$p$-AUTOMORPHIC $p$-GROUPS AND HOMOGENEOUS ALGEBRAS

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A p-group was called p-automorphic by Boen, if its automorphism group is transitive on elements of order p. Boen conjectured that if p is odd, then such a p-group is abelian. Let P be a nonabelian p-automorphic p-group, p odd, generated by n elements. Boen proved that n > 3, and in joint work with Rothaus and Thompson proved that n > 5. Kostrikin then showed that n > p + 6, as a corollary of results on homogeneous algebras. In this paper it is shown that n > 2p + 3, using Kostrikin’s methods, and his proof is somewhat simplified by eliminating special case considerations for small values of p.

The above results and the following terminology may be found in [1], [2], and [4]. Let A be a finite-dimensional algebra over the field K, where if x, y ∈ A and λ ∈ K, we assume bilinearity and the law (λx) ∘ y = λ(x ∘ y) = x ∘ (λy), but associativity is not assumed. Following [4], A is said to be homogeneous if the automorphism group Γ of A is transitive on A* = A — {0}, anticommutative if x ∘ y + y ∘ x = 0, and nil if all endomorphisms Kα: x → x ∘ a are nilpotent.

For a fixed odd prime p, suppose that P is a nonabelian p-automorphic p-group with minimal number n of generators. It is shown in [1] that P has a p-automorphic quotient group P̂ with the same number of generators, where the Frattini subgroup Φ(P̂) is central and is the direct product of n cyclic groups of equal order p^m. If we consider A = P̂/Φ(P̂) as a vector space over GF(p), we define a multiplication in A as follows: for x = aΦ(P̂), y = bΦ(P̂) in A, a coset z = cΦ(P̂) is uniquely determined, such that [a, b] = c^p^m. Define x ∘ y = z. Then it is clear that A becomes an anticommutative homogeneous algebra, and Theorem 1 of [2] asserts that A is nil.

It is proved in [4] that if A is a finite-dimensional homogeneous algebra with nontrivial multiplication over a field K of characteristic not 2, then A is an anticommutative nil algebra and K is a finite field of q elements, where q < dim A − 6. In this paper we shall prove:

**Theorem.** Let A be a homogeneous anticommutative nil algebra with nontrivial multiplication of dimension n over the field K of q elements, q odd. Then n > 2q + 3.

This result immediately implies the corresponding result for p-
automorphic p-groups.

2. In proving the following notation. $A$ is a homogeneous anticommutative nil algebra of dimension $n$ over the field $K$ of $q$ elements, $q$ odd, and $\Gamma$ its automorphism group. We choose integers $m$ and $r$ such that

$$\dim AR_x = m, R_x = 0, R_x^{-1} \neq 0, \text{ all } x \neq 0 \text{ in } A.$$ 

Of course $r \leq m + 1$. Since $\Gamma$ is transitive on $A - \{0\}$, $q^n - 1$ divides the order of $\Gamma$. Let $s$ be a prime dividing $q^n - 1$, but not dividing $q^t - 1$ for any $t < n$; the existence of $s$ is proved in [3]. (We may assume $n > 2$; for the case $n = 2$, the theorem follows from the relation $r > q$, soon to be proved.) Let $\sigma \in \Gamma$ have order $s$; then $\sigma$ is irreducible on the vector space $A$. Fix a nonzero element $e \in A$. Then $A$ is spanned by $e, e\sigma, \cdots, e\sigma^{s-1}$; let

$$e\sigma^n = \sum_{j=1}^{n} a_j e\sigma^{n-j}, a_j \in K,$$

where $\sigma$ satisfies the irreducible polynomial $P(X) = X^n - \sum_{j=1}^{n} a_j X^{n-j}$.

Consider the vectors $e\sigma^i \cdot e, 0 \leq i \leq n - 1$. We see that

$$(e\sigma^i \cdot e)\sigma^{n-i} = e\sigma^n \circ e\sigma^{-i} = \left(\sum_{j=1}^{n} a_j e\sigma^{n-j}\right) \circ e\sigma^{-i}$$

$$= \sum_{j=1}^{n} a_j (e\sigma^{n-j} \circ e\sigma^{-i}) = \sum_{j=1}^{n} a_j (e\sigma^{i-j} \circ e)\sigma^{-i}$$

$$- \sum_{j>i} a_j (e\sigma^{j-i} \circ e)\sigma^{n-j} = \sum_{0 \leq k < i} a_{i-k} (e\sigma^k \circ e)\sigma^{n-i-k}$$

Transferring all terms to the right-hand side, we have a relation

$$AR_x B = 0,$$

where $B = (b_{ij})_{0 \leq i, j \leq n-1}$, as a matrix over $K = K(\sigma)$, say, with row index $j$ and column index $i$, is given as follows: Define $a_0 = -1, a_k = 0$ if $k < 0$ or $k > n$. Then

$$b_{ij} = a_{i-j} \sigma^{n-i} - a_{i+j} \sigma^{n-i-j}.$$ 

We look at this matrix $B$ quite closely. If $n$ is even, let $B_1$ be the lower right-hand $(n/2) \times (n/2)$ minor. $B_1$ is a triangular matrix with

$$\text{Det } B_1 = (-1)^{n/2} \sigma^{1+2+\cdots+(n-2)/2} (\sigma^{n/2} + a_n) \neq 0,$$

so rank $B \geq n/2$. If $n$ is odd, let $B_1$ be the lower right-hand
The next step in the proof is to show that $r > q + 1$; this is done in [4], but we repeat it here, as the final case simplifies.

First suppose $r \leq q$. Then we can linearize the identity
\[(R_x + \alpha R_e)^r = R_x^{r+\alpha} = 0,
\]
all $\alpha \in K$, obtaining
\[\sum_{i=0}^{r-1} R_x^i R_e R_x^{r-1-i} = 0.
\]
Applying to $y \in A$ and using anticommutativity,
\[y \cdot \sum_{i=0}^{r-1} R_x^i R_e R_x^{r-1-i} = -\sum_{i=0}^{r-1} z R_{yR_x} R_x^{r-1-i} = 0,
\]
and hence
\[\sum_{i=0}^{r-1} R_{yR_x} R_x^{r-1-i} = 0.
\]

The equation $e = a \circ e$ is not possible, since otherwise $e R_x^k = (-1)^k e \neq 0$, and $R_x$ is not nilpotent. Hence $\alpha \in AR_e$. We choose a basis $\{e_1, \ldots, e_{r_1}, e_{r_1+1}, \ldots, e_{r_1+r_2}, \ldots, e_{r_n}\}$, $e = e_n$, such that the nilpotent transformation $R_x$ is in Jordan canonical form. Thus we have
\[r = r_1 \geq r_2 \geq \cdots; e_i R_{e_n} = e_{i+1} \text{ if}
\]
\[r_1 + \cdots + r_{k-1} + 1 \leq i < r_1 + \cdots + r_k, \text{ some } k; e_{r_1+\cdots+r_k} R_{e_n} = 0.
\]
Setting $y = e_1$, $x = e_n$ in the last identity, we have
\[R_{e_x} + \left(\sum_{i=0}^{r-1} R_x^{i+1} R_{e_n}^{r-2-i}\right)R_{e_n} = 0.
\]
Hence $AR_{e_x} \subseteq AR_{e_n}$; but dim $AR_{e_x} = \dim AR_{e_n}$, so $AR_{e_x} = AR_{e_n}$. Thus $e_x = e_i R_{e_n}^{r-1} \in AR_{e_n} = AR_x$, a contradiction. We conclude $r > q$.

Now suppose $r = q + 1$. The identity $R_x^r = 0$ cannot be linearized, but the linearization process does enable us to prove
where \( f \) and \( g \) are homogeneous polynomials, linear in \( R_y \) and \( R_z \). (Expand \((R_x + \alpha R_y + \beta R_z)^{r+1} = 0\), use \( \alpha = \alpha' \), \( \beta = \beta' \) to combine two terms, and then use van der Monde determinants as in the usual linearization to show all terms are 0. The coefficient of \( \alpha \beta \) is the left side of the desired equation.) Applying this to \( x \) and using anti-commutativity,

\[
0 = z R_y R_i - z R_i R_y - z f(R_x, R_y) R_z, \text{ some } f,
\]

showing that

\[
R_y R_i - R_i R_y - f(R_x, R_y) R_z = 0.
\]

We choose a canonical basis for \( R_\epsilon_n \) as before and set \( x = e_n, y = e_i \) in the last identity, obtaining

\[
R_{\epsilon_r} = R_{\epsilon_n} R_{\epsilon_l} + f(R_{\epsilon_n}, R_{\epsilon_l}) R_{\epsilon_n}.
\]

For \( i \in \{1, r_1 + 1, r_1 + r_2 + 1, \ldots\} \), we see

\[
e_i R_{\epsilon_r} = e_i f(R_{\epsilon_n}, R_{\epsilon_l}) R_{\epsilon_n} \in A R_{\epsilon_n}.
\]

Also,

\[
e_i R_{\epsilon_r} = e_i R_{\epsilon_l} + e_i f(R_{\epsilon_n}, R_{\epsilon_l}) R_{\epsilon_n},
\]

so since the characteristic is odd, \( e_i R_{\epsilon_r} \in A R_{\epsilon_n} \). If \( r_2 < r_1 \), then \( e_i R_{\epsilon_n} = 0 \) for \( i \geq r \), and we conclude that \( A R_{\epsilon_n} = A R_{\epsilon_n} \), which we know to be impossible. Hence \( r_2 = r_1 = r \). Then \( n \geq 2r + 1 = 2q + 3 \). If we have equality, then the canonical form shows \( m = \dim A R_{\epsilon_n} = 2r - 2 = 2q > (n/2) \), a contradiction. Hence \( n > 2q + 3 \), and we are done in this case.

Thus we now may assume \( r \geq q + 2, r \leq m + 1, m \leq n/2 \). If \( n \) is even, we have \( q + 2 \leq r \leq m + 1 \leq (n/2) + 1 \), or \( n \geq 2q + 2 \), so we may assume \( n = 2q + 2 \); then equality holds everywhere, and \( r = q + 2, m = q + 1 \). If \( n \) is odd, we have

\[
q + 2 \leq r \leq m + 1 \leq \frac{n - 1}{2} + 1, \text{ or } n \geq 2q + 3,
\]

so we may assume \( n = 2q + 3 \); then equality holds everywhere, and \( r = q + 2, m = q + 1 \). In either case, we note \( n \leq 2m + 1 \).

Since \( q \) is odd and \( q^n - 1 \) divides the order of \( \Gamma \), we can choose an element \( \tau \in \Gamma \) of order 2. Define

\[
B = \{ a \in A \mid \tau(a) = a \}, C = \{ a \in A \mid \tau(a) = -a \}.
\]

Then \( A \) is a direct sum \( A = B \oplus C \) of its subspaces \( B \) and \( C \). Certainly
If \( B = 0 \), choose \( C_1, C_2 \in C \) with \( c_1 \circ c_2 \neq 0 \). Then \( c_1 \circ c_2 = (-c_2) \circ (c_2) = \tau(c_1) \circ \tau(c_2) = \tau(c_1 \circ c_2) = -c_1 \circ c_2 \), a contradiction. Define \( \dim B = k > 0 \), \( \dim C = n - k \). It is clear that \( B \circ B \subseteq B, C \circ C \subseteq B, B \circ C \subseteq C \). Hence if \( b \in B \), then \( B \circ b \subseteq B, C \circ b \subseteq C \); of course the nilpotency index \( r \) of \( R_6 \) is the maximum of its nilpotency indexes on the subspaces \( B \) and \( C \).

Suppose first \( B \circ C = 0 \). Then for any \( b \in B^* \), \( AR_b = BR_b \) has dimension \( m \); \( b \in BR_b \), so \( \dim B \geq m + 1 \), proving \( \dim C \leq m \). For any \( c \in C^* \), \( c \circ c = 0 \), so since \( AR_c = CR_c \), we have

\[
\dim AR_c = \dim CR_c < \dim C \leq m
\]
a contradiction.

We have thus proved \( B \circ C \neq 0 \). Pick \( b \in B \) with \( CR_b \neq 0 \); \( CR_b \subseteq C \), so \( AR_b = BR_b \oplus CR_b \), and \( \dim BR_b \leq m - 1 \). We look at the canonical form of \( R_6 \) on \( B \) and on \( C \), and use the fact

\[
r = m + 1; \dim BR_b \leq m - 1
\]
implies \( (R_6 \mid B)^m = 0 \), so \( (R_6 \mid C)^m \neq 0 \), and \( \dim CR_b \geq m \). Hence \( \dim CR_b = m \), \( \dim C \geq m + 1 \), \( \dim B \leq m \). This means that for any \( b' \in B^* \), \( \dim BR_b' < m \), so \( CR_b' \neq 0 \); the same argument then applies for \( b' \) as for \( b \). We conclude that \( B \circ B = 0 \).

Let \( c \) be any element of \( C^* \). Since \( R_6^c \neq 0 \) and \( \dim AR_c = m \), we have \( \dim AR_c^c = m - 1 \). Since \( BR_c \subseteq C \) and \( CR_c \subseteq B \), we have

\[
\dim AR_c = m = \dim BR_c + \dim CR_c
\]
Also,

\[
AR_c^c = (BR_c + CR_c)R_c \subseteq CR_c + BR_c
\]
Let \( \beta_i = \dim BR_c^i, \gamma_i = \dim CR_c^i, i = 1, 2 \). We see that \( \beta_1 + \gamma_1 = m, \beta_2 + \gamma_2 = m - 1, \beta_2 \leq \gamma_1, \gamma_2 \leq \beta_1 \), and of course \( \beta_2 \leq \beta_1, \gamma_2 \leq \gamma_1 \). Since \( m = q + 1 \) is even, let \( m = 2l \); the only solutions for the \( \beta_i \) and \( \gamma_i \) have \( \beta_1 = \gamma_1 = l \). So \( \dim BR_c = l \), for any \( c \in C^* \).

We now consider separately the cases \( n = 2q + 2 \) and \( n = 2q + 3 \). Let \( S \) denote the set of all ordered pairs \( \langle b, c \rangle, b \in B, c \in C \), with \( b \circ c = 0 \). In each case we compute the order \( |S| \) in two different ways to obtain a contradiction.

When \( n = 2q + 2 = 2m = 4l \), we know that for any

\[
b \in B^* \text{, } \dim CR_b = m
\]
so

\[
\dim \{ c \in C \mid b \circ c = 0 \} = (n - k) - m = m - k
\]
and for any
Let $c \in C^t$, $\dim BR_c = l$, so $\dim \{b \in B \mid b \circ c = 0\} = k - l$.

Hence
\[ |S| = (q^k - 1)q^{m-k} + q^{n-k} \]
and
\[ |S| = (q^{n-k} - 1)q^{k-l} + q^k. \]

Therefore
\[ q^{n-k} + q^m - q^{m-k} = q^{n-l} + q^k - q^{k-l}. \]

We know $\dim C = n - k \geq m + 1$, so $k < m$. Equating highest terms, the equation must imply $k = l$. But now the left side is divisible by $q$ and the right is not, a contradiction.

When $n = 2q + 3 = 2m + 1 = 4l + 1$, then for any
\[ b \in B^t, \quad \dim \{c \in C \mid b \circ c = 0\} = (n - k) - m = m - k + 1, \]
and for any
\[ c \in C^t, \quad \dim \{b \in B \mid b \circ c = 0\} = k - l. \]

Hence
\[ |S| = (q^k - 1)q^{m-k+1} + q^{n-k} \]
and
\[ |S| = (q^{n-k} - 1)q^{k-l} + q^k, \]
showing that
\[ q^{m+1} - q^{m+1-k} + q^{n-k} = q^{n-l} - q^{k-l} + q^k. \]

The largest terms on the two sides are necessarily equal, so $n - k = n - l, k = l$. But then the left side is divisible by $q$ and the right is not, the final contradiction.

**Remark.** Following [5], one can also consider semi-$p$-automorphic $p$-groups, in which the automorphism group is transitive on subgroups of order $p$, and the corresponding notion of spa-algebras, in which the automorphism group is transitive on one-dimensional subspaces. The arguments above then show $n > 2p + 1$. To prove $n > 2p + 3$, we require the involution $\tau$ in the automorphism group $\Gamma$; $\tau$ does exist, since otherwise $\Gamma$ would be of odd order and hence solvable, and the case of a solvable $\Gamma$ is treated in [5].

*Added in proof.* Ernest Schult has announced a complete solution of Boen's problem in Bull. Amer. Math. Soc. 74 (1968), 268–270.
REFERENCES


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