Pacific Journal of Mathematics

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Vol. 26, No. 2 December 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 2, 1968

BOUNDARY VALUE PROBLEMS FOR ELLIPTIC CONVOLUTION EQUATIONS OF WIENER-HOPF TYPE IN A BOUNDED REGION

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Let A be an elliptic convolution operator of order α on a bounded open set G of $R^n, \alpha > 0$. Let A_j be the principal part of A in a local coordinates system and $\tilde{A}_j(x^j, \xi)$ be its symbol with a Wiener-Hopf type of factorization with respect to ξ_n: $\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$ for $x_n^j = 0$. $\tilde{A}_j^+$ is analytic in $\operatorname{Im} \xi_n > 0$, is homogeneous of order k in ξ, k is a positive integer, $k<\alpha$. $\tilde{A}_j^-$ is analytic in $\operatorname{Im} \xi_n \leqq 0$. Let B_r; $r=1,\cdots,k$ be a system of convolution operators on ∂G, of orders α_r; $0 \leqq \alpha_r < \alpha$ and let B_{rj} be the principal part of B_r in a local coordinates system. The $\tilde{A}_j^+, \tilde{B}_{rj}$ are assumed to satisfy a Shapiro-Lopatinskii type of condition for each j.

Visik and Eskin have shown that the operator U from $H_+^s(G)$ into

$$H^{s-\alpha}(G, \partial G) = H^{s-\alpha}(G)x \times \prod_{r=1}^{k} H^{s-\alpha_r-1/2}(\partial G)\,; \qquad \alpha \leqq s\ ,$$

defined by: $Uu = \{Au, B_1u, \cdots, B_ku\}$ is of Fredholm type. In this paper, we show the smoothness in the interior of the solutions of $Uu = (f, g_1, \cdot, g_k)$. We prove that if $\tilde{A}_j^+, \tilde{B}_{rj}$ satisfy a strengthened form of the Shapiro-Lopatinskii condition, then the operator $U_\lambda u = \{(A + \lambda^\alpha)u, B_1u, \cdots, B_ku\}$ is one-to-one and onto. The nonlinear problem:

$$U_\lambda u = \{f(x, S_0u, \cdots, S_{\alpha-1}u), g_1, \cdots, g_k\}$$

has a solution in $H_+^\alpha(G)$. $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ is continuous in all the variables and has at most a linear growth in $(\zeta_0, \cdots,\zeta_{\alpha-1})$. If the set $\Omega = \{u: u \in H_+^\alpha(G), B_ru = 0$ on $\partial G, r = 1, \cdots, k\}$ is dense in $L^2(G)$, then the completeness in $L^2(G)$ of the generaliled eigenfunctions of the operator A_2 associated with $Uu = \{f, 0, \cdots, 0\}$ is established.

Boundary-value problems for elliptic convolution operators have been considered recently by Visik-Eskin [4].

In § I, we give the notation and terminology which are those of Visik-Eskin and state the assumptions. The main results are given without proofs in § 2. The proofs are carried out in § 3.

1. Let s be an arbitrary real number and $H^s(R^n)$ be the Sobolev Slobodetskii space of (generalized) functions f such that:

$$\|f\|_s^2 = \int_{E^n}(1 + |\xi|^2)^s\,|\tilde{f}(\xi)|^2\, d\xi$$

$\tilde{f}(\xi)$ is the Fourier transform of f.

By $H^s(R^n_+)$, we denote the space consisting of functions defined on $R^n_+ = \{x: x_n > 0\}$ and which are the restrictions to R^n_+ of functions in $H^s(R^n)$. Let lf be an extension of f to R^n. Then:

$$\|f\|^+_s = \|f\|_{H^s(R^n_+)} = \inf \|lf\|_s .$$

The infimum is taken over all the extensions lf of f.

Let $\theta(x_n)$ be the function equal to 1 if $x_n > 0$ and to 0 if $x_n \leqq 0$. Every function f in $L^2(R^n)$ may be written as $f = \theta f + (1 - \theta)f$. Hence $L^2(R^n)$ has the following orthogonal decomposition:

$$L^2(R^n) = \overset{\circ}{H}{}^+_0 + \overset{\circ}{H}{}^-_0 .$$

We denote by H_s^+, the space of functions f_+ with f_+ in $\overset{\circ}{H}{}^+_0$ and such that f_+ belongs to $H^s(R^n_+)$. $\overset{\circ}{H}{}_s^+$ is the subspace of $H^s(R^n)$ consisting of functions with supports in $\mathrm{cl}\,(R^n_+)$. $\widetilde{H}{}^+_s, \widetilde{H}_s, \widetilde{\overset{\circ}{H}}{}^+_s$ denote respectively the spaces which are the Fourier images of $H^+_s, H_s, \overset{\circ}{H}{}^+_s$.

Let $\tilde{f}(\xi)$ be a smooth decreasing function (i.e. $\tilde{f}(\xi) \leqq M|\xi_n|^{-1-\varepsilon}$ for large $|\xi_n|$ and $\varepsilon > 0$). The operator Π^+ is defined as:

$$\Pi^+\tilde{f}(\xi) = \frac{1}{2}\tilde{f}(\xi', \xi_n) + i(2\pi)^{-1}\,\mathrm{v.p.}\int \tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1}d\eta_n ,$$

$$\xi' = (\xi_1, \cdots, \xi_{n-1})$$

For any $\tilde{f}$, then the above relation is understood as the result of the closure of the operator Π^+ defined on the set of smooth and decreasing functions.

Π^+ is a bounded mapping from $\widetilde{H}_s$ into $\widetilde{\overset{\circ}{H}}{}^+_s$ if $0 \leqq s < \frac{1}{2}$ and a mapping from $\widetilde{H}_s$ into $\widetilde{H}{}^+_s$ if $\frac{1}{2} \leqq s$. Π^- is defined similarly.

Set: $\xi_- = \xi_n - i|\xi'|$; $(\xi_- - i)^s$ is analytic in $\mathrm{Im}\,\xi_n < 0$. Then:

$$\|f\|^+_s = \|\Pi^+(\xi_- - i)^s\widetilde{lf}(\xi)\|_0$$

where lf is any extension of f to R^n (Cf. [4], p. 93, relation (8.1))

Let G be a bounded open set of R^n with a smooth boundary ∂G. We denote by $H^s(G)$ the restriction to G of functions in $H^s(R^n)$ with the norm: $\|f\|_s = \inf \|g\|_{H^s(R_n)}$; $g = f$ on G; $s \geqq 0$.

By $H^s_+(G)$, we denote the space of functions f defined on all of R^n, equal to 0 on $R^n/\mathrm{cl}\,(G)$ and coinciding in $\mathrm{cl}\,(G)$ with functions in $H^s(G)$.

$H^s(\partial G)$ is defined as the completion of $C^\infty(\partial G)$ with respect to:

$$\|f\|'_s = \left\{\sum_j \|\varphi_j f\|^2_{H^s(R^{n-1})}\right\}^{1/2}; \; s \geqq 0$$

where $\| \varphi_j f \|_{H^s(R^{n-1})}$ is taken in local coordinates and the φ_j are those functions of a finite partition of unity corresponding a covering of $\mathrm{cl}\, G$, whose supports intersect the boundary G. We may show that different partitions of unity give rise to equivalent norms (cf. [3]).

DEFINITION 1. $\tilde{A}(\xi)$ is in 0_α if and only if:
(i) $\tilde{A}$ is homogeneous in ξ of order α.
(ii) $\tilde{A}$ is continuous for $\xi \neq 0$.

DEFINITION 2. $\tilde{A}_+(\xi', \xi_n)$ is in 0_α^+ if and only if:
(i) $\tilde{A}_+$ is in 0_α.
(ii) $\tilde{A}_+(\xi', \xi_n)$ has an analytic continuation with respect to ξ_n in $\mathrm{Im}\, \xi_n > 0$ for each ξ'.
Similar definition for 0_α^-.

DEFINITION 3. $\tilde{A}(\xi)$ is in E_α if and only if:
(i) $\tilde{A}(\xi)$ is in 0_α.
(ii) $\tilde{A}(\xi)$ satisfies the ellipticity condition, i.e. $\tilde{A}(\xi) \neq 0$ for $\xi \neq 0$.
(iii) $\tilde{A}(\xi)$ has for $\xi' \neq 0$, continuous first order derivatives, bounded if $|\xi| = 1, \xi' \neq 0$.

DEFINITION 4. $\tilde{A}_+(\xi)$ is in C_k^+ if and only if:
(i) $\tilde{A}_+(\xi)$ is in 0_k^+ and $\tilde{A}_+(\xi) \neq 0$ for $\xi \neq 0$; k is a positive integer.
(ii) For any integer $p > 0$, there is an expansion:

$$\tilde{A}_+(\xi) = \sum_{s=0}^{p} c_s(\xi')\xi_+^{k-s} + R_{k,p+1-k}(\xi', \xi_n)$$

where $\xi_+ = \xi_n + i|\xi'|$; all the terms are in 0_k^+ and:

$$|R_{k,p+1-k}(\xi', \xi_n)| \leqq C|\xi'|^{p+1}(|\xi'| + |\xi_n|)^{k-p-1} .$$

DEFINITION 5. $\tilde{A}(\xi)$ is in D_α if and only if:
(i) $\tilde{A}(\xi)$ is in 0_α.
(ii) For each $s \geqq \alpha$; there is a decomposition:

$$\xi_-^s \tilde{A}(\xi) = \tilde{A}_-(\xi) + R_{s+\alpha,-1}(\xi)$$

where $\tilde{A}_-(\xi)$ is in $0_{\alpha+s}^-$, $|R_{s+\alpha,-1}(\xi)| \leqq C|\xi'|^{s+1+\alpha}(|\xi'| + |\xi_n|)^{-1}$.

DEFINITION 6. $\tilde{A}(\xi)$ is in $D_{\alpha,1}$ if and only if:
(i) $\tilde{A}(\xi)$ is in D_α.
(ii) $\tilde{A}_-(\xi)$ and $R_{s+\alpha,-1}(\xi)$ are continuously differentiable for $\xi' \neq 0$.
(iii) $|\tilde{A}_-(\xi)| \leqq C|\xi|^{\alpha-1}$; $|R_{s+\alpha,-1}(\xi)| \leqq C|\xi'|^{s+\alpha}(|\xi'| + |\xi_n|)^{-1}$.

DEFINITION 7. Let A be a linear, bounded operator from H_s^+ into

$H^{s-\alpha}(R^n_+)$. Then any bounded, linear operator T from H^+_{s-1} into $H^{s-\alpha}(R^n_+)$ (or from H^+_s into $H^{s-\alpha+1}(R^n_+)$) is called a right (left) smoothing operator with respect to A.

T is a smoothing operator with respect to A if T is both a left and right smoothing operator.

Let $\tilde{A}(\xi)$ be in E_α for $\alpha > 0$ and u_+ be in H^+_s, $s \geqq 0$. Then we define: $Au_+ = F^{-1}(\tilde{A}(\xi)\tilde{u}_+(\xi))$ where the inverse Fourier transform is understood in the sense of the theory of distributions. Au_+ is well-defined.

Let $\tilde{A}(x, \xi)$ be in E_α for x in $\operatorname{cl} G$ and $\tilde{A}(x, \xi)$ be infinitely differentiable with respect to x and to ξ. We extend $\tilde{A}(x, \xi)$ with respect to x, to all of R^n by setting $\tilde{A}(x, \xi) = 0$ for $|x| \geqq p - \varepsilon, \varepsilon > 0$. The homogeneity with respect to ξ is preserved. We expand $\tilde{A}(x, \xi)$ into a Fourier series:

$$\tilde{A}(x, \xi) = \sum_{k=-\infty} \psi_0(x) \exp(ikx\pi/p)\tilde{L}_k(\xi) ; \qquad k = (k_1, \cdots, k_n)$$

and:

$$\tilde{L}_k(\xi) = (2p)^{-n}\int_{-p}^{p} \exp(-ikx\pi/p)\tilde{A}(x, \xi)dx$$

$\psi_0(x) \in C^\infty_c(R^n)$ with $\psi_0(x) = 1$ for $|x| \leqq p - \varepsilon$; $\psi_0(x) = 0$ for $|x| \geqq p$. For u_+ in $H^s_+(G)$, we define:

$$P^+Au_+ = P^+\Big(\sum_{k=-\infty} \psi_0(x) \exp(ikx\pi/p)L_k * u_+\Big) .$$

P^+ is the restriction operator of functions defined on R^n to G, $L_k u_+$ is defined as before since its symbol $\tilde{L}_k(\xi)$ is independent of x and $|\tilde{L}_k(\xi)| \leqq (1 + |k|)^{-M} |\xi|^\alpha$ for large positive M.

DEFINITION 8. $\tilde{A}(x, \xi)$ is in D^0_α if and only if:

(i) $\tilde{A}(x, \xi)$ is infinitely differentiable with respect to x and $\xi \neq 0$.

(ii) $\tilde{A}(x, \xi)$ is in 0_α for x in R^n.

(iii) $a_{k2}(x) = (\partial^k/\partial\xi'^k)\tilde{A}(x, 0, -1) = (-1)^k \exp(-i\pi\alpha)(\partial^k/\partial\xi'^k)\tilde{A}(x, 0, 1)$ x in R^n, $0 \leqq |k| < +\infty$.

DEFINITION 9. $\tilde{A}(x, \xi)$ is in $\hat{D}^1_{\alpha,1}$ if and only if:

(i) $|D^p_x\tilde{A}(x, \xi)| \leqq C_p(1 + |\xi|)^\alpha$; $0 \leqq |p| < +\infty$.

(ii) For each x in R^n and for any $s \geqq -\alpha$, there is a decomposition: $(\xi_- - i)^s\tilde{A}(x, \xi) = \tilde{A}_-(x, \xi) + R(x, \xi)$
$\tilde{A}_-(x, \xi)$ and $R(x, \xi)$ are infinitely differentiable with respect to x
$\tilde{A}_-(x, \xi)$ is analytic in $\operatorname{Im} \xi_n \leqq 0$ and:

$$| D_x^p \tilde{A}_-(x, \xi) | \leqq C_p(1 + | \xi |)^{s+\alpha}; | D_x^p D_\xi \tilde{A}_-(x, \xi) | \leqq c_p(1 + | \xi |)^{s-1+\alpha}$$
$$| D_x^p R(x, \xi) | \leqq C_p(1 + | \xi' |)^{s+1+\alpha}(1 + | \xi |)^{-1}$$
$$| D_x^p D_\xi R(x, \xi) | \leqq c_p(1 + | \xi' |)^{s+\alpha}(1 + | \xi |)^{-1} ; \qquad 0 \leqq | p | < +\infty .$$

Let B_r; $r = 1, \cdots, k$ be a system of convolution operators on ∂G. We introduce the definition of a regular elliptic convolution boundary value problem on G:

DEFINITION 10. Let G be a bounded open set of R^n and φ_j be a finite partition of unity corresponding to a covering N_j of $\mathrm{cl}\, G$. Let ψ_j be the infinitely differentiable functions with compact supports in N_j and such that: $\varphi_j \psi_j = \varphi_j$

(1) Let: $P^+A = \sum_j P^+\varphi_j A \psi_j + \sum_j P^+\varphi_j A(1 - \psi_j)$ be an elliptic convolution operator of order α on G with the following properties:

(a) The operator $\varphi_j A \psi_j$ transformed in local coordinates, is the sum of a convolution operator A_j and a smoothing operator. The symbol $\tilde{A}_j(x^j, \xi)$ is homogeneous of order α in ξ; $\alpha > 0$

(b) $\tilde{A}_j(x^j, \xi) \in E_\alpha$ and for $x_n^j = 0$ admits the factorization:

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi) \tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+$, $\tilde{A}_j^-$ belong to 0_k^+, $0_{\alpha-k}^-$ respectively and k is a positive integer.

(c) $\tilde{A}_j(x^j, \xi)$ is in $D_\alpha^0 \cap \hat{D}_{\alpha,1}^1$ for $x \in N_j \cap \partial G \neq 0$.

(2) Let γ denote a passage to the boundary ∂G and:

$$P^+B_r = \sum_j P^+\varphi_j B_r \psi_j + \sum_j P^+\varphi_j B_r(1 - \psi_j) ; \qquad r = 1, \cdots, k$$

be a system of convolution operators on ∂G with the following properties:

(a) The operator $\varphi_j B_r \psi_j$ taken in local coordinates, is the sum of a convolution operator B_{rj} with symbol $\tilde{B}_{rj}$, homogeneous of order α_r in ξ and a smoothing operator. $0 \leqq \alpha_r < \alpha - \frac{1}{2}$.

(b) $\tilde{B}_{rj}(x^j, \xi) \in D_{\alpha_r}^0 \cap \hat{D}_{\alpha_r,1}^1$ for $x \in N_j \cap \partial G \neq 0$.

The boundary-value problem: $\{P^+Au_+, \gamma P^+\beta_1 u_+, \cdots, \gamma P^+B_k u_+\}$ is said to be uniformly regular on G if:

$$\mathrm{Det}\, ((b_{rs}(x^j, \xi'))) \neq 0 \quad \text{for all} \quad x^j \in N_j \cap \partial G \neq 0$$

and:

$$\Pi^+ \tilde{B}_{rs}(x^j, \xi) \xi_n^{s-1} (\tilde{A}_j^+(x^j, \xi))^{-1} = i b_{rs}(x^j, \xi') \xi_+^{-1} + R_{rs}(x^j, \xi)$$
$$\mathrm{ord}\, (b_{rs}(\xi')) = \alpha_r + k - s , \qquad r, s = 1, \cdots, k .$$

Assumption (1); Let $\{P^+A, \gamma P^+B_1, \cdots, \gamma P^+B_k\}$ be a uniformly regular elliptic convolution boundary-value problem on G in the sense of Definition 10.

We assume there exists a ray $\arg\lambda = \theta$ such that:

(i) If: $\tilde{A}_j(x^j, \xi, \lambda) = \tilde{A}_j(x^j, \xi) + \lambda^\alpha = \tilde{A}_j^+(x^j, \xi, \lambda)\tilde{A}_j^-(x^j, \xi, \lambda)$; then: $\tilde{A}_j^+(x^j, \xi, \lambda)$ is in C_k^+.

(ii) $\operatorname{Det}((b_{rs}(x^j, \xi', \lambda))) \neq 0$ for all x^j with $N_j \cap \partial G \neq 0$ and $\arg\lambda = \theta, |\lambda| > \lambda_0 > 0$

$$\Pi^+\tilde{B}_{rj}(x^j, \xi)\xi_n^{s-1}(\tilde{A}_j^+(x^j, \xi, \lambda))^{-1} = ib_{rs}(x^j, \xi', \lambda)(\xi_+^\lambda)^{-1} + R_{rs}(x^j, \xi, \lambda)$$
$$\xi_+^\lambda = \xi_n + i(|\lambda| + |\xi'|)\,; \qquad r, s = 1, \cdots, k\,.$$

2. In this section, we shall state the results of the paper. First, we have an interior regularity theorem:

THEOREM 2.1. *Let* $\{P^+A, \gamma P^+B_1, \cdots, \gamma P^+B_k\}$ *be a uniformly regular elliptic convolution boundary-value problem on G in the sense of Definition* 10. *Let* u_+ *be an element of* $H_+^\alpha(G)$ *and* $Uu_+ = \{f, g_1, \cdots, g_k\}$ *with* $\{f, g_1, \cdots, g_k\}$ *in* $H^0(G, \partial G)$ *and* $\alpha \geqq 0$. *Suppose that f is in* $H^{s-\alpha}(G)$, $s \geqq \alpha$ *then* u_+ *is in* $H_+^\alpha(G) \cap H_{\mathrm{loc}}^s(G)$.

If f is in $C^\infty(cl\, G)$, *then*: u_+ *is in* $C_{\mathrm{loc}}^\infty(G)$.

With an additional hypothesis, we show that the operator associated with the problem is one-to-one and onto:

THEOREM 2.2. *Let* $\{P^+A, \gamma P^+B_r; r = 1, \cdots, k\}$ *be a uniform uniformly regular elliptic convolution boundary value problem on G in the sense of Definition* 10. *Suppose that Assumption* (1) *is satisfied. Then for every* $(f, g_1, \cdots, g_k)$ *in* $H^{s-\alpha}(G, \partial G)$, *there exists a unique solution* u_+ *in* $H_+^s(G)$ *of*:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } G, \qquad \gamma P^+B_r u_+ = g_r \text{ on } \partial G; r = 1, \cdots, k$$

$s \geqq \alpha$ *and* s, α, α_r *are all assumed to be nonnegative integers.*

Moreover, there exists a positive constant M independent of λ, u_+ f, g_r *such that*:

$$\|u_+\|_s + |\lambda|^s \|u_+\|_0 \leqq M\Big\{\|f\|_{s-\alpha} + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^k \|g_r\|'_{s-\alpha_r-(1/2)} + |\lambda|^{s-\alpha_r-(1/2)}\|g_r\|'_0\Big\}$$

for all u_+ *in* $H_+^s(G), \arg\lambda = \theta; |\lambda| \geqq \lambda_0 > 0$.

Now, we have a global regularity theorem for the solutions of $Uu_+ = (f, g_1, \cdots, g_k)$.

THEOREM 2.3. *Suppose the hypotheses of Theorem 2.2 are satisfied. Let u_+ be a solution in $H_+^\alpha(G)$ of $Uu_+ = (f, g_1, \cdots, g_k)$. If $(f, g_1, \cdots, g_k)$ is in $H^{s-\alpha}(G, \partial G)$, $s \geqq \alpha$, then: $u_+ \in H_+^s(G)$. More generally if f is in $C^\infty(\mathrm{cl}\, G)$, g_r are in $C^\infty(\partial G)$; then u_+ is in $C^\infty(G)$.*

We shall now consider problems related to the spectral theory of the operator associated with $Uu_+ = (f, 0, \cdots, 0)$.

COROLLARY 2.1. (i) *Suppose the hypotheses of Theorem 2.2 are satisfied. Let*

$$\Omega = \{u_+ : u_+ \in H_+^\alpha(G), \gamma P^+ B_r u_+ = 0 \text{ on } \partial G; r = 1, \cdots, k\}$$

Suppose that Ω is dense in $L^2(G)$. Let A_2 be the operator on $L^2(G)$ with $D(A_2) = \Omega$; $A_2 u_+ = P^+ A u_+$ on G.

Then: $(A_2 + \lambda^\alpha I)^{-1}$ exists, is defined on all of $L^2(G)$ and is a compact operator. The spectrum of A_2 is discrete.

(ii) *Suppose further that Assumption (1) is satisfied by rays $\arg \lambda = \theta_r$; $r = 1, \cdots, N$ and that the plane is divided by those rays into angles less than $2\alpha\pi/n$. Then the generalized eigenfunctions of A_2 are complete in $L^2(G)$.*

COROLLARY 2.2. *Suppose that the hypotheses of Theorem 2.2 are satisfied. Let S_r; $r = 0, \cdots, \alpha - 1$ be bounded linear operators from $H_+^r(G)$ into $L^2(G)$. Let $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ be a function measurable in x on G, continuous in all the other variables and such that:*

$$|f(x, \zeta_0, \cdots, \zeta_{\alpha-1})| \leqq M\left\{1 + \sum_{j=0}^{\alpha-1} |\zeta_j|\right\}.$$

Then for $(g_1, \cdots, g_k)$ in $\prod_{r=1}^k H^{\alpha-\alpha_r-(1/2)}(\partial G)$ and $|\lambda| \geqq \lambda_0 > 0$, $\arg \lambda = \theta$ there exists a solution u_+ in $H_+^\alpha(G)$ of:

$$P^+(A + \lambda)u_+ = f(x, S_0 u_+, \cdots, S_{\alpha-1} u_+) \text{ on } G;$$
$$\gamma P^+ B_r u_+ = g_r \text{ on } \partial G; r = 1, \cdots, k$$

3. *Proof of Theorem* 2.1. (1) First, we show the existence of a left regularizer of U.

From Theorem 2.9 of [4], the operator U has a right regularizer S, i.e. $US = I + R$, where S is a bounded linear mapping from $H^{s-\alpha}(G, \partial G)$ into $H_+^s(G)$ and R is a bounded linear mapping from $H^{s-\alpha}(G, \partial G)$ in $H^{s+1-\alpha}(G, \partial G)$.

Let R_1 be the operator from $H_+^s(G)$ into itself defined by the ralation: $R_1 u_+ = SUu_+ - u_+$.

We show that: $\| R_1 u_+ \|_{s+1} \leqq C \| u_+ \|_s$ for all u_+ in $H_+^{s+1}(G)$.

Consider:

$$UR_1u_+ = USUu_+ - Uu_+ = Uu_+ + RUu_+ - Uu_+ = RUu_+ .$$

From Theorem 2.9 of [4], we have:

$$\| R_1u_+ \|_{s+1} \leqq M\Big\{\| R_1u_+ \|_0 + \| P^+AR_1u_+ \|_{s+1-\alpha} + \sum_{r=1}^{k} \| \gamma P^+B_rR_1u_+ \|'_{s-\alpha_r+(1/2)}\Big\} .$$

But $RUu_+ = UR_1u_+$ and R is a bounded mapping from $H^{s-\alpha}(G, \partial G)$ into $H^{s+1-\alpha}(G, \partial G)$. Therefore:

$$\| R_1u_+ \|_{s+1} \leqq M\Big\{\| R_1u_+ \|_0 + \| P^+Au_+ \|_{s-\alpha} + \sum_{r=1}^{k} \| \gamma P^+B_ru_+ \|'_{s-\alpha_r-(1/2)}\Big\} .$$

Since we assume that in all the local coordinates system, the principal parts of A, B_r have symbols belonging to $\hat{D}^1_{\alpha,1}; \hat{D}^1_{\alpha_r,1}$ respectively with $0 \leqq \alpha_r < \alpha$; we have:

$$\| P^+Au_+ \|_{s-\alpha} \leqq C \| u_+ \|_s \quad \text{and} \quad \| \gamma P^+B_ru_+ \|'_{s-\alpha_r-1/2} \leqq C \| u_+ \|_s$$

(Cf. [4], Th. 1.4; p. 104).

Hence: $\| R_1u_+ \|_{s+1} \leqq M \| u_+ \|_s$ for all u_+ in $H^{s+1}_+(G)$.

(2) (a) We show that: $\| R_1(\varphi u_+) \|_{s+1} \leqq M \| \varphi u_+ \|_s$ for all u_+ in $H^s_+(G)$ and φ in $C^\infty_c(G)$.

Let $\zeta(x)$ be an infinitely differentiable function with compact support in G and such that: $0 \leqq \zeta(x) \leqq 1$; $\zeta(x) = 1$ on G_1, $\zeta(x) = 0$ outside of G_0 with $\mathrm{cl}\, G_1 \subset G_0 \subset \mathrm{cl}\, G_0 \subset G$.

Let u_+ be an element of $H^s_+(G)$. Then u_+ is in $H^s(G)$ and there exists a sequence φ_n of elements in $C^\infty(\mathrm{cl}\, G)$ such that:

$$\varphi_n \longrightarrow u_+ \quad \text{in} \quad H^s(G) .$$

One can check easily that: $\zeta\varphi_n \to \zeta u_+$ in $H^s_+(G)$; $s \geqq 0$. Consider $\zeta\varphi_n$ It is an element of $H^{s+1}_+(G)$, so from the first part we get:

$$\| R_1(\zeta\varphi_n) \|_{s+1} \leqq M \| \zeta\varphi_n \|_s .$$

M is independent of n. Hence $R_1(\zeta\varphi_n) \to v$ in $H^{s+1}(G)$. Since $\zeta\varphi_n \to \zeta u_+$ in $H^s_+(G)$ and R_1 is a bounded linear mapping from $H^s_+(G)$ into itself, we obtain: $v = R_1(\zeta u_+)$.

Therefore: $\| R_1(\zeta u_+) \|_{s+1} \leqq C \| \zeta u_+ \|_s$ for all u_+ in $H^s_+(G)$.

(b) We shall deduce the smoothness in the interior of the solutions of $Uu_+ = (f, g_1, \cdots, g_k)$ from the above argument.

Let u_+ be a solution in $H^\alpha_+(G)$ of $Uu_+ = (f, g_1, \cdots, g_k)$ where $(f, g_1, \cdots, g_k)$ is in $H^0(G, \partial G)$ and f is in $H^1(G)$.

Consider:

$$P^+A(\zeta u_+) = \sum_j P^+\varphi_j A(\zeta\varphi_j u_+^+) + \sum_j P^+\varphi_j A(1 - \psi_j)(\zeta u_+) .$$

Transforming $\varphi_j A(\zeta\psi_j u_+)$ in local coordinates and applying Lemma 4.D.1 of [4], (p. 145), we get:

$$\varphi_j A(\zeta\psi_j u) = \zeta\varphi_j A_j(\psi_j u_+) + T_j^1(\psi_j u_+) + \zeta T_j^2(\psi_j u_+)$$

where T_j are smoothing operators with respect to A_j, i.e. with respect to a bounded linear mapping from $H_+^s(B_+)$ into $H^{s-\alpha}(B_+)$.

On the other hand, since the kernel of A has a point singularity and $\varphi_j(1 - \psi_j) = 0$, the operator $\varphi_j A(1 - \psi_j)u_+$ has an infinitely differentiable kernel and hence may be estimated in any norm (Cf. [4], p. 125).

So:

$$P^+A(\zeta u_+) = \zeta A u_+ + T_0 u_+$$

where T_0 is a smoothing operator with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$.

Doing in a similar fashion for $B_r(\zeta u_+)$, we obtain:

$$\gamma P^+B_r(\zeta u_+) = \zeta B_r u_+ + T_r u_+ ; \qquad r = 1, \cdots, k$$

where T_r are smoothing operators with respect to a bounded, linear mapping from $H_+^s(G)$ into $H^{s-\alpha_r-(1/2)}(\partial G)$.

Combining the results and taking into account the fact that ζ has compact support in G_0 whose closure is in G, we get:

$$U(\zeta u_+) = (\zeta f + T_0 u_+, \gamma T_r u_+; r = 1, \cdots, k) .$$

We have: $SU(\zeta u_+) = \zeta u_+ + R_1(\zeta u_+)$.

Consider $U(\zeta u_+)$. Since u_+ is in $H_+^\alpha(G)$ and the T_s are all smoothing operators, $U(\zeta u_+)$ is in $H^1(G, \partial G)$. Therefore $SU(\zeta u_+)$ is in $H_+^{\alpha+1}(G)$.

From the first part of the proof, we get: $R_1(\zeta u_+) \in H_+^{\alpha+1}(G)$. Hence ζu_+ is in $H_+^{\alpha+1}(G)$.

(c) We prove by induction for the general case.

Suppose that ζu_+ is in $H_+^{s-1}(G)$, $s - 1 \geqq \alpha$. We show that it is true for s.

Let η be an infinitely differentiable function with compact support in G and such that: $0 \leqq \eta(x) \leqq 1$; $\eta(x) = 1$ on G_3, $\eta(x) = 0$ outside of G_2 with

$$\text{cl}\, G_3 \subsetneqq G_2 \subsetneqq \text{cl}\, G_2 \subsetneqq G_1 \subsetneqq \text{cl}\, G_1 \subsetneqq G_0$$

and $\text{cl}\, G_0 \subsetneqq G$. Consider $U(\zeta\eta u_+)$. We have:

$$P^+A(\zeta\eta u_+) = \sum_j P^+\varphi_j A(\zeta\eta\psi_j u_+) + \sum_j P^+\varphi_j A(1 - \psi_j)(\zeta\eta u_+) .$$

We express $\varphi_j A(\zeta\eta\psi_j)$ in local coordinates and applying Lemma 4.D.1 of [4], we obtain:

$$\begin{aligned}\varphi_j A(\zeta\eta\psi_j u_+) &= \eta\varphi_j A_j(\zeta\psi_j u_+) + T_0^1(\zeta u_+) \\ &= \zeta\eta\varphi_j A_j(\psi_j u_+) + \eta T_0^2(u_+) + T_1^0(\zeta u_+) .\end{aligned}$$

So:

$$P^+\varphi_j A(\zeta\eta\psi_j u_+) = \zeta\eta A u_+ + \eta T_0^3 u_+ + T_0^4(\zeta u_+)$$

where T_0^3, T_0^4 are smoothing operators with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$.

Since $\zeta u_+ \in H_+^{s-1}(G)$, $T_0^4(\zeta u_+)$ lies in $H^{s-\alpha}(G)$ and:

$$\| \eta T_0^3 u_+ \|_{s-\alpha} \leqq M \| T_0^3 u_+ \|_{H^{s-\alpha}(G_2)} \leqq M \| u_+ \|_{H^{s-1}(G_2)} .$$

So, $P^+A(\zeta\eta u_+)$ is in $H^{s-\alpha}(G)$.

We do in a similar fashion for $\gamma P^+B_r(\zeta\eta u_+)$.

An argument as above shows that $U(\zeta\eta u_+)$ is in $H^{s-\alpha}(G, \partial G)$. Therefore $SU(\zeta\eta u_+)$ belongs to $H_+^s(G)$. Moreover, since ζu_+ is in $H_+^{s-1}(G)$, $R_1(\zeta\eta u_+)$ lies in $H_+^s(G)$. Hence $\zeta\eta u_+$ belongs to $H_+^s(G)$.

(d) If f is in $C^\infty(G)$, then by repeated use of the Scbolev imbedding theorem, we get: $u_+ \in C_{\text{loc}}^\infty(G)$.

Proof of Theorem 2.3 *using Theorem* 2.2. Let u be a solution in $H_+^\alpha(G)$ of: $Uu = (f, g_1, \cdots, g_k)$ where $(f, g_1, \cdots, g_k)$ is an element of $H^{s-\alpha}(G, \partial G)$ for $s \geqq \alpha$.

From Theorem 2.2, there exists a unique element v in $H_+^s(G)$, solution of:

$$U(\lambda)v = (f, g_1, \cdots, g_k)$$

where

$$U(\lambda)v = (P^+(A + \lambda^\alpha)v, \gamma P^+B_1 v, \cdots, \gamma P^+B_k v) .$$

Consider:

$$U(\lambda)(v - u) = (-\lambda^\alpha u, 0, \cdots, 0) .$$

Since $\lambda^\alpha u$ is in $H^\alpha(G)$, it follows from Theorem 2,2 that the unique solution $w = v - u$ of $U(\lambda)w = (-\lambda^\alpha u, 0, \cdots, 0)$ is in $H_+^{2\alpha}(G)$. Therefore $u = v - w$ belongs to $H_+^{\min(s,2\alpha)}(G)$.

If $\min(s, 2\alpha) = s$, then we are through. If $2\alpha < s$, then since u is in $H_+^{2\alpha}(G)$, w is in $H_+^{3\alpha}(G)$, so u $H_+^{\min(s,3\alpha)}(G)$.

Repeating this boot-strap argument, we get finally u in $H_+^s(G)$.

Proof of Corollary 2.1. (1) Let A_2 be the linear operator from $D(A_2) = \Omega$ into $L^2(G)$ with $A_2 u = P^+Au$ if $u \in D(A_2)$.

With the hypotheses of the corollary, it follows from the theorem that $(A_2 + \lambda^\alpha I)^{-1}$ exists, is defined on all of $L^2(G)$ and maps $L^2(G)$ into $H_+^\alpha(G)$. Since G is bounded, the injection mapping from $H_+^\alpha(G)$ into $L^2(G)$ is compact. So $(A_2 + \lambda^\alpha I)^{-1}$ is a compact mapping of $L^2(G)$ into itself and therefore the spectrum of A_2 is discrete, and the eigenspaces are of finite dimension.

(2) We have the following estimate on the growth of $(A_2+\lambda^\alpha I)^{-1}$:

$$\|(A_2 + \lambda^\alpha I)^{-1}\| \leqq M/|\lambda|^\alpha .$$

If Assumption (1) is valid for rays $\arg\lambda = \theta_j; j = 1, \cdots, N$ and the plane is divided by these rays into angles less than $2\alpha\pi/n$, then it follows from Theorem 3.2 of Agmon [1] (p. 128–129) that the generalized eigenfunctions of A_2 are complete in $L^2(G)$. Indeed in the proof of the theorem, only the compactness of $(A_2 + \lambda^\alpha I)^{-1}$ and an estimate on the growth of the resolvent operator as in this paper are needed.

Proof of Corollary 2.2. Taking into account Theorem 2.2, we may prove without much modification Corollary 2.2 as in [2].

Proof of Theorem 2.2. The proof is long. It is technically simpler than in the case when $\lambda = 0$. First, we have the lemma:

LEMMA 3.1. *Let* $\{P^+A; \gamma P^+B_r; r = 1, \cdots, k\}$ *be a regular elliptic convolution boundary-value problem on* R_+^n *in the sense of Definition* 10, *with constant symbols* $\tilde{A}(\xi), \tilde{B}_r(\xi)$, *homogeneous of orders* α, α_r *repectively.* α, α_r *are positive integers. Suppose that Assumption* (1) *is satisfied. Then for every* $(f, g_1, \cdots, g_k)$ *in* $H^{s-\alpha}(R_+^n, R^{n-1})$, $s \geqq \alpha$, *there exists a unique solution* u_+ *in* H_s^+ *of*: $P^+(A + \lambda^\alpha)u_+ = f$ *on* R_+^n; $\gamma P^+B_r u_+ = g_r$ *on* R^{n-1}; $r = 1, \cdots, k$ *Moreover*:

$$\|u_+\|_s^+ + |\lambda|^s \|u_+\|_0^+ \leqq M\Big\{\|f\|_{s-\alpha}^+ + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^k \|g_r\|'_{s-\alpha_r-(1/2)} + |\lambda|^{s-\alpha_r-(1/2)}\|g_r\|'_0\Big\} .$$

M is independent of λ, u_+, f, g_r, u_+ *is the inverse Fourier transform of* $\tilde{u}_+(\xi)$ *with*:

$$\tilde{u}_+(\xi) = (\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\lambda)(\tilde{A}_-(\xi, \lambda))^{-1} + \sum_{r=1}^k \tilde{D}_r(\xi, \lambda)(\tilde{g}_r(\xi') - \tilde{f}_r(\xi', \lambda))$$

where:

$$\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda)$$

$$\tilde{D}_r(\xi, \lambda) = \sum_{m-1}^{k} b^1_{rm}(\xi', \lambda)\xi_n^{m-1}(\tilde{A}_+(\xi, \lambda))^{-1}$$

b^1_{rm} are the elements of the transpose of the inverse of the matrix $((b_{rm}(\xi', \lambda)))$. lf is any extension of f to R^n and

$$\tilde{f}_r(\xi', \lambda) = \Pi'\Pi^+\tilde{B}_r(\xi)(\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

Proof. Set $\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha$. It is homogeneous of order α in (ξ, λ) and belongs to E_α. Since $\tilde{A}(\xi)$ is in E_α; it has a factorization of the form: $\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$ with $\tilde{A}_+ \in C_k^+$, $\tilde{A}_- \in 0^-_{\alpha-k}$. The factorization is unique up to a constant multiplier. The same proof as in Theorem 1.2 of [4], p. 95 with ξ_+ replaced by $\xi_+^\lambda = \xi_n + i(|\lambda| + |\xi'|)$ and $\xi_-^\lambda = \xi_n - i(|\lambda| + |\xi'|)$ gives:

$$\tilde{A}(\xi, \lambda) = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda) .$$

Moreover if $\tilde{A}_+ \in C_k^+$; then: $\tilde{A}_+(\xi, \lambda) \in C_k^+$, (with respect to ξ, λ). Similarly $\tilde{A}_-(\xi, \lambda) \in 0^-_{\alpha-k}$.

(1) First, we show that $\tilde{u}_+(\xi) \in \widetilde{\mathring{H}}{}_0^+$ so that $\Pi^+\tilde{u}_+(\xi) = \tilde{u}_+(\xi)$ (Cf. [4], p. 93, relation 10.1). $\tilde{u}_+(\xi)$ is analytic in $\operatorname{Im}\xi_n > 0$ for $|\lambda| \neq 0$. It suffices to show that:

$$\int|\tilde{u}_+(\xi', \xi_n + i\tau)|^2 d\xi' d\xi_n \leqq C .$$

C is independent of $\tau > 0$.

(i) We write:

$$\tilde{u}_+(\xi) = \tilde{v}_+(\xi) + \tilde{w}_+(\xi) .$$

We have:

$$\int|\tilde{v}_+(\xi', \xi_n + i\tau)|^2 d\xi' d\xi_n \leqq C\int(|\xi| + |\lambda| + \tau)^{-2k}|\tilde{g}(\xi', \xi_n + i\tau, \lambda)|^2 d\xi' d\xi_n$$
$$\leqq C\int|\tilde{g}(\xi', \xi_n + i\tau, \lambda)|^2 d\xi' d\xi_n$$

where:

$$\tilde{g}(\xi, \lambda) = \Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

But $\widetilde{lf}(\tilde{A}_-)^{-1}$ is in $\tilde{H}_0$, so $\Pi^+\widetilde{lf}(\tilde{A}_-)^{-1} = \tilde{g}$ is in $\widetilde{\mathring{H}}{}_0^+$, hence $\tilde{v}_+ \in \widetilde{\mathring{H}}{}_0^+$.

(ii) Since $\tilde{A}_+(\xi, \lambda) \in C_k^+$, $(\tilde{A}_+(\xi, \lambda))^{-1} \in C_{-k}^+ \subset D_{-k}$ (Lemma 2.4 of [4]). So:

$$\tilde{D}_r(\xi, \lambda) \in D_{-1-\alpha_r} .$$

We have:

$$(\xi_-^\lambda)^M \tilde{D}_r(\xi, \lambda) = \tilde{P}_r(\xi, \lambda) + \tilde{R}_r(\xi, \lambda)$$

with:

$$\tilde{P}_r \in 0^-_{-1-\alpha_r+M} \quad \text{and} \quad |\tilde{R}_r| \leqq C(|\xi'| + |\lambda|)^{M-\alpha_r}(|\xi| + |\lambda|)^{-1} .$$

Therefore:

$$\tilde{P}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r)$$

is in $\widetilde{\mathring{H}}{}_0^-$, and:

$$\Pi^+ \tilde{P}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r) = 0 .$$

It remains to show that:

$$\tilde{R}_r(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r) \in \tilde{H}_0 .$$

We take M large enough and the proof is trivial.

So:

$$\Pi^+ \tilde{R}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r(\xi') - \tilde{f}_r(\xi')) \in \widetilde{\mathring{H}}{}_0^+ .$$

Therefore:

$$\tilde{w}_+ \in \widetilde{\mathring{H}}{}_0^+ .$$

(2) Consider:

$$\| u_+ \|_s^+ = \| \Pi^+(\xi_- - i)^s \tilde{u}_+(\xi) \|_0 = \| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{u}_+(\xi) \|_0 .$$

It is majorized by:

$$\| \Pi^+(\xi_- - i)^s \Pi^+\{(\tilde{A}_+)^{-1} \Pi^+ \widetilde{lf}(\tilde{A}_-)^{-1}\} \|_0 + \sum_{r=1}^{k} \| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{D}_r(\tilde{g}_r - \tilde{f}_r) \|_0 .$$

(i) Consider the first expression. It follows from [4] (footnote of p. 113) that the expression is equal to:

$$\| \Pi^+(\xi_- - i)^s (\tilde{A}_+)^{-1} \Pi^+ \widetilde{lf}(\tilde{A}_-)^{-1} \|_0$$

which is majorized by:

$$C \| (\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_s .$$

Since $\tilde{A}_+(\xi, \lambda)$ is in 0_k^+, we may write:

$$\tilde{A}_+(\xi, \lambda) = (|\xi| + |\lambda|)^k \tilde{A}_+(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)) .$$

Let $c = \operatorname{Min} |\tilde{A}_+(\xi, \lambda)|$ for $|\xi| + |\lambda| = 1$. Then $c > 0$ and is independent of ξ, λ. We obtain:

$$\| (\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_s \leqq c^{-1} \| \Pi^+\widetilde{lf}(\tilde{A}_-)^{-1} \|_{s-k}$$

which is majorized by: $c^{-1} \| \widetilde{lf}(\tilde{A}_-)^{-1} \|_{s-k}$ (Cf. Remark 2 of [4], p. 105). We also have:

$$\| (\tilde{A}_+)^{-1}\Pi^+\widetilde{lf}(\tilde{A}_-)^{-1} \|_0 \leqq C \,|\, \lambda \,|^{-k} \| \widetilde{lf}(\tilde{A}_-)^{-1} \|_0 .$$

Since:

$$\tilde{A}_-(\xi, \lambda) \in 0^-_{\alpha-k} .$$

We have:

$$\tilde{A}_-(\xi, \lambda) = (|\, \xi \,| + |\, \lambda \,|)^{-k}\tilde{A}_-(\xi/(|\, \xi \,| + |\, \lambda \,|), \lambda/(|\, \xi \,| + |\, \lambda \,|)) .$$

So as before, we get:

$$\| \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \leqq C \| \widetilde{lf}(\xi) \|_{s-\alpha} \leqq C \| f \|^+_{s-\alpha}$$

and:

$$|\, \lambda \,|^{-k} \| \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_0 \leqq C \,|\, \lambda \,|^{-\alpha} \| f \|^+_0 .$$

Therefore:

$$\| \Pi^+(\xi_- - i)^s\Pi^+\{(\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\} \|_s$$
$$+ \,|\, \lambda \,|^s \| \Pi^+\{(\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\} \|_0$$

is majorized by:

$$C\{\| f \|^+_{s-\alpha} + |\, \lambda \,|^{s-\alpha} \| f \|^+_0\} .$$

(ii) Consider:

$$\| \Pi^+(\xi_- - i)^s\tilde{D}_r(\xi, \lambda)\tilde{f}_r(\xi', \lambda) \|_0 + |\, \lambda \,|^s \| \Pi^+\tilde{D}_r(\xi, \lambda)\tilde{f}_r \|_0 .$$

From this first part, we know that $\tilde{D}_r(\xi, \lambda) \in D_{-1-\alpha_r}$. Let M be a large positive integer. We have from the definition of $D_{-1-\alpha_r}$:

$$(\xi^\lambda_-)^M\tilde{D}_r(\xi, \lambda) = \tilde{P}_r(\xi, \lambda) + R_r(\xi, \lambda)$$

with:

$$\tilde{P}_r(\xi, \lambda) \in 0^-_{-1-\alpha_r+M} ; \qquad |\, R_r(\xi, \lambda) \,| \leqq C(|\, \xi' \,| + |\, \lambda \,|)^{M-\alpha_r}(|\, \xi \,| + |\, \lambda \,|)^{-1} .$$

We can show easily that: $(\xi^\lambda_-)^{-M}\tilde{P}_r(\xi, \lambda) \in \tilde{\mathring{H}}^-_0$, so: $\Pi^+(\xi^\lambda_-)^{-M}\tilde{P}_r = 0$. From [4] (footnote of p. 113), we get:

$$\Pi^+(\xi_- - i)^s\Pi^+(\xi^\lambda_-)^{-M}\tilde{P}_r(\xi, \lambda) = 0 .$$

Hence:

$$\begin{aligned}\| \Pi^+(\xi_- - i)^s \Pi^+ \widetilde{D}_r(\xi, \lambda)\widetilde{f}_r(\xi', \lambda) \|_0 &= \| \Pi^+(\xi_- - i)^s \Pi^+(\xi_-^\lambda)^{-M} R_r \widetilde{f}_r \|_0 \\ &= \| \Pi^+(\xi_- - i)^s(\xi_-^\lambda)^{-M} R_r \widetilde{f}_r \|_0 \\ &\leqq C \| (\xi_- - i)^s(\xi_-^\lambda)^{-M} R_r \widetilde{f}_r \|_0 .\end{aligned}$$

Consider:

$$\begin{aligned}&\int | (\xi_- - i) |^{2s} | \xi_-^\lambda |^{-2M} | R_r(\xi, \lambda)\widetilde{f}_r(\xi', \lambda) |^2 d\xi_n d\xi' \\ &\leqq C\int (| \xi' | + | \lambda |)^{2M-2\alpha_r}(| \xi | + | \lambda |)^{2s-2M-2} | \widetilde{f}_r(\xi', \lambda) |^2 d\xi_n d\xi' \\ &\leqq C\int (| \xi' | + | \lambda |)^{2s-2\alpha_r-1} | \widetilde{f}_r(\xi', \lambda) |^2 d\xi'\end{aligned}$$

for M sufficiently large.
So:

$$\| \Pi^+(\xi_- - i)^s \Pi^+ \widetilde{D}_r \widetilde{f}_r \|_0 \leqq C\{\| \widetilde{f}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \widetilde{f}_r \|'_0\}$$

and:

$$\| \Pi^+ \widetilde{D}_r \widetilde{f}_r \|_0 \leqq C | \lambda |^{-\alpha_r-(1/2)} \| \widetilde{f}_r \|'_0 .$$

(iii) Similarly, we have:

$$\begin{aligned}&\| \Pi^+(\xi_- - i)^s \Pi^+ \widetilde{D}_r \widetilde{g}_r \|_s + | \lambda |^s \| \Pi^+ \widetilde{D}_r \widetilde{g}_r \|_0 \\ &\leqq C\{\| \widetilde{g}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \widetilde{g}_r \|'_0\} .\end{aligned}$$

(iv) Since s, α, α_r are positive integers, we have from [3] (relation 1.14, p. 63):

$$\| \widetilde{f}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \widetilde{f}_r \|'_0 \leqq M\{\| \widetilde{f}_r \|_{s-\alpha_r} + | \lambda |^{s-\alpha_r} \| \widetilde{f}_r \|_0\}$$

with

$$\widetilde{f}_r = \Pi^+ \widetilde{B}_r(\xi)(\widetilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\widetilde{A}_-(\xi, \lambda))^{-1} .$$

Since $\widetilde{B}_r(\xi)$ is homogeneous of order α_r in ξ with $\alpha_r \geqq 0$; we get:

$$\| \widetilde{f}_r \|_{s-\alpha_r} \leqq C \| (\widetilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\widetilde{A}_-(\xi, \lambda))^{-1} \|_s .$$

Again as before, we write:

$$\widetilde{A}_+(\xi, \lambda) = (| \xi | + | \lambda |)^k \widetilde{A}_+(\xi/(| \xi | + | \lambda |, \lambda/(| \xi | + | \lambda |)) .$$

So:

$$\begin{aligned}\| f_r \|_{s-\alpha_r} &\leqq C \| \Pi^+ \widetilde{lf}(\xi)(\widetilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \\ &\leqq C \| \widetilde{lf}(\xi)(\widetilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \\ &\leqq C \| f \|^+_{s-\alpha} .\end{aligned}$$

Similarly, we obtain:

$$\| \tilde{f}_r \|_0 \leqq C |\lambda|^{-\alpha-\alpha_r} \| f \|_0^+ .$$

Combining all the results, we get the *a priori* estimate

(3) A direct verification shows that u_+ is a solution of the problem. It remains to show that the solution is unique.

Let v_+ be a solution of the problem with $v_+ \in H_s^+$. Then $\tilde{v}_+(\xi)$, its Fourier transform has the same form as $\tilde{u}_+(\xi)$ with $\widetilde{lf}(\xi)$ replaced by $\widetilde{l_1 f}(\xi)$. $l_1 f$ is an extension of f to R^n.

Set $l_2 f = lf - l_1 f$. Then $l_2 f \in \overset{\circ}{H_0^-}$, so $\widetilde{l_2 f(\xi)} \in \widetilde{\overset{\circ}{H_0^-}}$.

Now a verification as in the first part shows that:

$$\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \in \widetilde{\overset{\circ}{H_0^-}} ,$$

hence:

$$\Pi^+ \widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} = 0 .$$

Taking into account the ellipticity of $\tilde{A}(\xi, \lambda)$, we get: $\tilde{u}_+(\xi) = \tilde{v}_+(\xi)$.

Let:

$$A_0 u_+ = \sum_k \psi_0(x_0) \exp(ikx_0\pi)/pL_k u$$

$$A_1 u_+ = \sum_k \psi_0(x) \exp(ikx\pi)/pL_k u_+$$

where $\psi_0(x)$, L_k are as in § 1.

We have the Lemma:

LEMMA 3.2. *Let $\psi(x)$ be in $C_c^\infty(R^n)$, $\psi(x) = 0$ outside of $|x - x_0| \leqq \delta$ $|\psi(x)| \leqq K$ where K is independent of δ. Suppose that $\tilde{A}_1(x, \xi)$ is in D_α^1. Then:*

$$\| \psi(A_1 - A_0)u_+ \|_{s-\alpha}^+ \leqq C\delta \| u_+ \|_{s-\alpha}^+ + C(\delta) \| u_+ \|_{s-1-\alpha}^+$$

$C(\delta) = 0$ *if* $s = \alpha$.

Proof. Cf. Lemma 4.7 of [4] (p. 119).

Proof of Theorem 2.2 (*continued*).

(1) First, we establish the *a priori* estimate.

Let N_j be a finite open covering of cl G with diam (N_j) sufficiently small; φ_j be a finite partition of unity corresponding to N_j and ψ_j be the infinitely differentiable functions with compact supports in N_j and such that: $\varphi_j \psi_j = \varphi_j$.

Let: $F = (f, g_1, \cdots, g_k)$ be an element of $H^{s-\alpha}(G, \partial G)$; $s \geqq \alpha$.

By definition, we have:

$$U(\lambda)u_+ = \sum_j P^+\varphi_j U(\lambda)(\psi_j u_+) + Tu_+ = F\,.$$

We express $\varphi_j U(\lambda)\psi_j$ in local coordinates. From Appendix 2 of [4], we get:

$$\varphi_j U(\lambda)\psi_j u_+ = \sum \varphi_j U_j(\lambda)(\psi_j u_+) + T_j u_+$$

where T_j is a smoothing operator with respect to $U_j(\lambda)$.
So:

$$\varphi_j U_j^0(\lambda)(\psi_j u_+) = \varphi_j F + T_j u_+ + \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j u_+)\,.$$

$U_j^0(\lambda)$ corresponds to the case when A_j, B_{rj} have constant symbols. From Lemma 4.D.1 of [4] (p. 145), we have:

$$\varphi_j U_j^0(\lambda)(\psi_j u_+) = U_j^0(\lambda)(\varphi_j\psi_j u_+) + T_j^2 u_+$$

where T_j^2 is again a smoothing operator.
Hence:

$$U_j^0(\lambda)(\varphi_j u_+) = \varphi_j F + \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j u_+) + T_j^2 u_+\,.$$

Applying Lemma 3.1, we obtain:

$$\begin{aligned} ||\varphi_j u_+||_s^+ + |\lambda|^s ||\varphi_j u_+||_0^+ \leqq M\Big\{ & ||\varphi_j f||_{s-\alpha}^+ + |\lambda|^{s-\alpha} ||\varphi_j f||_0^+ \\ & + ||\varphi_j(A_j - A_{j0})(\psi_j u_+)|| + |\lambda|^{s-\alpha} ||\varphi_j(A_j - A_{j0})(\psi_j u_+)||_0^+ \\ & + ||u_+||_{s-1} + |\lambda|^{s-\alpha} ||u_+||_{\alpha-1} + \sum_{r=1}^{k} ||\varphi_j g_r||'_{s-\alpha_r-(1/2)} \\ & + |\lambda|^{s-\alpha_r-(1/2)} ||\varphi_j g_r||'_0 + ||\gamma P^+\varphi_j(B_{rj} - B_{rj0})(\psi_j u_+)||'_{s-\alpha^r-(1/2)} \\ & + |\lambda|^{s-\alpha_r-(1/2)} ||\gamma P^+\varphi_j(B_{rj} - B_{rj0})(\psi_j u_+)||'_0\Big\}\,. \end{aligned}$$

Using Lemma 3.2, we get:

$$\begin{aligned} ||\varphi_j u_+||_s^+ + |\lambda|^s ||\varphi_j u_+||_0^+ \leqq M\Big\{ & ||u_+||_{s-1} + |\lambda|^{s-\alpha} ||u_+||_{\alpha-1} + ||\varphi_j f||_{s-\alpha}^+ \\ & + |\lambda|^{s-\alpha} ||\varphi_j f||_0^+ + \delta ||\varphi_j u_+||_{s-\alpha}^+ + \delta |\lambda|^{s-\alpha} ||\varphi_j u_+||_0^+ \\ & + \sum_{r=1}^{k} ||\varphi_j g_r||'_{s-\alpha_r-(1/2)} + |\lambda|^{s-(1/2)-\alpha_r} ||\varphi_j g_r||'_0\Big\} \end{aligned}$$

(by using an inequality in [3] p. 63).
Summing with respect to j, we have:

$$\begin{aligned} ||u_+||_s + |\lambda|^s ||u_+||_0 \leqq M\Big\{ & ||u_+||_{s-1} + |\lambda|^{s-\alpha} ||u_+||_{\alpha-1} + ||f||_{s-\alpha} \\ & + |\lambda|^{s-\alpha} ||f||_0 + \delta ||u_+||_{s-\alpha} + \delta |\lambda|^{s-\alpha} ||u_+||_0 \\ & + \sum_{r=1}^{k} ||g_r||'_{s-(1/2)-\alpha_r} + |\lambda|^{s-(1/2)-\alpha_r} ||g_r||'_0\Big\}\,. \end{aligned}$$

Taking δ small and $|\lambda|$ large, we obtain by taking into account an interpolation inequality of Visik-Agranovich [3] (p. 64, relation 1.21):

$$\|u_+\|_s + |\lambda|^s \|u_+\|_0 \leqq M\Big\{\|f\|_{r-\alpha} + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^{k} \|g_r\|'_{s-(1/2)-\alpha_r} + |\lambda|^{s-\alpha_r-(1/2)}\|g_n\|'_0\Big\} .$$

(2) It follows from the *a priori* estimate that if there exists a solution, then it is unique.

It remains to show the existence of a solution.

We know from Lemma 3.1 that $U_j^0(\lambda)$ has a right inverse R_j Let $\hat{R}_j$ be the operator R_j expressed in the global coordinates system of G.

Set:

$$RF = \sum_j P^+\varphi_j\hat{R}_j(\psi_j F) .$$

We have:

$$U(\lambda)RF = \sum_j U(\lambda)\varphi_j\hat{R}_j(\psi_j F) = \sum_j U(\lambda)\varphi_j\psi_j\hat{R}_j(\psi_j F) .$$

Passing into local coordinates (using Appendix 2 of [4]) and applying Lemma 4.D.1 of [4], we obtain:

$$\begin{aligned} U(\lambda)\varphi_j\psi_j R_j(\psi_j F) &= \varphi_j U_j(\lambda)(\psi_j R_j(\psi_j F)) + T_j^2 F \\ &= \varphi_j U_j^0(\lambda)(\psi_j R_j(\psi_j F)) + T_j^2 F \\ &\quad + \varphi_j(U_j(\lambda) - U_j^0(\lambda))(\psi_j R_j(\psi_j F)) \end{aligned}$$

where T_j^2 is a smoothing operator.

Applying again Lemma 4.D.1 of [4], we have:

$$\begin{aligned} \varphi_j U_j^0(\lambda)\psi_j R_j(\psi_j F) &= \varphi_j\psi_j U_j^0(\lambda)R_j(\psi_j F) + T_j^2 RF \\ &= \varphi_j\psi_j F + T_j^2 RF . \end{aligned}$$

Therefore:

$$U(\lambda)RF = F + T'RF + \sum_j \varphi_j\hat{T}'_j F$$

where T' is a smoothing operator with respect to $U(\lambda)$; i.e. with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$; and $\hat{T}'_j$ is the operator T'_j defined by:

$$T'_j F = (U_j^0(\lambda) - U_j(\lambda))(\psi_j R_j(\psi_j F))$$

expressed in the global coordinates system of G.

So: $U(\lambda)RF = (I + \mathscr{C}R)F$.

Denote by:

$$|||\cdot|||_{s-\alpha} = ||\cdot||_s + |\lambda|^{s-\alpha}||\cdot||_0$$
$$|||\cdot|||'_{s-\alpha} = ||\cdot||'_{s-\alpha-1/2} + |\lambda|^{s-\alpha-1/2}||\cdot||'_0$$
$$|||\cdot|||_{H^{s-\alpha}(G,\partial G)} = |||\cdot|||_{s-\alpha} + |||\cdot|||'_{s-\alpha} .$$

Since T' is a smoothing operator, we get by taking into account the first part of the proof:

$$|||\, T'RF\,|||_{H^{s-\alpha}(G,\partial G)} \leqq C\, |||\, F\,|||_{H^{s-1-\alpha}(G,\partial G)} .$$

Using Lemma 3.2, we obtain:

$$|||\, \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j\hat{R}_j(\psi_j F))\,|||_{H^{s-\alpha}(G,\partial G)} \leqq |||\, F\,|||_{H^{s-1-\alpha}(G,\partial G)}$$
$$+ C(\delta)/\lambda\, |||\, F\,|||_{H^{s-\alpha}(G,\partial G)} .$$

So for small δ, large $|\lambda|$, by using an interpolation inequality of [3], we have:

$$|||\, \mathscr{C}RF\,||_{H^{s-\alpha}(G,\partial G)} < 1/2\, |||\, F\,|||_{H^{s-\alpha}(G,\partial G)} .$$

Hence: $(I + \mathscr{C}R)^{-1}$ exists and $U(\lambda)^{-1} = R(I + \mathscr{C}R)^{-1}$.

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Received April 14, 1967.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics
Vol. 26, No. 2 December, 1968