A REMARK ON INTEGRAL FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Let $R_\nu$, $\nu = I, II, III, IV$, be the 4 types of the classical Cartan domains and let $\mathcal{C}(R_\nu)$ denote the class of solutions $u$ of the Laplace's equation $\Delta u = 0$ corresponding to the Bergman metric of $R_\nu$ which satisfy certain regularity conditions specified below.

In this note we give a distortion theorem for functions which are holomorphic in $R_\nu$ and omit the value 0 there, and an application which leads to an interesting property of integral functions omitting the value 0. The tools used here are the generalized Harnack inequality for functions in the class $\mathcal{C}(R_\nu)$ and the classical theorem of Liouville for integral functions.

Let $D$ be a bounded domain in the space $C^p$ of $p$ complex variables $z = (z', \cdots, z^p)$. The Laplace-Beltrami operator corresponding to the Bergman metric of $D$ is

$$\Delta_D = T^{a\overline{\beta}} \partial a \partial \overline{b} \partial \overline{c} \partial \overline{d};$$

here $T^{a\overline{b}}$ are the contravariant components of the metric tensor $T_{a\overline{b}} = \partial a \log K_D/\partial z^a \partial \overline{z}^b$ and $K_D = K_D(z, \overline{z})$ is the Bergman kernel function of $D[1]$. Let $\mathcal{E}(D)$ be the class of real functions $u$ satisfying:

(a) $u$ is continuous in $D$. (b) In $\overline{D} - b(D)$, $u$ is of $C^2$ and satisfies $\Delta_D u = 0$, where $b(D)$ is the Bergman-Šilov boundary of $D$. It is well-known that the class $\mathcal{E}(D)$ solves the Dirichlet problems for certain types of bounded symmetric domains $D$ ([3], [4]). These are the classical Cartan domains. Let $z$ be a matrix of complex entries, $z'$ its transpose, $z^*$ its conjugate transpose and $I$ the identity matrix. By $H > 0$ we mean that a hermitian matrix $H$ is positive definite.

The first 3 types are defined by $R_\nu = [z : I - zz^* > 0], \nu = I, II, III,$ where $z$ is an $m \times n$ matrix ($m \leq n$) for $R_I$, an $n \times n$ symmetric matrix for $R_{II}$ and an $n \times n$ skew symmetric matrix for $R_{III}$. The fourth type $R_{IV}$ is the set of all $1 \times n$ matrices satisfying the conditions:

$$1 + |zz'|^2 - 2zz^* > 0, |zz'| < 1,$$

or

$$1 > \overline{zz'} + |(\overline{zz'})^2 - |zz'|^2|^{1/2}. $$

By $\|z\|_e$ we denote the norm of the matrix $z \in R_\nu$, i.e., $\|z\|_e = \sup_{|x| = 1} |xz|$, where $x$ is an $n$-dimensional vector and $|x|$ the length.
of $x$. It can be shown that $\|z\|_v$ is the largest among the positive square roots of the characteristic roots of the hermitian matrix $zz^*$, and $R_v = [z: \|z\|_v < 1]$ ([2]). For any $r > 0$ we write

$$R_v(r) = [z: r^2 I - zz^* > 0] = [z: \|z\|_v < r].$$

2. Distortion theorems. A generalization of Harnack’s inequality to functions of the class $\mathcal{E}$ for the classical Cartan domains has been obtained in [6] and it is contained in the following lemma.

**Lemma 1.** If $u \in \mathcal{E}(R_v(r))$ is nonnegative on $b(R_v(r))$ then on $R_v(r)$

$$u(0)Q_v(r, z) \leq u(z) \leq u(0)Q_v(r, z)^{-1}, Q_v(r, z) = \prod_{k=1}^{n_v} \left( \frac{r - \lambda_k}{r + \lambda_k} \right)^{N_v},$$

where

$$n_1 = m, \quad n_{II} = n, \quad n_{III} = [n/2], \quad n_{IV} = 2;$$

$$N_1 = n, \quad N_{II} = (n + 1)/2, \quad N_{III} = n - 1$$

if $n$ is even and $= n$ if $n$ is odd, $N_{IV} = n/2$; $\lambda_1, \lambda_2, \ldots, \lambda_{n_v}$ are the nonnegative square roots of the characteristic roots of the hermitian matrix $zz^*$ for $z \in R_v(r)$, and $r > \lambda_1 \geq \cdots \geq \lambda_{n_v} \geq 0$.

We remark that $n_v$ is the rank of the domain $R_v$, and $p_v = n_vN_v$ gives the (complex) dimension of $R_v$.

A simple application of the above lemma leads to the following distortion theorem for holomorphic functions.

**Theorem 1.** Let $f(z)$ be a holomorphic function in $\overline{R_v(r)}$ which omits there the value 0. Then on $R_v(r)$

$$|f(0)||Q_v(r, z)m_v(r, f)^{-1-Q_v(r, z)} \leq |f(z)| \leq |f(0)||Q_v(r, z)M_v(r, f)^{-1-Q_v(r, z)}$$

where $m_v(r, f) = \min_{|z|=r} |f(z)|$, $M_v(r, f) = \max_{|z|=r} |f(z)|$ and $Q_v(r, z)$ is given in Lemma 1.

**Proof.** Since $f(z)$ is holomorphic and omits the value 0 in $\overline{R_v(r)}$ the maximum principle of a holomorphic function yields:

$$m_v(r, f) \leq |f(z)| \leq M_v(r, f), \quad z \in \overline{R_v(r)}.$$

Let $g_1(z) = f(z)/m_v(r, f)$ and $g_2(z) = M_v(r, f)/f(z)$. Since $m_v(r, f) \neq 0$ $g_1(z)$ is holomorphic in $\overline{R_v(r)}$ and $|g_2(z)| \equiv 1$ in $\overline{R_v(r)}$. Therefore, $u_1(z) = \log |g_1(z)|$ belongs to $\mathcal{E}(R_v(r))$ and satisfies all the hypotheses of Lemma 1. Applying the first inequality of (2) to $u_1(z)$ and the second inequality to $u_2(z)$ we have inequalities (3).
Specializing Theorem 1 to the hypersphere $H(r) = \{ z : |z| < r \}$, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, which can be obtained from $R_1(r)$ by taking $m = 1$, we obtain

**COROLLARY 1.** Let $f(z)$ be a function which is holomorphic in $H(r)$ and continuous in $\overline{H(r)}$. If $f(z)$ omits the value 0 on $H(r)$ then on $H(r)$.

\[ (4) \quad |f(0)|^{Q(r,z)} m(r,f)^{1-Q(r,z)} \leq |f(z)| \leq |f(0)|^{Q(r,z)} M(r,f)^{1-Q(r,z)}, \]

where $m(r,f) = \min_{|z|=r} |f(z)|$, $M(r,f) = \max_{|z|=r} |f(z)|$ and $Q(r,z) = (r - |z|)^n/(r + |z|)^n$.

A slight modification of the above theorem is the following.

**THEOREM 2.** Let $f(z)$ be a holomorphic function in $R_1(r)$ which omits there the value 0. Then for any $\delta > 0$

\[ (5) \quad \|[f(0) | m_r(r,f)^q]^{1/(1+\delta)} \leq |f(z)| \leq \|[f(0) | M_r(r,f)^q]^{1/(1+\delta)} \]

holds for all $z \in R_1(r_\delta)$, where

\[ r_\delta = \frac{t_\delta - 1}{t_\delta + 1} r, \quad t_\delta = (1 + \delta)^{\nu^{-1}}. \]

**Proof.** For any $\delta > 0$ $f(z)$ is holomorphic in $\overline{R_1(r_\delta)}$ and omits the value 0. By Theorem 1,

\[ (6) \quad |f(0) | m_r(r,f)^q M_r(r,f)^{1-q} \leq |f(z)| \leq |f(0) | m_r(r,f)^q M_r(r,f)^{1-q} \]

for $z \in R_1(r_\delta)$. Let $\delta_0 > 0$ be fixed arbitrarily. Since $r_\delta(\delta) \to r$ as $\delta \to \infty$, we have

\[ (7) \quad |f(0) | m_r(r,f)^q M_r(r,f)^{1-q} \leq |f(z)| \leq |f(0) | m_r(r,f)^q M_r(r,f)^{1-q} \]

for $z \in R_1(r_\delta)$, $r_\delta = (t_\delta - 1)r/(t_\delta + 1), t_\delta = (1 + \delta_0)^{\nu^{-1}}$. On the other hand, if $z \in R_1(r_\delta)$ then $||z||_\nu < r_\delta$ or $\{ (r - ||z||_\nu)/(r + ||z||_\nu) \}^{\nu} > 1/(1 + \delta_0)$. Since $||z||_\nu \geq c_k, k = 1, \ldots, n, Q_z(r,z) > 1/(1 + \delta_0)$. Combining this with (7) and the inequalities: $m_r(r,f) \leq |f(z)| \leq M_r(r,f)$, we obtain the theorem.

3. Main theorem. The following lemma is a simple application of Theorem 2.

**LEMA 2.** Let $\{ f_k \}$ be a sequence of holomorphic functions in $R_1(r)$ such that $f_k$ omits the value 0 there. Suppose that for some $\delta > 0$ there exists a $A > 0$ such that
Then for \( z \) \( \nu^2 (t_\nu - 1)r/(t_\nu + 1), \) \( t_\nu = (1 + \delta)^{\nu^{-1}}, \)

\[ |f_k(z)| \leq A^{(1+\delta)^{-1}}, \quad k = 1, 2, \ldots. \]

We observe that the hypothesis that each \( f_k \) omits the value 0 is essential for the validity of Lemma 2, as is shown by the following example in \( C^2. \) Let

\[ f_k(z^1, z^2) = k(z^1 + z^2 + 1/k^2), \quad k = 1, 2, \ldots \]

be a sequence of holomorphic functions in the unit hypersphere \( H. \) A formal computation shows that

\[ M(1, f_k) = ((3k + 1/k)(k + 1/k))^{1/2}, \quad f_k(0) = 1/k. \]

For \( \delta = 1 \) we find \( A = 8^{1/2}. \) But no \((z^1, z^2)\) with \( |z^1|^2 + |z^2|^2 < 1\) satisfies (9).

Using Lemma 2 and the classical theorem of Liouville on integral functions we prove:

**Theorem 3.** Let \( f \) be an integral function in the space \( C^\nu \) omitting the value 0, where \( \nu = mn, n(n+1)/2, n(n-1)/2, n \) if \( \nu = I, II, III, IV, \) respectively. If there exists a \( \delta > 0 \) and a monotonically increasing \( \{s_k\} \) of positive numbers without bound such that for

\[ \tau > 2(1 + \delta)^{\nu^{-1}}/((1 + \delta)^{\nu^{-1}} - 1) \]

\[ \lim_{s_k \to \infty} m(s_k, f)M(s_k, f)^\delta < \infty, \]

then \( f \) is constant.

**Proof.** Let \( z_k \) be a point on \( ||z||_\nu = s_k \) such that

\[ \zeta = \zeta_k(z) = (z - z_k)/(\tau - 1)s_k, \quad k = 1, 2, \ldots \]

Then (12) defines a biholomorphic mapping of \( C^\nu \) for each \( \delta > 0. \) Hence, \( g_k(\zeta) = f(\zeta_e^{-1}(\zeta)) \) is again an integral function in \( C^\nu \) which omits the value 0. Further, the set \([z: ||z - z_k||_\nu < s_k(\tau - 1)]\) is contained in \( R(\tau s_k), \) and hence,

\[ M(1, g_k) \leq M(s_k, f), \quad k = 1, 2, \ldots, \]

Since \( |g_k(0)| = |f(z_k)| = m(s_k, f), \) from (11) we have

\[ \lim_{k \to \infty} |g_k(0)| M(1, g_k)^\delta < \infty. \]

Hence there exists a number \( A > 0 \) such that
By Lemma 2,

\[ |g_k(\zeta)| \leq A^{(1+\delta)^{-1}}, \quad k = 1, 2, \ldots \]

for all

\[ \zeta \in R_{\nu}\left(\frac{t_{\nu} - 1}{t_{\nu} + 1}\right), \quad t_{\nu} = (1 + \delta)^{\nu^{-1}}. \]

This together with (12) implies that \( f(z) \) is bounded by \( A^{(1+\delta)^{-1}} \) in \( R_{\nu}(s_{\nu}\sigma_{\nu}) \) for each \( k \), where \( \sigma_{\nu}(\delta) = (t_{\nu} - 1)\left(\tau - 1\right)/(t_{\nu} + 1) - 1 \). Since \( \sigma_{\nu}(\delta) > 0 \) for \( \tau > 2t_{\nu}/(t_{\nu} - 1) \), \( \{R_{\nu}(s_{\nu}\sigma_{\nu})\} \) covers the entire space \( C^{p_{\nu}} \).

The theorem now follows from the theorem of Liouville.

REFERENCES


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