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**A CHARACTERIZATION OF INTEGRAL OPERATORS ON THE  
SPACE OF BOREL MEASURABLE FUNCTIONS BOUNDED  
WITH RESPECT TO A WEIGHT FUNCTION**

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# A CHARACTERIZATION OF INTEGRAL OPERATORS ON THE SPACE OF BOREL MEASURABLE FUNCTIONS BOUNDED WITH RESPECT TO A WEIGHT FUNCTION

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Let  $I$  be a Borel set of the real line  $R$ ,  $C$  the space of complex numbers,  $V$  a  $\sigma$ -algebra of Borel subsets of  $I$ ,  $\mu$  a fixed measure on  $V$  such that for any bounded set  $Q \in V$ ,  $\mu(Q) < \infty$ ,  $g(\lambda, p)$  a nonvanishing complex valued function  $\mu$ -measurable in  $\lambda \in I$  such that  $|g(\lambda, p)| \uparrow$  in  $p$  where  $p$  belongs to a fixed open interval  $(a, b)$ , and  $S$  the set of  $\mu$ -measurable functions  $u$  from  $I$  into  $C$  such that  $|u(\lambda)g(\lambda, p)| \leq m$  for some  $p$  depending on  $u$ ,  $p \in (a, b)$ ,  $m \geq 0$  and  $m$  depending on  $u$ , and for all  $\lambda \in I$ . The purpose of this paper is to prove the following:

**THEOREM 1.** Let  $c(\lambda, \delta)$  be a  $\mu \times \mu$ -measurable function on  $I \times I$ . For every function  $u \in S$  the function

$$y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta), (\lambda, \delta) \in I \times I$$

is well defined and  $y \in S$  if and only if for every  $p \in (a, b)$  there exists a  $q \in (a, b)$  such that

$$\int_I |g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1}| d\mu(\delta) \leq m$$

for all  $(\lambda, \delta) \in I \times I$  and some  $m \geq 0$ .

Two examples of the space  $S$  are:

(1) Let  $I$  be the Borel set  $[0, \infty]$ ,  $C$  and  $V$  as before,  $\mu$  the Lebesgue measure, and  $g(\lambda, p) = e^{\lambda p}$  where  $p$  is some real number and  $\lambda \in I$ . Then  $S$  is the set of all functions  $u$  which are  $\mu$ -measurable from  $I$  into  $C$  and whose Laplace tranforms  $\int_I e^{\lambda x}u(x)d\mu(x)$  exist.

(2) A complex sequence  $u = \{u_n\}_{n=0}^\infty$  is analytic if and only if there exists some constant  $M > 0$  such that  $|u_n| \leq M^{n+1}$  for  $(n = 0, 1, 2, \dots)$  if and only if the  $\sup_n |p^n u_n| \leq N$  for some  $p > 0$ , constant  $N > 0$ , and  $(n = 0, 1, 2, \dots)$ . Now let  $I$  be the set of nonnegative integers,  $C$  and  $V$  as before,  $\mu(Q) =$  number of elements of a set  $Q \in V$  and  $g(\lambda, p) = p^\lambda$  where  $p \in (0, \infty)$ ,  $\lambda \in I$ . Then  $S$  is the space of all complex functions analytic at zero, or the space of analytic sequences, which will be henceforth denoted by  $A$ .

In light of Example (2), it is clear that Theorem 1 gives as a corollary a necessary and sufficient condition for infinite complex matrices to map  $A$  into itself. At the end of the paper it is shown

that this corollary is equivalent to I. Heller's characterization [3, Th. 1, p. 154], namely,

**PROPOSITION 1.** The transformation  $y_\lambda = \sum_{\delta=0}^\infty c_{\lambda\delta}u_\delta$  maps  $A$  into  $A$  if and only if for every  $p > 0$  there exists a  $q > 0$  and a constant  $M > 0$  such that  $|c_{\lambda\delta}| \leq Mp^\delta/q^\lambda$  for all  $(\lambda, \delta = 0, 1, 2, \dots)$ .

In [4] an alternative proof of Heller's result was given. And now the functional analysis techniques developed therein will be used to gain insight into the structure of  $S$  as a countable union of Banach spaces, and thereby to prove Theorem 1.

For every fixed  $p \in (a, b)$  let  $S_p = \{u \in S \mid |u(\lambda)g(\lambda, p)| \leq m\}$  for all  $\lambda \in I$  and  $\|u\|_p = \sup_{\lambda \in I} \{|u(\lambda)g(\lambda, p)|\}$ .

Let  $BM$  denote the set of all bounded  $\mu$ -measurable functions  $u^*$  from  $I$  into  $\mathbb{C}$  with  $\|u^*\|_{BM} = \sup_{\lambda \in I} |u^*(\lambda)|$ .

**THEOREM 2.** (1)  $S = \bigcup_{n=0}^\infty S_{p_n}$  where  $\{p_n\}_{n=0}^\infty$  is a sequence of numbers from  $(a, b)$  such that  $p_n \downarrow a$ , and

(2) for every  $p \in (a, b)$ ,  $(S_p, \|u\|_p)$  is a Banach space.

*Proof.* If  $r < s$  where  $r$  and  $s \in (a, b)$ , then  $S_r \subset S_s$ . A set theoretic argument completes the proof.

(2) It suffices to observe that  $(S_p, \|u\|_p)$  is isometrically isomorphic with the Banach space  $(BM, \|u^*\|_{BM})$ . The operator  $E_p$  from  $S_p$  into  $BM$  establishing this maps  $u$  into  $u^*$  where  $u^*(\lambda) = u(\lambda)g(\lambda, p)$  for all  $\lambda \in I$ .

**THEOREM 3.** Let  $c^*(\lambda, \delta)$  be defined on  $I \times I$  such that

$$(C^*(u^*))(\lambda) = y^*(\lambda) = \int_I c^*(\lambda, \delta)u^*(\delta)d\mu(\delta)$$

is well defined for all  $u^* \in BM$ ,  $(\lambda, \delta) \in I \times I$  and the obtained function  $y^* = C^*(u^*) \in BM$ . Then (1)  $C^*$  is a linear continuous operator from  $BM$  into  $BM$ , and

$$(2) \quad \|C^*\| = \sup_{\lambda \in I} \int_I |c^*(\lambda, \delta)| d\mu(\delta) < \infty.$$

*Proof.* (1) For each  $\lambda \in I$ , let  $h_\lambda(u^*) = \int_I c^*(\lambda, \delta)u^*(\delta)d\mu(\delta)$ . It is now shown that for each  $\lambda \in I$ ,  $h_\lambda$  is a continuous linear functional from  $BM$  into  $\mathbb{C}$  with  $\|h_\lambda\| = \int_I |c^*(\lambda, \delta)| d\mu(\delta)$ . For each  $\lambda \in I$ , and non-negative integer  $n$  define  $c_n^*(\lambda, \delta) = c_n(\lambda, \delta)\chi_{Q_n}(\delta)$  where

$$c_n(\lambda, \delta) = \begin{cases} c^*(\lambda, \delta) & \text{if } |c^*(\lambda, \delta)| \leq n \\ 0 & \text{if } |c^*(\lambda, \delta)| > n, \end{cases}$$

where  $Q_n = I \cap [-n, n]$ , and  $\chi_{Q_n}$  is the characteristic function of  $Q_n$ . Define for each  $\lambda \in I$ ,  $(h_\lambda(u^*))_n = \int_I c_n^*(\lambda, \delta) u^*(\delta) d\mu(\delta)$ . Clearly for each  $\lambda \in I$ ,  $(h_\lambda(u^*))_n$  is a continuous linear functional on  $BM$ . And now as the hypotheses of the Dominated Convergence Theorem are satisfied,  $(h_\lambda(u^*))_n \xrightarrow{n \rightarrow \infty} h_\lambda(u^*)$ . That  $h_\lambda(u^*)$  is linear and continuous follows from [2, Th. 17, p. 54].

From this last property, it follows that  $|h_\lambda(u^*)| \leq \|h_\lambda\| \cdot \|u^*\|_{BM}$  for all  $u^* \in BM$ . In particular, for each  $\lambda \in I$ , let

$$u_{0_\lambda}^*(\delta) = \frac{c^*(\lambda, \delta)!}{|c^*(\lambda, \delta)|} \chi_B$$

where  $B = I - \{\delta \in I \mid c^*(\lambda, \delta) = 0\}$  and ! denotes complex conjugation. So  $u_{0_\lambda}^*$  is a bounded  $\mu$ -measurable function such that  $\|u_{0_\lambda}^*\| \leq 1$ . Substituting  $u_{0_\lambda}^*$  for  $u^*$  yeilds  $|h_\lambda(u_{0_\lambda}^*)| = \int_I |c^*(\lambda, \delta)| d\mu(\delta) \leq \|h_\lambda\|$ .

Conversely, for any  $u^* \in BM$ ,

$$|h_\lambda(u^*)| \leq \|u^*\|_{BM} \int_I |c^*(\lambda, \delta)| d\mu(\delta)$$

or  $\|h_\lambda\| \leq \int_I |c^*(\lambda, \delta)| d\mu(\delta)$ . And so  $\|h_\lambda\| = \int_I |c^*(\lambda, \delta)| d\mu(\delta)$ .

Moreover, for all  $\lambda \in I$  and all  $u^* \in BM$ ,  $|h_\lambda(u^*)| = |y^*(\lambda)| \leq \|y^*\|_{BM}$ . By the Uniform Boundness Theorem,  $\|h_\lambda\| \leq P$  for all  $\lambda \in I$  and so  $a = \sup_{\lambda \in I} \left\{ \int_I |c^*(\lambda, \delta)| d\mu(\delta) \right\} \leq P$ . But  $|h_\lambda(u^*)| = |y^*(\lambda)| \leq a \|u^*\|_{BM}$ . And thus for all  $u^* \in BM$ ,  $\|y^*\|_{BM} = \|C^*(u^*)\|_{BM} \leq a \|u^*\|_{BM}$ . This implies that  $C^*$  is continuous from  $BM$  into itself and that  $\|C^*\| \leq a$ .

(2) As  $C^*$  is a linear continuous operator from  $BM$  into  $BM$   $\left| \int_I c^*(\lambda, \delta) u^*(\delta) d\mu(\delta) \right| \leq \|C^*\| \cdot \|u^*\|_{BM}$  for all  $u^* \in BM$ . Substituting  $u_{0_\lambda}^*$  (defined above in (1)) for  $u^*$  yeilds  $\int_I |c^*(\lambda, \delta)| d\mu(\delta) \leq \|C^*\|$  for each  $\lambda \in I$ . Thus  $a \leq \|C^*\|$ . And so  $\|C^*\| = a$ .

**THEOREM 4.** Let  $c(\lambda, \delta)$  be a function defined on  $I \times I$  such that for all  $u \in S$ ,  $y(\lambda) = \int_I c(\lambda, \delta) u(\delta) d\mu(\delta)$  is well defined and  $y \in S$ . Put  $y = C(u)$ . For each  $p$  and  $q$  fixed and belonging to  $(a, b)$ , let  $S_{pq} = \{u \in S_p \mid C(u) \in S_q\}$ . Then

(1)  $S_p = \bigcup_{n=0}^\infty S_{pq_n}$  where  $q_n \downarrow a$  for any  $p \in (a, b)$ , and

(2)  $(S_{pq}, \|u\|_{pq} = \|u\|_p + \|C(u)\|_q)$  is a Banach space.

*Proof.* (2) If the graph of  $C$  is closed in  $S_p \times S_q$ , then

$$(Z, \|u\|_p + \|C(u)\|_q)$$

where  $Z = \{(u, C(u)) \mid u \in S_p\}$  is a Banach space. And as the mapping

from  $S_{pq}$  into  $Z$  defined by  $u \rightarrow (u, C(u))$  establishes an isometric isomorphism between  $S_{pq}$  and  $Z$ , it suffices to prove that the graph of  $C$  is closed in  $S_p \times S_q$ .

For each  $\lambda \in I$ , let

$$k_\lambda(u) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta) = (C(u))(\lambda) = k_\lambda(E_p^{-1}(u^*)) .$$

Here  $E_p: u \rightarrow u^*$  is the isometric isomorphism from  $S_p$  into  $BM$ , and  $u^*(\lambda) = u(\lambda)g(\lambda, p)$  for all  $\lambda \in I$ . As  $k_\lambda E_p^{-1}$  is a linear continuous functional on  $BM$ , it follows that  $k_\lambda$  is a linear continuous functional on  $S_p$  for each  $\lambda \in I$ . This with the uniqueness of limits in  $S_q$  and the Closed Graph Theorem, prove that  $C$  is closed in  $S_p \times S_q$ .

**THEOREM 5.** *Let  $c(\lambda, \delta)$  be a function defined on  $I \times I$  such that for all  $u \in S$ ,  $y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta)$  is well defined and  $y \in S$ . Put  $y = C(u)$ . Then*

(1) *for every  $p \in (a, b)$  there exists a  $q \in (a, b)$  such that  $u \in S_p$  implies  $C(u) \in S_q$ .*

*The operator  $C$  from  $S_p$  into  $S_q$  generated by  $c(\lambda, \delta)$  is*

(2) *linear and continuous, and*

(3) *its norm,  $\|C\| = \sup_{\lambda \in I} \int_I |g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1}| d\mu(\delta) < \infty$ .*

*Proof.* (1) From Theorem 2. (1),  $S = \bigcup_{n=0}^\infty S_{q_n}$  where  $\{q_n\}_{n=0}^\infty \in (a, b)$  and  $q_n \downarrow a$ . As  $C$  maps  $S$  into itself, for any  $p \in (a, b)$ ,  $C$  maps  $S_p$  into  $\bigcup_{n=0}^\infty S_{q_n}$ . But  $S_p = \bigcup_{n=0}^\infty S_{pq_n}$  by Theorem 4. (1). Now as the injective maps from  $S_{pq_n}$  into  $S_p$  are continuous for all  $p$  and  $q_n$ , by [5, Corollary 6, p. 205] or [6, Satz 4.6, p. 472] there exists an index  $q_k \in (a, b)$  such that  $S_p = S_{pq_k}$ . So  $q_k$  is the desired number.

(2) The linearity of  $C$  is clear. And by definition of the Banach norm  $\|u\|_{pq}$  on  $S_{pq}$ ,  $C$  is continuous from  $S_p$  into  $S_q$ .

(3) Map  $S_p$  into  $BM$  by the operator  $E_p: u \rightarrow u^*$  where  $u^*(\lambda) = u(\lambda)g(\lambda, p)$  for all  $\lambda \in I$ . Define the operator  $C^*$  to be  $E_q C E_p^{-1}$  where  $p, q \in (a, b)$ .  $C^*$  is a linear and continuous operator from  $BM$  into itself whose norm is given by Theorem 3 (2). But  $\|C^*\| = \|C\|$ .

*Proof of Theorem 1.* Necessity follows immediately from (1) and (3) of Theorem 5.

Conversely, let  $u \in S$  and  $y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta)$ . Now for any  $p, q \in (a, b)$

$$\begin{aligned} |y(\lambda)g(\lambda, q)| &\leq \int_I |g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1}| \cdot |g(\delta, p)u(\delta)| d\mu(\delta) \\ &\leq \|C\| \cdot \|u(\delta)g(\delta, p)\|_{BM} \leq M . \end{aligned}$$

Moreover as  $(I, V, \mu)$  is a totally  $\sigma$ -finite measure space, the  $\mu$ -measurable function  $u(\delta)$  defined to be  $u''(\lambda, \delta)$  is  $\mu \times \mu$ -measurable. An application of Tonelli's Theorem completes the proof that  $y(\lambda)$  is  $\mu$ -measurable. And so  $y(\lambda) \in S$ .

If  $N$  is the set of nonnegative integers,  $c(\lambda, \delta)$ , where  $(\lambda, \delta) \in N \times N$ , can be identified with an infinite complex matrix  $(c_{\lambda\delta})$ . Clearly  $(c_{\lambda\delta})$  is  $\mu \times \mu$ -measurable on  $N \times N$ .

**COROLLARY TO THEOREM 1.** *The transformation  $C$  generated by an infinite complex matrix  $(c_{\lambda\delta})$ ,  $(\lambda, \delta) \in N \times N$  defined by  $y_\lambda = \sum_{\delta=0}^\infty c_{\lambda\delta} u_\delta$  maps the space  $A$  of analytic sequences into itself if and only if for every  $p > 0$  there exists a  $q > 0$  such that*

$$\sup_{\lambda \in I} \sum_{\delta=0}^\infty q^\lambda |c_{\lambda\delta}| p^{-\delta} \leq k \quad \text{for } (\lambda, \delta) \in N \times N, \text{ constant } k > 0 .$$

The next proposition shows that this corollary is equivalent to Heller's characterization, Proposition 1.

**PROPOSITION 2.** For each  $p > 0$  there exists a  $q > 0$  and a constant  $k > 0$  such that  $\sup_{\lambda \in I} \sum_{\delta=0}^\infty q^\lambda |c_{\lambda\delta}| p^{-\delta} \leq k$  if and only if for each  $p > 0$  there exists a  $r > 0$  and a constant  $m > 0$  such that  $|c_{\lambda\delta}| \leq m p^\delta / r^\lambda$ .

*Proof.* Sufficiency. Let  $p > 0$  and let  $p'$  be such that  $0 < p' < 1$ . Then  $pp' > 0$ . Given there exists a  $r > 0$  such that  $|c_{\lambda\delta}| \leq m(pp')^\delta / r^\lambda$  for all  $(\lambda, \delta) \in N \times N$ , and so  $\sum_{\delta=0}^\infty r^\lambda |c_{\lambda\delta}| p^{-\delta} \leq m(1 - p')^{-1}$ , for each  $\lambda \in N$ .

In conclusion, it is natural to ask: (1) which analytic functions  $f$  in the half plane  $\text{Re}(z) \leq r$  can be represented by the integral  $f(z) = \int_I u(\lambda)e^{z\lambda} d\mu(\lambda)$  where the determining function  $u \in S$ ,  $I$  is a Borel set of the real line and  $\mu$  is the Lebesgue measure; and (2) to which classes of measurable functions can Theorem 1 be generalized? It is thought that Theorem 1 can be generalized to (a) Bochner measurable functions bounded with respect to a weight function simply by using a Fubini theorem in place of a Tonelli theorem in the sufficiency proof of Theorem 1, and (b) Borel measurable functions essentially bounded with respect to a weight function, where two functions are equal if and only if they coincide everywhere, by using the lifting property of A. and C. Ionescu-Tulcea.

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