

Pacific Journal of Mathematics

BASES IN HILBERT SPACE

WILLIAM JAY DAVIS

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A sequence (x_i) of elements of a Hilbert space, \mathcal{H} , is a *basis* for \mathcal{H} if every $h \in \mathcal{H}$ has a unique, norm-convergent expansion of the form $h = \sum a_i x_i$, where (a_i) is a sequence of scalars. The sequence is *minimal* if there exists a sequence $(y_i) \subset \mathcal{H}$ such that $(x_i, y_j) = \delta_{ij}$. Every basis is minimal, and the sequence (a_i) in the expansion of h (above) is given by $a_i = (h, y_i)$. In this paper, we restrict our attention to *real* Hilbert space.

We derive, from classical characterizations of bases in B -spaces, criteria for (x_i) to be a basis for \mathcal{H} , as well as for (x_i) to be minimal in \mathcal{H} . We show that the sequence is minimal if and only if there are sequences $(g_i) \subset \mathcal{H}$ whose Gram matrices have a prescribed form. Similar conditions are obtained for (x_i) to be a basis for \mathcal{H} .

Let (x_i) be a linearly independent sequence of elements of \mathcal{H} . Using the Gram-Schmidt process, one finds an orthonormal basis, (w_i) , for the closed span, $[x_i]$ of the sequence (x_i) . We assume throughout that $[x_i] = \mathcal{H}$. Then, we may write

$$x_i = \sum_{j=0}^i p_{ij} w_j,$$

and

$$w_i = \sum_{j=0}^i q_{ij} x_j.$$

If we let P and Q denote the matrices (p_{ij}) and (q_{ij}) , respectively, then each is lower triangular, and $PQ = QP = I = (\delta_{ij})$. It is a classical result that Q is the unique inverse of P .

For (x_i) to be minimal, we need a sequence (y_i) such that $(x_i, y_j) = \delta_{ij}$. It is easy to see that, formally, $y_i = \sum_{j=i}^{\infty} q_{ji} w_j$. Further, the sequence is minimal if and only if the distance from x_k to $[x_j]$, $j \neq k$ is positive. Using these facts, we get the following theorem. The second part is similar to the characterization of minimality due to Foias and Singer [2].

THEOREM 1. *Let $H = (h_{ij})$ denote the Gram matrix of (x_i) , i.e., $h_{ij} = (x_i, x_j)$. Then the sequence is minimal if and only if any of the following conditions holds:*

- (a) *The matrix $R = Q^r Q$ exists.*
- (b) *There exists a sequence, (δ_i) , with $\delta_i > 0$ for all i , such*

that for all real vectors $A = (a_0, a_1, \dots, a_n, 0, \dots)$, $AHA^T \geq \sum \delta_i a_i^2$.

(c) There exists a sequence (ε_i) with $\varepsilon_i > 0$ for all i such that $ARA^T \geq \sum \varepsilon_i a_i^2$, with A as in (b).

Proof. (a) Follows from the formal relation $y_i = \sum q_{ji} w_j$. For (b), notice that $AHA^T \geq \|\sum a_i x_i\|^2$. If (x_i) is minimal, then $AHA^T \geq \lambda_i \|x_i\|^2 a_i^2$, where $\lambda_i^{1/2}$ is the distance from $x_i/\|x_i\|$ to $[x_j], j \neq i$. Therefore, for each permutation (n_i) of the nonnegative integers,

$$AHA^T \geq \sum a^{-(n_i+1)} \lambda_i a_i^2 \|x_i\|^2.$$

So $\delta_i = 2^{-(n_i+1)} \lambda_i \|x_i\|^2$ works. On the other hand, if $AHA^T \geq \sum \delta_i a_i^2$, then $AHA^T \geq \delta_i a_i^2 = \lambda_i \|x_i\|^2 a_i^2$ for each i . Part (c) follows since (y_i) is minimal if and only if (x_i) is minimal.

2. Here we derive further criteria, for minimal and basic sequences, which depend upon the existence of certain Gram matrices. First, we recall that a fundamental sequence (x_i) in a B -space is minimal if and only if, for each n , there exists a constant $K_n \geq 1$ such that, for all m and all sequences (a_j) ,

$$\left\| \sum_{j=0}^n a_j x_j \right\| \leq K_n \left\| \sum_{j=0}^{n+m} a_j x_j \right\|.$$

Further, such a sequence is basic if and only if (K_n) is bounded (that is, if and only if a bounded sequence (K_n) can be chosen) [1]. In either case, K_n is to be chosen in such a way that

$$\left\{ K_n^2 \left\| \sum_{j=0}^{n+m} a_j x_j \right\|^2 - \left\| \sum_{j=0}^n a_j x_j \right\|^2 \right\}$$

defines a positive definite form on the collection of all finite real sequences. Associated with this form is the matrix $S = S(n, K_n)$, defined as follows:

$$S_{ij} = \begin{cases} (K_n^2 - 1)(x_i, x_j); & 1 \leq i, j \leq n \\ K_n^2(x_i, x_j); & \text{otherwise.} \end{cases}$$

The positive definiteness of the form ASA^T will be achieved over the finite vectors $A = (a_1, a_2, \dots, a_n, 0, \dots)$ if and only if each principal $k \times k$ submatrix, $S^{(k)}$ of S is positive definite. Each $S^{(k)}$ is positive definite if and only if there exists a real, nonsingular, lower triangular matrix T such that $S^{(k)} = T^{(k)} T^{(k)T}$. A routine calculation shows that

$$T_{ij} = \begin{cases} \sqrt{K_n^2 - 1} p_{ij}; & 1 \leq i, j \leq n \\ \frac{K_n^2}{\sqrt{K_n^2 - 1}} p_{ij}; & i > n, 1 \leq j \leq n. \end{cases}$$

Thus, we must solve, in the reals, the equations

$$\sum_{j=n+1}^i T_{ij} T_{kj} = K_n^2(x_i, x_k) - \frac{K_n^4}{K_n^2 - 1} (\pi_n x_i, x_k),$$

where $\pi_n x_i = \sum_{j=1}^n p_{ij} w_j$. If these equations are solvable, then S is positive definite (over finite A), if and only if $T_{ii} \neq 0$. Now let $(f_i)_{i=n+1}^\infty$ be any linearly independent sequence in \mathcal{H} for which

$$(f_i, f_j) = K_n^2(x_i, x_j) - \left(\frac{K_n^4}{K_n^2 - 1} \right) (\pi_n x_i, x_j),$$

if it exists. If we orthonormalize (f_i) , we get a sequence $(g_i)_{i=n+1}^\infty$ and

$$f_i = \sum_{j=n+1}^i T_{ij} g_j.$$

Linear independence of (f_i) gives $T_{ii} \neq 0$. On the other hand, if the equations above are solvable, for (T_{ij}) , we may set $f_i = \sum_{j=n+1}^i T_{ij} w_j$.

We have the following theorem:

THEOREM 2. *The sequence (x_i) is*

(a) *minimal if and only if, for each n , there exists $K_n \geq 1$ and a linearly independent sequence $(f_i)_{i=n+1}^\infty$ such that*

$$(f_i, f_j) = K_n^2(x_i, x_j) - \frac{K_n^4}{K_n^2 - 1} (\pi_n x_i, x_j).$$

(b) *a basis if and only if it is minimal, and the sequence (K_n) may be chosen so that it is bounded.*

The sequence (x_j) is minimal if and only if, for each n , there exists $C_n \geq 1$ such that, for all m and sequences (a_i) ,

$$\left\| \sum_{i=n+1}^{n+m} a_i x_i \right\| \leq C_n \left\| \sum_{i=1}^{n+m} a_i x_i \right\|.$$

It is basic if and only if (C_n) may be chosen as a bounded sequence (see, e.g., [4]). Using these facts, and arguments similar to those for Theorem 2, we obtain,

THEOREM 3. *The sequence (x_i) is*

(a) *minimal if and only if, for each n , there exists $C_n \geq 1$ and a linearly independent sequence $(g_i)_{i=n+1}^\infty$ such that, for $i, j > n$,*

$$(g_i, g_j) = (C_n^2 - 1)(x_i, x_j) - C_n^2(\pi_n x_i, x_j),$$

and

(b) *basic if and only if it is minimal and (C_n) may be chosen as a bounded sequence.*

In deriving Theorem 3, one must determine the positive definiteness of the matrices S defined by

$$S_{ij} = \begin{cases} C_n^2(x_i, x_j); & 1 \leq i \leq n \text{ or } 1 \leq j \leq n \\ (C_n^2 - 1)(x_i, x_j); & i, j > n. \end{cases}$$

An interesting characterization of minimal sequences and bases is the following.

PROPOSITION. *The sequence (x_i) is*

(a) *minimal if its Gram matrix, H , is strictly diagonally dominant, and*

(b) *a basis if its Gram matrix is uniformly diagonally dominant.*¹

Proof. If H is strictly diagonally dominant, for each n there exists $\gamma_n \in (0, 1)$ such that $\gamma_n |(x_n, x_n)| < \sum_{j \neq n} |(x_n, x_j)|$. Then, for $C_n^2 = 1/\gamma_n$, the matrix S is strictly diagonally dominant, and hence positive definite over finite $A[5]$. Part (b) follows in the same manner.

Using the same method of proof, Theorems 3 and 4, and the fact that the positive definite $n \times n$ matrices define a cone in the linear space of all $n \times n$ matrices, we obtain the most general form of our characterization of minimal sequences and bases in \mathcal{H} .

THEOREM 4. *The sequence (x_i) is*

(a) *minimal if and only if, some (and hence all) $\alpha, \beta > 0$ and all n , there exist $K_n, C_n \geq 1$ and $(g_i)_{i=n+1}^\infty$ such that, for $i, j > n$,*

$$(g_i, g_j) = (\alpha K_n^2 + \beta C_n^2 - \beta)(x_i, x_j) - \left(\frac{(\alpha K_n^2 + \beta C_n^2)^2}{\alpha K_n^2 + \beta C_n^2 - \alpha} \right) (\pi_n x_i, x_j),$$

and

A symmetric matrix A is *strictly diagonally dominant* [5] if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i , and is *uniformly diagonally dominant* if there exists $\gamma \in (0, 1)$ such that $\gamma |a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for each i .

(b) *a basis if and only if (K_n) and (C_n) may be chosen as bounded sequences.*

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Received December 13, 1966. This work was supported by National Science Foundation Grant number GP-6152.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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