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FRACTIONAL INTEGRATION AND INVERSION FORMULAE ASSOCIATED WITH THE GENERALIZED WHITTAKER TRANSFORM

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In the present note, we invoke the theories of the Mellin transform as well as fractional integration to investigate a solution of the integral equation

$$(*)\qquad \int_0^\infty (xt)^{\sigma-(1/2)} e^{-(1/2)xt} W_{k+(1/2),m}(xt) f(t)dt = W_{k,m}^{(\sigma)}\{f: x\}\,, \qquad x > 0\,,$$

which defines a generalized Whittaker transform of the unknown function $f \in L_2(0, \infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f: x\}$.

It is shown that under certain constraints **$(*)$ can be reduced to the form of a Laplace integral which is readily solvable by familiar techniques.**

Two well-known generalizations of the classical Laplace transform (cf., e.g., [12])

$$(1)\qquad L[f: x] = \int_0^\infty e^{-xt} f(t)dt\,, \qquad x > 0\,,$$

are due to Meijer [6] and Varma [11]. The object of the present note is to investigate a solution of the integral equation

$$(2)\qquad \int_0^\infty (xt)^{\sigma-(1/2)} e^{-(1/2)xt} W_{k+(1/2),m}(xt) f(t)dt = W_{k,m}^{(\sigma)}\{f: x\}\,, \qquad x > 0\,,$$

which defines a generalized Whittaker transform [5, p. 23] of the unknown function $f(t) \in L_2(0, \infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f(t): x\}$, so that by appropriately specializing the parameter σ our results would readily enable us to invert the integral transforms of Meijer (cf., [1], [7]) and Varma (cf., [8], [9]).

In what follows we shall make a free use of the existing theories of (i) fractional integration due to Kober [4] and Erdélyi [2], and (ii) the Mellin transform detailed in [10, p. 94]. In the familiar notation, the operator of fractional integration that we need in our analysis is defined as follows:—

$$(3)\qquad K_{\zeta,\alpha,n}^{(-)} f(x) = \frac{n}{\Gamma(\alpha)} x^{\zeta} \int_x^\infty (u^n - x^n)^{\alpha-1} u^{-\zeta-n\alpha+n-1} f(u)du\,,$$

where $f \in L_p(0, \infty)$, $p^{-1} + q^{-1} = 1$, if $1 < p < \infty$, and q^{-1} or $p^{-1} = 0$

according as p or $q = 1; \alpha > 0, n > 0, \zeta > -p^{-1}$.

Confining ourselves to the L_2-space theory, for simplicity of the conditions involved, and invoking Fox's lemma (see [3], p. 458), we can establish the following theorems in the usual manner.

THEOREM 1. *Let $f \in L_2(0, \infty)$ be a solution of the integral equation* (2). *Then*

$$f(x) = L^{-1}[K^{(-)}_{\sigma-k,\sigma+k,1} W^{(\sigma)}_{k,m}\{f: x\}] , \tag{4}$$

provided (i) $x > 0$, (ii) $\sigma + k \geqq 0$ *and* (iii) $1/2 + \sigma - k > 0$.

THEOREM 2. *Let $f(x)$ be a solution of* (1) *that belongs to* $L_2(0, \infty)$. *Then*

$$K^{(-)}_{\sigma+m,\alpha,1} x^{\sigma-m} L[t^{\sigma-m} f(t): x] = W^{(\sigma)}_{-m-\alpha,m}\{f: x\} , \tag{5}$$

provided (i) $x > 0$, (ii) $\alpha \geqq 0$ *and* (iii) $1/2 + \sigma + m > 0$.

It may be of interest to remark here that when $\alpha = -k - m$, (5) is reduced to the interesting relationship

$$W^{(\sigma)}_{k,m}\{f: x\} = K^{(-)}_{\sigma+m,-k-m,1} x^{\sigma-m} L[t^{\sigma-m} f(t): x] , \tag{6}$$

which leads us to the construction of a table of generalized Whittaker transforms from that of the classical Laplace transform, provided $k + m \leqq 0, 1/2 + \sigma + m > 0$ and $x > 0$.

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