FRACTIONAL INTEGRATION AND INVERSION FORMULAE ASSOCIATED WITH THE GENERALIZED WHITTAKER TRANSFORM

H. M. (HARI MOHAN) SRIVASTAVA
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H. M. SRIVASTAVA

In the present note, we invoke the theories of the Mellin transform as well as fractional integration to investigate a solution of the integral equation

\[
\left( x \right) \int_0^\infty (xt)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2}, m}(xt)f(t)dt = W_{k,m}^{(\sigma)}{f: x},
\]

\[
x > 0 ,
\]

which defines a generalized Whittaker transform of the unknown function \( f \in L_e(0, \infty) \) to be determined in terms of its image \( W_{k,m}^{(\sigma)}{f: x} \).

It is shown that under certain constraints \((*)\) can be reduced to the form of a Laplace integral which is readily solvable by familiar techniques.

Two well-known generalizations of the classical Laplace transform (cf., e.g., [12])

\[
(1) \quad L[f: x] = \int_0^\infty e^{-xt}f(t)dt , \quad x > 0 ,
\]

are due to Meijer [6] and Varma [11]. The object of the present note is to investigate a solution of the integral equation

\[
(2) \quad \int_0^\infty (xt)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2}, m}(xt)f(t)dt = W_{k,m}^{(\sigma)}{f: x}, \quad x > 0 ,
\]

which defines a generalized Whittaker transform [5, p. 23] of the unknown function \( f(t) \in L_e(0, \infty) \) to be determined in terms of its image \( W_{k,m}^{(\sigma)}{f(t): x} \), so that by appropriately specializing the parameter \( \sigma \) our results would readily enable us to invert the integral transforms of Meijer (cf., [1], [7]) and Varma (cf., [8], [9]).

In what follows we shall make a free use of the existing theories of (i) fractional integration due to Kober [4] and Erdélyi [2], and (ii) the Mellin transform detailed in [10, p. 94]. In the familiar notation, the operator of fractional integration that we need in our analysis is defined as follows:

\[
(3) \quad K_{x, a, n}^{(\sigma)}f(x) = \frac{n}{\Gamma(\alpha)}x^{\sigma-\frac{1}{2}}\int_x^\infty (u^n - x^n)^{\alpha-1}u^{\sigma-n\alpha+\frac{1}{2}}f(u)du ,
\]

where \( f \in L_p(0, \infty) \), \( p^{-1} + q^{-1} = 1 \), if \( 1 < p < \infty \), and \( q^{-1} \) or \( p^{-1} = 0 \)
according as \( p \) or \( q = 1; \alpha > 0, n > 0, \zeta > -p^{-1} \).

Confining ourselves to the \( L_2 \)-space theory, for simplicity of the conditions involved, and invoking Fox’s lemma (see [3], p. 458), we can establish the following theorems in the usual manner.

**Theorem 1.** Let \( f \in L_2(0, \infty) \) be a solution of the integral equation (2). Then

\[
f(x) = L^{-1}[K_{\sigma-k, \sigma+k, 1} W_{\sigma, m}(f; x)],
\]

provided (i) \( x > 0 \), (ii) \( \sigma + k \geq 0 \) and (iii) \( 1/2 + \sigma - k > 0 \).

**Theorem 2.** Let \( f(x) \) be a solution of (1) that belongs to \( L_2(0, \infty) \). Then

\[
K_{\sigma+m, \sigma, 1} w^{\sigma-m} L[t^{\sigma-m} f(t); x] = W_{-m-\sigma, m}(f; x),
\]

provided (i) \( x > 0 \), (ii) \( \alpha \geq 0 \) and (iii) \( 1/2 + \sigma + m > 0 \).

It may be of interest to remark here that when \( \alpha = -k - m \), (5) is reduced to the interesting relationship

\[
W_{k, m}(f; x) = K_{\sigma+m, -k-m, 1} w^{\sigma-m} L[t^{\sigma-m} f(t); x],
\]

which leads us to the construction of a table of generalized Whittaker transforms from that of the classical Laplace transform, provided \( k + m \leq 0, 1/2 + \sigma + m > 0 \) and \( x > 0 \).

The author wishes to thank Professor Richard F. Arens and the referee for helpful suggestions.

**References**


Received November 27, 1967.

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The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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