A GENERALIZED COROLLARY OF THE BROWDER-KIRK FIXED POINT THEOREM

William Grady Dotson, Jr. and W. R. Mann
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By W. G. Dotson, Jr. and W. Robert Mann

This paper generalizes a corollary, due to W. A. Kirk, of the F. E. Browder-W. A. Kirk fixed point theorem for nonexpansive self-mappings of closed, bounded, convex sets in uniformly convex Banach spaces.

F. E. Browder [1] and W. A. Kirk [4] have independently proved that if $F$ is a closed, bounded, convex subset of a uniformly convex Banach space, and if $T$ is a nonexpansive mapping from $F$ into $F$, then $T$ has a fixed point in $F$. The following corollary was proved by Kirk [4] and also by Browder and Petryshyn [2]: If $E$ is a uniformly convex Banach space, and $T: E \to E$ is a nonexpansive mapping, and if for some $x_i \in E$ the sequence $(T^n x_i)$ of Picard iterates of $T$ is bounded, then $T$ has a fixed point in $E$. Browder and Petryshyn also observed that if the nonexpansive mapping $T$ has a fixed point in $E$, then for any $x_i \in E$ the sequence $(T^n x_i)$ will be bounded. Outlaw and Groetsch [6] have recently announced the following extension of this corollary: If $E$ is a uniformly convex Banach space, and $T: E \to E$ is a nonexpansive mapping, and $S_\lambda = \lambda I + (1 - \lambda)T$ for a given $\lambda$, $0 < \lambda < 1$, then $T$ has a fixed point in $E$ if and only if the sequence $(S_\lambda^n x_i)$ of Picard iterates of $S_\lambda$ is bounded for each $x_i \in E$. The purpose of this note is to show that this corollary and its extension are both special cases of a considerably more general corollary of the Browder-Kirk theorem.

W. R. Mann [5] introduced the following general iterative process: Suppose $A = [a_{np}]$ is an infinite real matrix satisfying (1) $a_{np} \geq 0$ for all $n, p, a_{np} = 0$ for $p > n$; (2) $\sum_{p=1}^{\infty} a_{np} = 1$ for each $n$; (3) $\lim a_{np} = 0$ for each $p$. If $F$ is a closed convex subset of a Banach space $E$, and $T: F \to F$ is a continuous mapping, and $x_1 \in F$, then the process $M(x_1, A, T)$ is defined by

$$v_n = \sum_{p=1}^{n} a_{np} x_p, \quad x_{n+1} = T v_n, \quad n = 1, 2, 3, \ldots.$$ 

Various choices of the matrix $A$ yield many interesting iterative processes as special cases. With $A$ the infinite identity matrix, one gets the Picard iterates of $T$: $v_{n+1} = x_{n+1} = T v_n$, whence $v_{n+1} = T^n x_1 = T^n x_i$. With $0 < \lambda < 1$ and $A = [a_{np}]$ defined by $a_{np} = \lambda^{n-p}(1 - \lambda)$ if $p \leq n$, $a_{np} = 0$ if $p > n$, $n = 1, 2, 3, \ldots$, one gets $v_{n+1} = \lambda v_n + (1 - \lambda) T v_n = S_\lambda^n v_1$, whence $v_{n+1} = S_\lambda^n x_1$. If $T$ is linear then an appropriate choice of $A$ yields
\[ v_{n+1} = (x_1 + Tx_1 + \cdots + T^s x_1) / (n+1) \],

thus providing a connection with mean ergodic theorems for linear operators. Another choice of A yields an iterative process recently investigated by Halpern [3], provided \( x_1 = 0 \). Many other choices are possible, of course. Our main theorem is as follows.

**Theorem 1.** If \( E \) is a uniformly convex Banach space, and if \( T: E \to E \) is a nonexpansive mapping, and if there exist \( x_1 \in E \) and a process \( M(x_1, A, T) \) such that either of the sequences \( \{x_n\}, \{v_n\} \) is bounded, then \( T \) has a fixed point in \( E \).

To prove this, we will make use of the following lemma which is a straightforward consequence of uniform convexity.

**Lemma 1.** Suppose \( E \) is a uniformly convex Banach space, and suppose \( r > 0 \). For each \( \varepsilon > 0 \) let \( p_\varepsilon = \sup \{s: s = \| u - v \| \) where \( u, v \in E, \| u \| = 2r, 2r < \| v \| \leq 2r + \varepsilon, \) and \( \| (1-t)u + tv \| > 2r \) for all \( t \in (0, 1) \). Given any \( c > 0 \), there exists \( \varepsilon > 0 \) such that \( p_\varepsilon = c \).

**Proof of Theorem 1.** We first observe that if either of the sequences \( \{x_n\}, \{v_n\} \) in the process \( M(x_1, A, T) \) is bounded, then the other is also bounded. For if \( \| x_n \| \leq b \) for all \( n \), then

\[ \| v_n \| = \left\| \sum_{p=1}^n a_{np} x_p \right\| \leq \sum_{p=1}^n a_{np} \| x_p \| \leq b \sum_{p=1}^n a_{np} = b \]

for all \( n \); and if \( \| v_n \| \leq b \) for all \( n \), then

\[ \| x_{n+1} - T(0) \| = \| T(v_n) - T(0) \| \leq \| v_n - 0 \| \leq b \]

for all \( n \). So, given \( x_1 \in E \) and a process \( M(x_1, A, T) \) in which both of the sequences \( \{x_n\}, \{v_n\} \) are bounded, we wish to show that \( T \) has a fixed point. This will be done by showing that \( T \) maps a certain bounded, closed, convex set into itself. We use the notation \( D_r(p) = \{x: \| x - p \| \leq r\} \), \( r > 0 \), \( p \in E \). Let \( r > 0 \) be such that \( x_n \in D_r(0) \) and \( v_n \in D_r(0) \) for all \( n \). For each \( i = 1, 2, 3, \ldots \), define sets \( C_i \) and \( G_i \) by

\[ C_i = \bigcap_{n=1}^\infty D_{2r}(x_n), \quad G_i = \bigcap_{n=i}^\infty \{D_{2r}(x_n) \cap D_{2r}(v_n)\} \].

For each \( i \), we have

\[ D_r(0) \subset G_i \subset C_i \subset D_{2r}(x_i) \subset D_{2r}(0). \]

Each \( C_i \) and each \( G_i \) is a nonempty bounded, closed, convex set, and it is clear that \( C_i \subset C_{i+1} \) and \( G_i \subset G_{i+1} \). We now show \( T(G_i) \subset C_{i+1} \): \( x \in G_i \) implies \( \| x - v_n \| \leq 2r \) for all \( n \geq i \), which gives \( \| Tx - T v_n \| \leq 2r \) for all \( n \geq i \), which gives \( T(G_i) \subset C_{i+1} \).
\[ \| x - v_n \| \leq 2r \text{ for all } n \geq i; \text{ but, since } x_{n+1} = Tx_n, \text{ this can be written} \]
\[ \| Tx - x_{n+1} \| \leq 2r \text{ for all } n \geq i, \text{ so that } Tx \in C_{i+1}. \]
Define sets \( C \) and \( G \) by
\[ C = \bigcup_{i=1}^{\infty} C_i, \quad G = \bigcup_{i=1}^{\infty} G_i. \]
Clearly, \( D_r(0) \subset G \subset C \subset D_{3r}(0); \) and \( \bar{G}, \bar{C} \) are bounded, closed, convex sets. Since \( T(G_i) \subset C_{i+1} \) for each \( i, \) we have \( T(G) \subset C. \) Since \( T \) is continuous, \( T(\bar{G}) \subset \bar{T(G)} \subset \bar{C}. \) The proof will be completed by showing \( C \subset \bar{G}, \) so that \( T(\bar{G}) \subset \bar{C} \subset \bar{G} \) (i.e., \( T \) maps the bounded, closed, convex set \( \bar{G} \) into itself). Since \( C = \bigcup_{i=1}^{\infty} C_i, \) it suffices to show that for each \( i, C_i \subset \bar{G}. \) Suppose \( i \) is a given positive integer, and \( x \in C_i. \)
We wish to show that \( x \in \bar{G}. \) The first step toward this end is set off as the following lemma.

**Lemma 2.** For each \( \varepsilon > 0 \) there exists a positive integer \( j, \geq i \) such that \( x \in \bigcap_{n=j}^{\infty} \{D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n)\} = F_{j,\varepsilon}. \)

**Proof of Lemma 2.** Since \( x \in C_i \) we have \( \| x - x_p \| \leq 2r \) for all \( p \geq i. \) For all \( n \geq i \) we have
\[ \| x - v_n \| = \left\| \sum_{p=1}^{n} a_{np}x - \sum_{p=1}^{n} a_{np}x_p \right\| = \left\| \sum_{p=1}^{n} a_{np}(x - x_p) \right\| \]
so that
\[ \| x - v_n \| \leq \sum_{p=1}^{n} a_{np}\| x - x_p \| = \sum_{p=1}^{n} a_{np}\| x - x_p \| + \sum_{p=1}^{n} a_{np}\| x - x_p \|, \]
whence, for all \( n \geq i, \)
\[ \| x - v_n \| \leq \left( \sum_{p=1}^{n} a_{np} \right) \cdot \max_{i \leq p \leq i-1} \| x - x_p \| + 2r. \]
Since \( i \) and \( x \) are fixed, and since \( \lim_{n} a_{np} = 0 \) for each \( p = 1, 2, \ldots, i-1, \) it is clear that for any \( \varepsilon > 0 \) there exists a positive integer \( j, \geq i \) such that \( \| x - v_n \| \leq 2r + \varepsilon \) for all \( n \geq j. \) But \( n \geq j, \geq i \) also implies \( \| x - x_n \| \leq 2r \) since \( x \in C_i. \) Hence \( n \geq j, \) implies
\[ x \in D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n), \]
and so \( x \in \bigcap_{n=j}^{\infty} \{D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n)\} = F_{j,\varepsilon}. \)

**Proof of Theorem 1 continued.** We return now to the final problem of showing \( x \in \bar{G} \) (see immediately before Lemma 2). Given any \( c > 0, \) choose \( \varepsilon > 0 \) such that \( p, \leq c \) (this can be done by Lemma 1, in which \( r > 0 \) is taken as the \( r \) we are using in this proof). For
this $\varepsilon$, there exists a positive integer $j_i \geq i$ such that $x \in F_{j_i}$ (by Lemma 2). We will show $G_{j_i} \cap D_i(x) \neq \emptyset$. Since $c$ is arbitrary, this will show $x \in \bar{G} = \bigcup_{i=1}^{\infty} G_i$. We suppose $G_{j_i} \cap D_i(x) = \emptyset$ and obtain a contradiction. Since $0 \in D_c(0)$ and $D_c(x)$ is closed, $0 \in \bigcap_{i=1}^{\infty} D_c(x)$. Let $t_1 = 1 - (c/||x||)$. Then $0 < t_1 < 1$ and $||t_1x - x|| = (1 - t_1)||x|| = c$.

Since $t_1x \in D_i(x)$, we have $t_1x \in G_{j_i}$. Now $x \in F_{j_i} \subset \bigcap_{i=1}^{\infty} D_c(x) = C_i$, and since $C_j$ also contains $0$ and is convex, $t_1x \in C_{j_2}$. Since $t_1x \in G_{j_i}$ and $t_1x \in C_{j_2}$, we have $t_1x \in \bigcap_{i=1}^{\infty} D_c(x)$. Let $n$ be a positive integer, $n \geq j_i$, such that $t_1x \in D_{2r}(v_n)$. Let

$$t_2 = \sup \{t: 0 < t < 1 \text{ and } tx \in D_{2r}(v_n)\}.$$ 

This set of $t$'s is nonempty since $D_c(0) \subset D_{2r}(v_n)$. Since $D_{2r}(v_n)$ is closed, we have $t_2x \in D_{2r}(v_n)$; and it is easily seen from the definition of $t_2$ that we must have $||tx - v_n|| = 2r$. If $t_2 \geq t_1$, then, since $0$ and $t_2x$ are in the convex set $D_{2r}(v_n)$, we would have $tx \in D_{2r}(v_n)$ which is not true. Hence $t_2 < t_1$. Similarly we have $||x - v_n|| > 2r$, since $0$ is in the convex set $D_{2r}(v_n)$ and $tx$ is not. Since $x \in F_{j_i}$ and since $n \geq j_i$, $x \in D_{2r+\varepsilon}(v_n)$, so we have $2r < ||x - v_n|| \leq 2r + \varepsilon$. Next we observe that if $t \in (0, 1)$

$$||(1 - t)(t_2x - v_n) + tv|| = ||[(1 - t)t_2 + t\cdot 1]x - v_n|| > 2r$$

since $t_2 < (1 - t)t_2 + t\cdot 1 < 1$ so that $[(1 - t)t_2 + t\cdot 1]x \in D_{2r}(v_n)$. With $u = t_2x - v_n$ and $v = x - v_n$ we now have $||u|| = 2r, 2r < ||v|| \leq 2r + \varepsilon$, and $||(1 - t)u + tv|| > 2r$ for all $t \in (0, 1)$. Hence $||u - v|| = ||t_2x - x|| \leq p$, (see Lemma 1). But $\varepsilon$ was chosen so that $p_i \leq \varepsilon$. So we have

$$||t_2x - x|| = (1 - t_2)||x|| \leq p, \leq c .$$

This gives $t_2 \geq 1 - (c/||x||) = t_1$, which is a contradiction.

For completeness, we include the following theorem which is somewhat stronger than the converse of Theorem 1.

**Theorem 2.** If $E$ is a normed linear space, and if $T: E \rightarrow E$ is a nonexpansive mapping, and if $T$ has a fixed point $p \in E$, then for any $x_i \in E$ and any process $M(x_i, A, T)$, the sequences $\{x_n\}, \{v_n\}$ are bounded.

**Proof.** For each $n = 1, 2, 3, \ldots$, we have

$$||x_{n+1} - p|| = ||Tv_n - Tp|| \leq ||v_n - p|| = \left| \sum_{j=1}^{n} a_{nj}(x_j - p) \right| \leq \sum_{j=1}^{n} a_{nj} ||x_j - p|| \leq \max_{j=1, \ldots, n} ||x_j - p|| .$$
Thus \( \| x_2 - p \| \leq \| x_1 - p \| \), \( \| x_3 - p \| \leq \max_{j=1,2} \| x_j - p \| = \| x_1 - p \| \), etc., so that we have \( \| x_j - p \| \leq \| x_1 - p \| \) for all \( j = 1, 2, 3, \ldots \); and hence with \( b = \| x_1 - p \| + \| p \| \) we get \( \| x_j \| = \| (x_j - p) + p \| \leq \| x_j - p \| + \| p \| \leq b \) for all \( j = 1, 2, 3, \ldots \). Finally,

\[
\| v_n \| = \left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \sum_{j=1}^{n} a_j \| x_j \| \leq b \cdot \sum_{j=1}^{n} a_j = b
\]

for all \( n = 1, 2, 3, \ldots \).

**Bibliography**


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