Bases in Hilbert Space

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A sequence \((x_i)\) of elements of a Hilbert space, \(\mathcal{H}\), is a basis for \(\mathcal{H}\) if every \(h \in \mathcal{H}\) has a unique, norm-convergent expansion of the form \(h = \sum a_i x_i\), where \((a_i)\) is a sequence of scalars. The sequence is minimal if there exists a sequence \((y_i) \in \mathcal{H}\) such that \((x_i, y_j) = \delta_{ij}\). Every basis is minimal, and the sequence \((a_i)\) in the expansion of \(h\) (above) is given by \(a_i = \langle h, y_i \rangle\). In this paper, we restrict our attention to real Hilbert space.

We derive, from classical characterizations of bases in \(B\)-spaces, criteria for \((x_i)\) to be a basis for \(\mathcal{H}'\), as well as for \((x_i)\) to be minimal in \(\mathcal{H}\). We show that the sequence is minimal if and only if there are sequences \((\gamma_i) \subseteq \mathcal{H}\) whose Gram matrices have a prescribed form. Similar conditions are obtained for \((x_i)\) to be a basis for \(\mathcal{H}'\).

Let \((x_i)\) be a linearly independent sequence of elements of \(\mathcal{H}\). Using the Gram-Schmidt process, one finds an orthonormal basis, \((w_i)\), for the closed span, \([x_i]\) of the sequence \((x_i)\). We assume throughout that \([x_i] = \mathcal{H}\). Then, we may write

\[x_i = \sum_{j=0}^{i} p_{ij} w_j,\]

and

\[w_i = \sum_{j=0}^{i} q_{ij} x_j.\]

If we let \(P\) and \(Q\) denote the matrices \((p_{ij})\) and \((q_{ij})\), respectively, then each is lower triangular, and \(PQ = QP = I = (\delta_{ij})\). It is a classical result that \(Q\) is the unique inverse of \(P\).

For \((x_i)\) to be minimal, we need a sequence \((y_i)\) such that \((x_i, y_j) = \delta_{ij}\). It is easy to see that, formally, \(y_i = \sum_{j=0}^{i} q_{ij} w_j\). Further, the sequence is minimal if and only if the distance from \(x_k\) to \([x_j]\), \(j \neq k\) is positive. Using these facts, we get the following theorem. The second part is similar to the characterization of minimality due to Foias and Singer [2].

**Theorem 1.** Let \(H = (h_{ij})\) denote the Gram matrix of \((x_i)\), i.e., \(h_{ij} = (x_i, x_j)\). Then the sequence is minimal if and only if any of the following conditions holds:

(a) The matrix \(R = Q'TQ\) exists.

(b) There exists a sequence, \((\delta_i)\), with \(\delta_i > 0\) for all \(i\), such
that for all real vectors $A = (a_0, a_1, \ldots, a_n, 0, \ldots)$, $AHA^T \geq \sum \delta_i a_i^t$.

(c) There exists a sequence $(\varepsilon_i)$ with $\varepsilon_i > 0$ for all $i$ such that $ARA^T \geq \sum \varepsilon_i a_i^t$, with $A$ as in (b).

Proof. (a) Follows from the formal relation $y_i = \sum q_j w_j$. For (b), notice that $AHA^T \geq \|\sum a_i x_i\|^2$. If $(x_i)$ is minimal, then $AHA^T \geq \lambda_i \|x_i\|^2 a_i^t$, where $\lambda_i^{1/2}$ is the distance from $x_i/\|x_i\|$ to $[x_j], j \neq i$. Therefore, for each permutation $(n_i)$ of the nonnegative integers,

$$AHA^T \geq \sum a^{-\|n_{i+1}\|\lambda_i a_i^t \|x_i\|^2 .$$

So $\delta_i = 2^{-n_{i+1}} \lambda_i \|x_i\|^2$ works. On the other hand, if $AHA^T \geq \sum \delta_i a_i^t$, then $AHA^T \geq \delta_i a_i^t = \lambda_i \|x_i\|^2 a_i^t$ for each $i$. Part (c) follows since $(y_i)$ is minimal if and only if $(x_i)$ is minimal.

2. Here we derive further criteria, for minimal and basic sequences, which depend upon the existence of certain Gram matrices. First, we recall that a fundamental sequence $(x_i)$ in a $B$-space is minimal if and only if, for each $n$, there exists a constant $K_n \geq 1$ such that, for all $m$ and all sequences $(a_j)$,

$$\left\| \sum_{j=0}^n a_j x_j \right\| \leq K_n \left\| \sum_{j=0}^{n+m} a_j x_j \right\| .$$

Further, such a sequence is basic if and only if $(K_n)$ is bounded (that is, if and only if a bounded sequence $(K_n)$ can be chosen) [1]. In either case, $K_n$ is to be chosen in such a way that

$$\left\{ K_n \left\| \sum_{j=0}^n a_j x_j \right\|^2 - \left\| \sum_{j=0}^n a_j x_j \right\|^2 \right\}$$

defines a positive definite form on the collection of all finite real sequences. Associated with this form is the matrix $S = S(n, K_n)$, defined as follows:

$$S_{ij} = \begin{cases} (K_n^2 - 1)(x_i, x_j); & 1 \leq i, j \leq n \\ K_n^2 (x_i, x_j); & \text{otherwise} \end{cases} .$$

The positive definiteness of the form $ASA^T$ will be achieved over the finite vectors $A = (a_0, a_1, \ldots, a_n, 0, \ldots)$ if and only if each principal $k \times k$ submatrix, $S^{(k)}$ of $S$ is positive definite. Each $S^{(k)}$ is positive definite if and only if there exists a real, nonsingular, lower triangular matrix $T$ such that $S^{(k)} = T^{(k)} T^{(k)T}$. A routine calculation shows that

$$T_{ij} = \begin{cases} \sqrt{K_n^2 - 1} p_{ij}; & 1 \leq i, j \leq n \\ \frac{K_n^2}{\sqrt{K_n^2 - 1}} p_{ij}; & i > n, 1 \leq j \leq n \end{cases} .$$
Thus, we must solve, in the reals, the equations
\[ \sum_{j=n+1}^{i} T_{ij} T_{kj} = K_{ij}(x_i, x_k) \]
\[ - \frac{K_{ij}}{K_{ii} - 1} (\pi_n x_i, x_k), \]
where \( \pi_n x_i = \sum_{j=1}^{n} p_{ij} w_j \). If these equations are solvable, then \( S \) is positive definite (over finite \( A \)), if and only if \( T_{ii} \neq 0 \). Now let \( (f_i)_{i=n+1}^{\infty} \) be any linearly independent sequence in \( \mathcal{H} \) for which
\[ (f_i, f_j) = K_{ij}(x_i, x_j) - \left( \frac{K_{ij}}{K_{ii} - 1} \right) (\pi_n x_i, x_j), \]
if it exists. If we orthonormalize \( (f_i) \), we get a sequence \( (g_i)_{i=n+1}^{\infty} \) and
\[ f_i = \sum_{j=n+1}^{i} T_{ij} g_j. \]
Linear independence of \( (f_i) \) gives \( T_{ii} \neq 0 \). On the other hand, if the equations above are solvable, for \( (T_{ij}) \), we may set \( f_i = \sum_{j=n+1}^{i} T_{ij} w_j \).

We have the following theorem:

**Theorem 2.** The sequence \( (x_i) \) is

(a) minimal if and only if, for each \( n \), there exists \( K_{ii} \geq 1 \) and a linearly independent sequence \( (f_i)_{i=n+1}^{\infty} \) such that
\[ (f_i, f_j) = K_{ij}(x_i, x_j) - \frac{K_{ij}}{K_{ii} - 1} (\pi_n x_i, x_j). \]

(b) a basis if and only if it is minimal, and the sequence \( (K_{ii}) \) may be chosen so that it is bounded.

The sequence \( (x_j) \) is minimal if and only if, for each \( n \), there exists \( C_n \geq 1 \) such that, for all \( m \) and sequences \( (a_i) \),
\[ \left\| \sum_{i=n+1}^{n+m} a_i x_i \right\| \leq C_n \left\| \sum_{i=1}^{n+m} a_i x_i \right\|. \]
It is basic if and only if \( (C_n) \) may be chosen as a bounded sequence (see, e.g., [4]). Using these facts, and arguments similar to those for Theorem 2, we obtain,

**Theorem 3.** The sequence \( (x_i) \) is

(a) minimal if and only if, for each \( n \), there exists \( C_n \geq 1 \) and a linearly independent sequence \( (g_i)_{i=n+1}^{\infty} \) such that, for \( i, j \geq n \),
(g_i, g_j) = (C_n^z - 1)(x_i, x_j) - C_n^z(\pi_n x_i, x_j),

and

(b) basic if and only if it is minimal and \( (C_n) \) may be chosen as a bounded sequence.

In deriving Theorem 3, one must determine the positive definiteness of the matrices \( S \) defined by

\[
S_{ij} = \begin{cases} 
C_n^z(x_i, x_j); & 1 \leq i \leq n \text{ or } 1 \leq j \leq n \\
(C_n^z - 1)(x_i, x_j); & i, j > n.
\end{cases}
\]

An interesting characterization of minimal sequences and bases is the following.

**PROPOSITION.** The sequence \( (x_i) \) is

(a) minimal if its Gram matrix, \( H \), is strictly diagonally dominant, and

(b) a basis if its Gram matrix is uniformly diagonally dominant.

**Proof.** If \( H \) is strictly diagonally dominant, for each \( n \) there exists \( \gamma_n \in (0, 1) \) such that \( \gamma_n |(x_n, x_n)| < \sum_{j \neq n} |(x_n, x_j)| \). Then, for \( C_n^z = 1/\gamma_n \), the matrix \( S \) is strictly diagonally dominant, and hence positive definite over finite \( A[5] \). Part (b) follows in the same manner.

Using the same method of proof, Theorems 3 and 4, and the fact that the positive definite \( n \times n \) matrices define a cone in the linear space of all \( n \times n \) matrices, we obtain the most general form of our characterization of minimal sequences and bases in \( \mathcal{H} \).

**THEOREM 4.** The sequence \( (x_i) \) is

(a) minimal if and only if, some (and hence all) \( \alpha, \beta > 0 \) and all \( n \), there exist \( K_n, C_n \geq 1 \) and \( (g_i)_{i=n+1}^\infty \) such that, for \( i, j > n \),

\[
(g_i, g_j) = (\alpha K_n^z + \beta C_n^z - \beta)(x_i, x_j) \\
- \left( \frac{(\alpha K_n^z + BC_n^z)^2}{\alpha K_n^z + \beta C_n^z - \alpha} \right) \left( \pi_n x_i, x_j \right),
\]

and

A symmetric matrix \( A \) is strictly diagonally dominant [5] if \( |a_{ii}| > \sum_{j \neq i} |a_{ij}| \) for all \( i \), and is uniformly diagonally dominant if there exists \( \gamma \in (0, 1) \) such that \( \gamma |a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \) for each \( i \).
(b) a basis if and only if \((K_n)\) and \((C_n)\) may be chosen as bounded sequences.

REFERENCES


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