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NORMAL STRUCTURE IN BANACH SPACES

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A bounded convex subset K of a Banach space B has normal structure if for each convex subset H of K which contains more than one point there is a point x in H which is not a diametral point of H. The concept of normal structure (introduced by Brodskii and Milman) and a strengthening of this concept called complete normal structure have been of fundamental importance in some recent investigations concerned with determining conditions on weakly compact K under which the members of any commutative family $\mathscr{T}$ of nonexpansive mappings of K into itself have a common fixed-point. A more thorough study of these concepts is initiated in the present paper. The theorems obtained primarily concern product spaces composed of spaces which possess normal structure.

Let K be a nonempty, closed, convex, weakly compact subset of the Banach space B. It was proved in [1] that if K has normal structure and if $\mathscr{T}$ is a *finite* family of commutative nonexpansive mappings on K then there is a point $x \in K$ such that $f(x) = x$ for each $f \in \mathscr{T}$. Although we do not know whether this theorem is true in general for infinite families, it is easily seen, as was pointed out in [2], that if the norm of B is strictly convex then the theorem is true if $\mathscr{T}$ is infinite. In [2] the existence of a common fixed-point was established for arbitrary families *without the assumption of strict convexity of the norm* by the introduction of a strengthening of normal structure called *complete normal structure*. It was shown in [2] that if B is uniformly convex, or if K is compact, then K has complete normal structure.

We are mainly interested here in providing examples of classes of sets which satisfy the hypotheses of the fixed point theorems of [1] (Theorem 3) and [2] (Theorem 2.1). Our purpose in § 2 is to exhibit such a class of sets K which are not compact and which do not have strictly convex norm. Fixed-point theorems for families of nonexpansive mappings established by De Marr [6] (for K compact) and by F. Browder [4] (for B uniformly convex) do not apply in this class of sets.

In [5], M. M. Day gave an example of a class of reflexive, strictly convex Banach spaces, none of which is isomorphic to *any* uniformly convex Banach space. We show in § 3 that these spaces do possess normal structure; thus they provide examples of spaces not isomorphic to any uniformly convex space which satisfy the hypothesis of the

fixed-point theorem of Kirk in [7].

2. **Normal structure and complete normal structure.** We say that a Banach space has normal structure if each of its bounded convex subsets has normal structure. In this section we first show that the direct sum (with the supremum norm) of two Banach spaces, each with normal structure, itself has normal structure.

THEOREM 2.1. *Let B_1 and B_2 be Banach spaces with norms $\|-\|_1$ and $\|-\|_2$, respectively. Let $B = B_1 \oplus B_2$ with the norm of B given by $\|-\| = \sup(\|-\|_1, \|-\|_2)$. If both B_1 and B_2 have normal structure, then B has normal structure.*

Proof. Let K be a bounded convex subset of B which contains more than one point and let p_i be the natural projection of B onto B_i, $i = 1, 2$. Let $K_i = p_i(K)$, $i = 1, 2$. Then K_i is a bounded, convex subset of B_i and thus possesses normal structure. For $i = 1, 2$, let x_i be a nondiametral point of K_i. (If K_i consists of a single point for some i, then the conclusion of the theorem is immediate.)

Select $u_i \in p_i^{-1}(x_i) \cap K$, $i = 1, 2$; then $u_1 = x_1 \oplus v$ and $u_2 = w \oplus x_2$ where $v = p_2(u_1) \in K_2$, $w = p_1(u_2) \in K_1$. Set $m = \frac{1}{2}(u_1 + u_2) = m_1 \oplus m_2$ where $m_1 = \frac{1}{2}(x_1 + w)$ and $m_2 = \frac{1}{2}(v + x_2)$. Let $z \in K$; then $z = z_1 \oplus z_2$ where $z_1 \in K_1$, $z_2 \in K_2$. Now $\|m - z\| = \sup(\|m_1 - z_1\|_1, \|m_2 - z_2\|_2)$. But $\|m_1 - z_1\|_1 = \|\frac{1}{2}(x_1 + w) - \frac{1}{2}(z_1 + z_1)\|_1 \leqq \frac{1}{2}(\|x_1 - z_1\|_1 + \|w - z_1\|_1)$. Letting $\delta(-)$ denote the diameter of a set and using the fact that x_1 is a nondiametral point of K_1 we see that $\|m_1 - z_1\|_1 \leqq \delta(K_1) - \varepsilon_1$ for some $\varepsilon_1 > 0$ which does not depend on z_1. Similarly we have that $\|m_2 - z_2\|_2 \leqq \delta(K_2) - \varepsilon_2$, $\varepsilon_2 > 0$ and independent of z_2. Hence

$$\|m - z\| \leqq \sup(\delta(K_1) - \varepsilon_1, \delta(K_2) - \varepsilon_2) ;$$

letting $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ we see that

$$\|m - z\| \leqq \sup(\delta(K_1) - \varepsilon , \quad \delta(K_2) - \varepsilon) = \delta(K) - \varepsilon .$$

Thus m is a nondiametral point for K and the theorem is proved.

Before giving the next definition, we introduce the following notation. For subsets H and K of a Banach space B, H bounded, let

$$r_x(H) = \sup\{\|x - y\| : y \in H\} ,$$
$$r(H, K) = \inf\{r_x(H) : x \in K\} ,$$
$$\mathscr{C}(H, K) = \{x \in K : r_x(H) = r(H, K)\} .$$

In general, $\mathscr{C}(H, K)$ may be empty, but if it is not, then it consists of precisely those points of K which serve as centers of balls

of minimal radius, $r(H, K)$, which contain H. We remark that if K is weakly compact and convex, and if H is bounded, then $\mathscr{C}(H, K)$ is a *nonempty* closed convex subset of K (see [2]).

DEFINITION 2.1. Let K be a bounded closed convex subset of a Banach space B. We say that K has *complete normal structure* (c.n.s.) if every closed convex subset W of K which contains more than one point satisfies the following condition:

(*) for every descending net $\{W_\alpha: \alpha \in A\}$ of subsets of W which have the property that $r(W_\alpha, W) = r(W, W)$ for each $\alpha \in A$, it is the case that the closure of $\bigcup_{\alpha \in A} \mathscr{C}(W_\alpha, W)$ is a nonempty proper subset of W.

By taking $W_\alpha \equiv W$ in the above definition, one sees that if K has complete normal structure then it has normal structure.

DEFINITION 2.2. Let K be a bounded closed convex subset of B. We say that a closed convex subset W of K satisfies *property* $(*)_q$ *with respect to* K if:

$(*)_q$ for every descending net $\{W_\alpha: \alpha \in A\}$ of subsets of W which have the property that $r(W_\alpha, W) = r(W, W)$ and $\mathscr{C}(W_\alpha, K) \cap W_\alpha \neq \emptyset$, for each $\alpha \in A$, then it is the case that the closure of $\bigcup \mathscr{C}(W_\alpha, W)$ is a nonempty proper subset of W.

Our reason for being interested in property $(*)_q$ is the following.

In property (*) of complete normal structure, it is not required that $\mathscr{C}(W_\alpha, K) \cap W_\alpha \neq \emptyset$. However, the condition (*) is applied in the proof of Theorem 2.1 of [2] as follows: Each set W_α is a fixed-point set of some nonexpansive mapping f of K into itself (the sets of [2] are labelled differently). It follows that $f(\mathscr{C}(W_\alpha, K)) \subset \mathscr{C}(W_\alpha, K)$ and, if K also has normal structure, by the theorem of [7], f has a fixed-point in $\mathscr{C}(W_\alpha, K)$. Thus $\mathscr{C}(W_\alpha, K) \cap W_\alpha \neq \emptyset$. *Thus if* K *has normal structure, Theorem* 2.1 *of* [2] *remains valid (without alteration of the proof) if complete normal structure is replaced by the assumption that every closed convex subset* W *of* K *has property* $(*)_q$ *with respect to* K.

We now investigate the preservation of property $(*)_q$ with respect to the taking of finite direct sums.

THEOREM 2.2. *Let* B *be the direct sum, with supremum norm, of the Banach spaces* $B_i, i = 1, 2$. *For each* i, *let* K_i *be a closed convex weakly compact subset of* B_i. *If every closed convex subset of* K_i *satisfies* $(*)_q$ *with respect to* K_i, $i = 1, 2$, *then every closed convex subset* W *of* $K = K_1 \oplus K_2$ *satisfies* $(*)_q$ *with respect to* K.

Proof. Let W be a closed convex subset of K which contains

more than one point, and let $\mathscr{W} = \{W_\alpha : \alpha \in A\}$ be a descending net of subsets of W for which $r(W_\alpha, W) = r(W, W)$ and

$$\mathscr{C}(W_\alpha, K) \cap W_\alpha \neq \emptyset , \qquad \alpha \in A .$$

Since $\mathscr{C}(W_\alpha, K) \cap W_\alpha \neq \emptyset$ it follows that $\mathscr{C}(W_\alpha, W) = \mathscr{C}(W_\alpha, K) \cap W$. Let $r = r(W_\alpha, W)$ and let $\alpha \in A$ be fixed. Choose $x \in \mathscr{C}(W_\alpha, W)$; then $x = x_1 \oplus x_2$ where $x_i \in K_i$, $i = 1, 2$. For $i = 1, 2$ let p_i denote the natural projection of B onto B_i. Since $x \in \mathscr{C}(W_\alpha, W)$ it must be the case that $W_\alpha \subseteqq \overline{\mathscr{S}}(x; r)$, $\overline{\mathscr{S}}(x, r)$ the closure of the ball centered at x with radius r. Since the norm in B is the supremum it follows that $p_i(W_\alpha) \subseteqq \overline{\mathscr{S}}(x_i, r)$, $i = 1, 2$. Therefore, $r(p_i(W_\alpha), p_i(K)) \leqq r$.

We now show that for some i, equality must hold. For assume we have $r(p_i(W_\alpha), p_i(K)) < r$ for $i = 1, 2$. Let

$$\bar{r} = \sup \{r(p_1(W_\alpha), p_1(K)), r(p_2(W_\alpha), p_2(K))\} .$$

Select $\bar{x}_i \in \mathscr{C}(p_i(W_\alpha), p_i(K))$ and set $\bar{x} = \bar{x}_1 \oplus \bar{x}_2 \in K$. It follows that $W_\alpha \subseteqq \overline{\mathscr{S}}(\bar{x}, \bar{r})$ since for $w = w_1 \oplus w_2 \in W_\alpha$,

$$|| \bar{x} - w || = \sup \{|| \bar{x}_1 - w_1 ||, || \bar{x}_1 - w_2 ||\} \leqq \bar{r} .$$

But this in turn implies $r(W_\alpha, K) = \bar{r} < r$ which is a contradiction. Therefore for each $\alpha \in A$ there is an i, $1 \leqq i \leqq 2$ such that

$$r(p_i(W_\alpha), p_i(K)) = r = r(W_\alpha, W) .$$

Now let $\mathscr{W}_i = \{W_\alpha \in \mathscr{W} \mid r(p_i(W_\alpha), p_i(K)) = r\}$, $i = 1, 2$. Suppose $\mathscr{W}_1$ is such that for some $W_\beta \in \mathscr{W}$, $W_\alpha \in \mathscr{W}_1$, there is no $W_\gamma \in \mathscr{W}_1$ with $W_\gamma \subseteqq W_\alpha \cap W_\beta$. Since $\mathscr{W}$ is a net there is then a $W_\gamma \in \mathscr{W}_2$ such that $W_\gamma \subseteqq W_\alpha \cap W_\beta$ and, moreover, for each $W_\delta \in \mathscr{W}$, if $W_\delta \subseteqq W_\gamma$ we must have $W_\delta \in \mathscr{W}_2$. Let $\mathscr{E}_2 = \{W_\alpha \in \mathscr{W}_2 \mid W_\alpha \subseteqq W_\gamma\}$; then $\mathscr{E}_2 \subseteqq \mathscr{W}_2$, $\mathscr{E}_2$ is a decreasing net and for all $W_\delta \in \mathscr{W}$ there is a $W_\alpha \in \mathscr{E}_2$ with $W_\alpha \subseteqq W_\delta$. Calling such an $\mathscr{E}$ a *filtered subnet* we see then that for some i, $1 \leqq i \leqq 2$, $\mathscr{W}_i$ contains a filtered subnet, $\mathscr{E}_i$. We may assume $i = 1$. Also we have, $\mathscr{C}(p_1(W_\beta), K_1) \cap p_1(W_\beta) \neq \emptyset$ for each $W_\beta \in \mathscr{E}_1$. This is clear since for some point $w = w_1 \oplus w_2 \in W_\beta$ we have $W_\beta \subseteqq \overline{\mathscr{S}}(w, r)$; hence $p_i(W_\beta) \subseteqq \overline{\mathscr{S}}(w_i, r)$ where

$$w_1 \in p_1(W_\beta) \cap \mathscr{C}(p_1(W_\beta), K_1) .$$

Thus we may apply $(*)_q$ to the net $p_1(\mathscr{E}_1) = \{p_1(W_\alpha) \mid W_\alpha \in \mathscr{E}_1\}$ in B_1, thereby concluding that the closure of $H_1 = \bigcup_{W_\beta \in \mathscr{E}_1} \mathscr{C}(p_1(W_\beta), p_1(W))$ is a proper subset of $p_1(W)$. Select a point $x_1 \in p_1(W)$ but not in the closure of H_1. Then there is a number $\varepsilon > 0$ such that

$$\inf_{W_\beta \in \mathscr{E}_1} \{|| x_1 - z_1 || \mid z_1 \in \mathscr{C}(p_1(W_\beta), p_1(W))\} > \varepsilon .$$

Choose $x \in W$ so that $p_1(x) = x_1$; then the above implies

$$\inf \{\| x - z \| \mid z \in \mathscr{C}(W_\alpha, W)\} > \varepsilon .$$

(Here we use the fact that $\mathscr{C}_1$ is a filtered subnet of $\mathscr{W}$.) From this we conclude that x is not in the closure of $\bigcup_{\alpha \in A} \mathscr{C}(W_\alpha, W)$, completing the proof.

Consequently we can now manufacture examples of convex sets in spaces without strictly convex norm to which the fixed-point theorems of [1], [2], [7] apply, but in which others do not. For example, let $M_2^{(1)}$ denote the two-dimensional Minkowski p-space for $p = 1$. Points of $M_2^{(1)}$ are pairs (x_1, x_2) of real numbers with

$$\| (x_1, x_2) \| = | x_1 | + | x_2 | .$$

This space is not strictly convex, but it is reflexive. Let K_1 denote the unit ball in $M_2^{(1)}$ and K_2 the unit ball in the Hilbert space l_2. It has already been proved [2] that bounded closed convex subsets of uniformly convex spaces and compact convex subsets of arbitrary Banach spaces have complete normal structure. Since complete normal structure implies normal structure, K_1 and K_2 also have normal structure. Thus by Theorems 2.1 and 2.2, $K = K_1 \oplus K_2$ is a closed convex *noncompact* subset of $B = M_2^{(1)} \oplus l_2$ which has normal structure and in which every closed convex subset satisfies $(*)_q$, *but in which the norm is not strictly convex.*

Finally, we remark that it follows immediately by induction that Theorems 2.1 and 2.2 hold for finite direct sums.

3. **Normal structure and uniform convexity.** As we mentioned in the preceding section, it follows from a result of [2] (Theorem 4.1) that every uniformly convex Banach space has normal structure. In this section we show that even if a Banach space is reflexive and has strictly convex norm, it may possess normal structure but not be isomorphic to any uniformly convex Banach space.

THEOREM 3.1. *There exists a Banach space which is reflexive, strictly convex, and which possesses normal structure, but which is not isomorphic to any uniformly convex Banach space.*

Proof. Let $p > 1$ and consider the space B of all sequences $x = (x^{(1)}, x^{(2)}, \cdots)$ where each element $x^{(i)}$ is a Banach space $B^{(i)}$ and

$$\sum_1^\infty \| x^{(i)} \|^p < \infty .$$

Take

$$\| x \| = \left(\sum_1^\infty \| x^{(i)} \|^p \right)^{1/p} .$$

We also assume that each of the spaces $B^{(i)}$ is finite dimensional, strictly convex, and restricted in such a way that B is not isomorphic to any uniformly convex Banach space. (This involves conditions on the dimensions of $B^{(i)}$ and on their respective norm functions which have no bearing on the following argument; see Day [5, Th. 2].)

Suppose K is a bounded convex subset of B which consists entirely of diametral points. We show that this assumption leads to a contradiction if $\delta(K) > 0$.

Let $\{x_n\}$ be a *diametral sequence* [3] in K (that is, $d(x_{n+1}, \text{conv } \{x_1, \cdots, x_n\}) \to \delta(K)$ as $n \to \infty$). Since K is bounded, $\{x_n^{(1)}\}$ is a bounded sequence in the finite dimensional space $B^{(1)}$ and thus it has a convergent subsequence $\{x_{1n}^{(1)}\}$ which has limit, say $y^{(1)}$. By induction one may select a subsequence $\{x_{kn}\}$ of $\{x_{(k-1)n}\}$ such that if $i \leqq k$,

$$\lim_{n\to\infty} x_{kn}^{(i)} = y^{(i)} .$$

Using the way the sequence $y = (y^{(1)}, y^{(2)}, \cdots)$ is defined, one may obtain a subsequence $\{y_n\}$ of $\{x_n\}$ which has property:

$$\lim_{n\to\infty} y_n^{(i)} = y^{(i)} , \qquad (i = 1, 2, \cdots) . \tag{1}$$

Furthermore, since B is reflexive this subsequence may be chosen so that it converges weakly, say to z. Because of (1) it follows that $z^{(i)} = y^{(i)}$ and thus that $z = y$. Hence y not only is in B, but it is in the closed convex hull of $\{x_n\}$.

Since $\{y_n\}$ converges weakly to y, $d(y, \text{conv } \{y_1, \cdots, y_n\}) \to 0$ as $n \to \infty$. But $\{y_n\}$ is a subsequence of the diametral sequence $\{x_n\}$ so $d(y_{n+1}, \text{conv } \{y_1, \cdots, y_n\}) \to \delta(K)$ as $n \to \infty$, and therefore

$$\| y_n - y \| \to \delta(K) \quad \text{as} \quad n \to \infty .$$

Now let $\varepsilon > 0$, $3\varepsilon < \delta(K)$. Integers $r, N > r, s > N$ may be chosen large enough that

$$\| y_r - y \| > \delta(K) - \varepsilon, \| y_s - y \| > \delta(K) - \varepsilon ,$$

$$\sum_1^N \| y_s^{(i)} - y^{(i)} \|^p < \varepsilon^p , \quad \text{and}$$

$$\sum_{N+1}^\infty \| y_r^{(i)} - y^{(i)} \|^p < \varepsilon^p .$$

Letting

$$|| x ||_{1,N} = \left(\sum_{1}^{N} || x^{(i)} ||^p \right)^{1/p},$$

$$|| x ||_{N+1} = \left(\sum_{N+1}^{\infty} || x^{(i)} ||^p \right)^{1/p},$$

it follows that

$$\begin{aligned} || y_s - y_r ||^p &= [|| y_s - y_r ||_{1,N}]^p + [|| y_s - y_r ||_{N+1}]^p \\ &\geqq [|| y_r - y ||_{1,N} - || y_s - y ||_{1,N}]^p \\ &\quad + [|| y_s - y ||_{N+1} - || y_r - y ||_{N+1}]^p \\ &\geqq [\delta(K) - 3\varepsilon]^p + [\delta(K) - 3\varepsilon]^p . \end{aligned}$$

Thus

$$\sup || y_s - y_r || \geqq 2^{1/p}\delta(K) > \delta(K)$$

and we have a contradiction.

Although reflexivity of B was useful in the above argument, normal structure is not implied by reflexivity or even by isomorphism with Hilbert space. For example, let H be Hilbert space and B the space isomorphic to H with norm defined (for $x \in H$) by

$$|| x || = \sup \{1/2 || x ||_H, | x_n | \} .$$

Let

$$K = \{x: || x || \leqq 1 \text{ and } x_i \geqq 0 \text{ for all } i\} .$$

If x and y are in K, then $|| x - y || \leqq \sup \{1, 1\} = 1$. Thus $\delta(K) \leqq 1$. Given x in K and $\varepsilon > 0$, choose y with a single nonzero component $y_k = 1$ with k chosen so that $| x_k | < \varepsilon$. Then

$$|| x - y || \geqq | x_k - y_k | > 1 - \varepsilon .$$

Thus each point of K is a diametral point of K.

We wish to thank the referee for the above example of a reflexive space which does not possess normal structure. (Another such example, due to R. C. James, was sent to us by Stanley Weiss.) We are also grateful to the referee for providing us with a much simpler proof of Theorem 3.1.

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