. 

So by the induction hypothesis we see that \( x \in p^\gamma L_\alpha(G) \) for all \( \gamma < \beta \), and thus \( x \in \bigcap_{\beta < \gamma} p^\gamma L_\alpha(G) = p^\beta L_\alpha(G) \).

We can now show the equivalence of conditions (1) and (2) of Theorem 2.4. Since \( L_\alpha(G) = \delta(G) + \ker \pi_\beta \), we see that every element of \( p^\beta(L_\alpha(G)/\delta(G)) \) is the image of an element of \( \ker \pi_\beta \). Lemma 2.6 then assures us that every element of \( p^\beta(L_\alpha(G)/\delta(G)) \) is the image of an element in \( p^\beta L_\alpha(G) \) under the homomorphism

\[
p^\beta L_\alpha(G) \longrightarrow (\delta(G) + p^\beta L_\alpha(G))/\delta(G).
\]

Since \( (\delta(G) + p^\beta L_\alpha(G))/\delta(G) \subseteq p^\beta(L_\alpha(G)/\delta(G)) \), it then follows that

\[
(\delta(G) + p^\beta L_\alpha(G))/\delta(G) = p^\beta(L_\alpha(G)/\delta(G))
\]

If \( L_\alpha(G)/\delta(G) \) is \( p \)-divisible, then \( p^\beta(L_\alpha(G)/\delta(G)) = L_\alpha(G)/\delta(G) \); and so \( L_\alpha(G) = \delta(G) + p^\beta L_\alpha(G) \). Conversely, if \( L_\alpha(G) = \delta(G) + p^\beta L_\alpha(G) \), then \( p^\beta(L_\alpha(G)/\delta(G)) = L_\alpha(G)/\delta(G) \). This completes the proof.

3. Some applications. The following definition is due to Harrison [4].

**Definition 3.1.** A g.p. group is called fully complete if \( L_\alpha G = G/\pi^\alpha G \) for all limit ordinals \( \alpha \) less than or equal to the length of \( G \).

Harrison [4] conjectured that a g.p. group is cotorsion if and only if \( G \) is fully complete. Using Theorems 1.3 and 2.4, we can find an example of a g.p. cotorsion group \( G \) which is not fully complete.

Let \( \Omega \) be the first uncountable ordinal. Nunke [11] has shown that \( p^\alpha \text{Ext} \) is not hereditary. Therefore, by Proposition 4.1, [11] and Theorem 13 we have that \( U_\alpha(G) \subseteq p^\alpha G \subseteq L_\alpha U_\alpha(G) \), for some group \( G \). The group \( U_\alpha(G) \) is a g.p. cotorsion group and is not fully complete.

Let \( \Omega \rightarrow G_\alpha \rightarrow H_\alpha \) define \( p^\alpha \). Let \( M_\alpha \) be the torsion subgroup of \( G_\alpha \). Nunke [11] has shown that \( M_\alpha \) is not \( p^\alpha \text{Ext} \)-projective. In showing that \( \alpha \) is hereditary if and only if \( U_\alpha(G) = p^\alpha(G) \) for all groups \( G \), Nunke actually showed that \( U_\alpha(G) = p^\alpha(G) \) if and only if \( p^\alpha \text{Ext}(M_\alpha, G) = 0 \), for \( G \) fixed.

**Lemma 3.2.** \( p^\alpha \text{Ext}(M_\alpha, \text{Tor}(M_\alpha, M_\alpha)) \neq 0 \).

**Proof.** In [11] it is shown that

\[
\begin{align*}
M_\alpha & \longrightarrow M_\alpha \\
\downarrow & \downarrow \\
Z \rightarrow & \rightarrow G_\alpha \rightarrow H_\alpha \\
\| & \downarrow & \downarrow \\
Z \rightarrow & Q_\alpha \rightarrow Z(p^\alpha)
\end{align*}
\]
is exact and the last column is $p^α$-pure. Here $Q_p = \{a/b \in Q \mid b = p^n$ for some $n\}$. From this we obtain

\[
\begin{array}{c}
(Tor M_\alpha, M_\alpha) \\
\downarrow \\
(Tor H_\alpha, M_\alpha) \longrightarrow M_\alpha \otimes Z \xrightarrow{\gamma} M_\alpha \otimes G_\alpha \\
\downarrow \\
M_\alpha = Tor (Z(p^\alpha), M_\alpha) \longrightarrow M_\alpha \otimes Z.
\end{array}
\]

Here $\beta$ is the zero map; for if $x \otimes n \in M_\alpha \otimes Z$, then $\beta(x \otimes n) = x \otimes n$. However, $n \in p^\alpha G_\alpha$. Thus $x \otimes n = 0$ in $M_\alpha \otimes G_\alpha$. Thus $\gamma$ is onto.

By Theorem 3.9 of [9], the sequence

\[E: Tor (M_\alpha, M_\alpha) \longrightarrow Tor (H_\alpha, M_\alpha) \longrightarrow Tor (Z(p^\alpha), M_\alpha) = M_\alpha\]

is $p^\alpha$-pure. Since $M_\alpha$ is not $p^\alpha$-projective, $M_\alpha$ is not a summand of $Tor (H_\alpha, M_\alpha)$, Theorem [3.1] of [9]. Thus $E \neq 0$, and

$p^\alpha Ext (M_\alpha, Tor (M_\alpha, M_\alpha)) \neq 0$.

This shows that $p^\alpha(Tor (M_\alpha, M_\alpha)) \neq U_\alpha(Tor (M_\alpha, M_\alpha))$. So, the group $U_\alpha(Tor (M_\alpha, M_\alpha))$ serves as a counter example to Harrison’s conjecture.

We are now in a position to examine condition (*) of Theorem 2.2. Let $G = U_\alpha(Tor (M_\alpha, M_\alpha))$. Then $L_\alpha G/G \neq 0$. Also, as $L_\alpha G$ and $G$ are cotorsion, $L_\alpha G/G$ is reduced. Theorem 2.4 now tells us that conditions (1), (2), and (4) of Theorem 2.2 do not hold. It follows that if $\alpha$ is not a countable limit of lesser ordinals, then $G$ need not be dense in $L_\alpha G$ in the natural topology. Also, the induced topology on $L_\alpha G$ need not be the natural topology on $L_\alpha G$.

**DEFINITION 3.3.** A g.p. group $G$ is called generally complete provided $L_\alpha(G)/\delta(G)$ is reduced for all limit ordinals $\alpha$ less than or equal to the length of $G$.

Notice that if the length of $G = \lambda(G)$ is less than $\Omega$ and if $G$ is generally complete, then $G$ is fully complete.

**THEOREM 3.4.** A necessary and sufficient condition for a g.p. group to be cotorsion is that it be generally complete.

**Proof.** Let $G$ be g.p. cotorsion group. Then $G/p^\alpha G$ is cotorsion for all $\beta$. By Theorem 5.3 of [9], $L_\alpha(G)$ is cotorsion. It follows that $L_\alpha(G)/\delta(G)$ is cotorsion and so reduced. Therefore, $G$ is generally complete.

Let $G$ be a g.p. generally complete group. Then $G/p^\alpha G$ is generally
complete for all $\beta$. We will show by transfinite induction on $\alpha$ that $G/p^\alpha G$ is cotorsion for all $\alpha$. If $\alpha = 0$, there is nothing to prove. Let $\alpha = \beta + 1$ for some ordinal $\beta$. The sequence $p^\beta G/p^\alpha G \to G/p^\alpha G \to G/p^\beta G$ is exact with ends cotorsion groups. Therefore, $G/p^\alpha G$ is cotorsion. Let $\alpha$ be a limit ordinal. Then, since $G$ is generally complete, $L(G)/\delta(G)$ is reduced. The group $L_\alpha(G)$ is cotorsion, since by the induction hypothesis it is an inverse limit of cotorsion groups by Theorem 5.3 of [9]. Therefore, $\delta(G) = G/p^\alpha G$ is cotorsion.

This last theorem answers Question 3 posed by Fuchs in [3].

In [11] Nunke showed that $p^\alpha \text{Ext}$ is hereditary, if $\alpha$ is a limit ordinal less than $\Omega$. In proving this he relied heavily upon Ulm's theorem. We now give a proof of this theorem which does not use Ulm's theorem.

**Theorem 3.5.** If $\alpha$ is an ordinal which satisfies condition (*) of theorem 2.4, then $p^\alpha \text{Ext}$ is hereditary.

**Proof.** Since $\alpha$ satisfies condition (*) of Theorem 2.4 $L_\alpha U_\alpha(G)/U_\alpha(G)$ is divisible. However, $L_\alpha U_\alpha(G)$ and $U_\alpha(G)$ are cotorsion groups; therefore, $L_\alpha U_\alpha(G)/U_\alpha(G)$ must be reduced. Thus, $L_\alpha U_\alpha(G) = U_\alpha(G)$, for all groups $G$.

Let $\beta$ be a hereditary ordinal; then $\beta + n$ is also hereditary Proposition 4.2 of [11]. If $\alpha < \Omega$, Proposition 4.1 of [11] and Theorem 1.3 give the desired result. If $\alpha \geq \Omega$, then $\alpha + \omega + n$ is hereditary if $n$ is any integer, by Proposition 4.2 of [11]. This fact together with Theorem 1.3 give the desired result.

We remark that for all other ordinals $\beta$ $p^\beta \text{Ext}$ is not hereditary. A proof of this fact may be found in [11].

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