

# Pacific Journal of Mathematics

**FULL CO-ORDINALS OF RETs**

ALFRED BERRY MANASTER

## FULL CO-ORDINALS OF RETS

ALFRED B. MANASTER

**Recursive analogues of cardinal and ordinal numbers have been developed by considering only subsets of the natural numbers and considering only one-to-one partial recursive functions as the maps or correspondences between sets. The recursive analogue of a cardinal is called a recursive equivalence type (RET) and that of an ordinal is called a co-ordinal. Using the RETs and the co-ordinals analogues of Cantor's number classes are defined and considered in this paper. The degree of indecomposability of an RET is seen to determine the set of classical ordinals represented in the RET's co-ordinal number class. If the RET is infinite this set of ordinals is always an initial segment (not necessarily proper) of Cantor's second number class.**

The basic reference for RETs is Dekker-Myhill [3]. The basic reference for co-ordinals is Crossley [1]. If  $\xi$  and  $\eta$  are subsets of  $E$  ( $E = \{0, 1, 2, \dots\}$ ),  $\eta$  is called recursively equivalent to  $\xi$  if there exists a one-to-one partial recursive function  $f$  whose domain includes  $\xi$  such that the  $f$ -image of  $\xi$  is  $\eta$ . The class of all sets recursively equivalent to  $\xi$  is called the recursive equivalence type (RET) of  $\xi$  and will be denoted by  $\langle \xi \rangle$ . If  $\langle_{\xi}$  is a well-ordering of  $\xi$  and  $\langle_{\eta}$  is a well-ordering of  $\eta$ , then  $(\xi, \langle_{\xi})$  is called recursively isotonic to  $(\eta, \langle_{\eta})$  if there exists a one-to-one partial recursive function  $f$  whose domain includes  $\xi$  and such that  $f$  is an order isomorphism of  $(\xi, \langle_{\xi})$  onto  $(\eta, \langle_{\eta})$ . The class of all orderings  $(\eta, \langle_{\eta})$  recursively isotonic to  $(\xi, \langle_{\xi})$  is called the co-ordinal of  $(\xi, \langle_{\xi})$  and will be denoted by  $\langle \xi, \langle_{\xi} \rangle$ . If  $Y = \langle \xi, \langle_{\xi} \rangle$  is a co-ordinal, the classical order type of  $(\xi, \langle_{\xi})$  is a countable ordinal which will be referred to as the order type of  $Y$  and denoted  $|Y|$  (Cf. Definition IV. 2.1 of Crossley [1]).

Addition is defined for RETs and co-ordinals in the following manner. The subsets  $\xi_1$  and  $\xi_2$  of  $E$  are called RE separable if there exists a pair of disjoint recursively enumerable (RE) sets,  $\theta_1$  and  $\theta_2$ , such that  $\xi_1 \subseteq \theta_1$  and  $\xi_2 \subseteq \theta_2$ . Assume that  $\xi_1$  and  $\xi_2$  are RE separable,  $\langle_i$  is a well-ordering of  $\xi_i$  for  $i = 1, 2$ , and  $X_i = \langle \xi_i, \langle_i \rangle$ ,  $Y_i = \langle \xi_i, \langle_i \rangle$  for  $i = 1, 2$ . Then  $X_1 + X_2 = \langle \xi_1 \cup \xi_2 \rangle$  and

$$Y_1 + Y_2 = \langle \xi_1 \cup \xi_2, \langle_1 \cup \langle_2 \cup (\xi_1 \times \xi_2) \rangle \rangle.$$

Using the definitions of addition, partial orderings  $\leq$  have been defined in both the RETs and the co-ordinals. For RETs  $X$  and  $X_1$  define  $X_1 \leq X$  if and only if there is a RET  $X_2$  such that  $X_1 + X_2 = X$ . Analogously, for co-ordinals  $Y_1$  and  $Y$  define  $Y_1 \leq Y$  if and only

if there is a co-ordinal  $Y_2$  such that  $Y_1 + Y_2 = Y$ .

Let  $Y_1 = \langle \xi_1, \langle_1 \rangle \rangle$  and  $Y_2 = \langle \xi_2, \langle_2 \rangle \rangle$  be co-ordinals.  $Y_2$  is called an initial segment of  $Y_1$  if  $(\xi_2, \langle_2)$  is recursively isotonic to an initial segment of  $(\xi_1, \langle_1)$ . Definition X. 4.3 of Crossley [2] may now be rephrased as follows. A co-ordinal  $Y$  is *full* if every initial segment of  $Y$  is a predecessor of  $Y$  in the sense of  $\leq$ . Note that if  $(\xi_2, \langle_2)$  is an initial segment of  $(\xi_1, \langle_1)$ , then  $\langle \xi_2, \langle_2 \rangle \rangle \leq \langle \xi_1, \langle_1 \rangle \rangle$  if and only if  $\xi_2$  and  $\xi_1 - \xi_2$  are RE separable. Thus  $\langle \xi_1, \langle_1 \rangle \rangle$  is full if and only if every initial segment of  $(\xi_1, \langle_1)$  is RE separable from its complement in  $\xi_1$ . Example IV. 5.1 of Crossley [1] shows the existence of co-ordinals which are not full. The existence of many full co-ordinals is proved in IV. 5.4 of Crossley [1].

There is a natural sense in which the field of a co-ordinal is an RET. To see this consider  $\langle \xi, \langle_\xi \rangle \rangle$ ; if  $(\xi, \langle_\xi)$  is recursively isotonic to  $(\eta, \langle_\eta)$  then  $\xi$  is recursively equivalent to  $\eta$  and  $\langle \xi \rangle = \langle \eta \rangle$ , conversely if  $\xi$  is recursively equivalent to  $\eta$  then there is an ordering  $\langle_\eta$  of  $\eta$  such that  $\langle \eta, \langle_\eta \rangle \rangle = \langle \xi, \langle_\xi \rangle \rangle$ . This observation justifies the following definition.

DEFINITION. The field of the co-ordinal  $Y = \langle \xi, \langle_\xi \rangle \rangle$  is the RET  $X = \langle \xi \rangle$ .

In this paper we consider the question of determining the order types of full co-ordinals with a given field  $X$ . For each RET  $X$  let  $\mathcal{F}(X)$  be the set of full co-ordinals  $Y$  whose field is  $X$ . Let

$$\|\mathcal{F}(X)\| = \{ |Y| : Y \in \mathcal{F}(X) \}$$

be the set of order types of full co-ordinals with field  $X$ .

It will be shown that for each infinite RET  $X$  either  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$  where  $\omega_1$  is the first uncountable ordinal or there is a countable positive ordinal  $\alpha$  and a finite  $n > 0$  such that  $\|\mathcal{F}(X)\| = [\omega, \omega^\alpha(n+1))$ . ( $[\alpha, \beta) = \{ \gamma : \alpha \leq \gamma < \beta \}$ .  $c$  is the cardinality of the continuum.) For each positive ordinal  $\alpha$  and each finite  $n > 0$  there exist  $c$  RETs  $X$  such that  $\|\mathcal{F}(X)\| = [\omega, \omega^\alpha(n+1))$ . It will also be shown that if the RET  $X$  is not an isol (See Chapter IV of Dekker-Myhill [3]) then  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$ .

A hierarchy structure of the RETs similar to that in Manaster [4] will be useful in demonstrating the results stated above. Note that although the terms and the symbols are similar to those in Definition 0.1 of [4], the definition is slightly different.

DEFINITION.  $I_0 = \{X : X \text{ is finite}\}$ . For each positive countable ordinal  $\alpha$  define

$$P_\alpha = \left\{ X: X = Y + Z \Rightarrow Y \in \bigcup_{\beta < \alpha} I_\beta \vee Z \in \bigcup_{\beta < \alpha} I_\beta \right\}$$

and

$$I_\alpha = \{ X: \text{there is an } n \text{ and RETs } X_1, \dots, X_n \text{ such that each } X_i \in P_\alpha \text{ and } X = X_1 + \dots + X_n \}.$$

Elements of  $P_\alpha$  are called  $\alpha$ -order indecomposable. Elements of  $\bigcup_{\beta < \alpha} I_\beta$  are called  $\alpha$ -small. Elements of  $P_\alpha$  which are not  $\alpha$ -small are called strictly  $\alpha$ -order indecomposable.

In spite of the difference between this definition and definition 0.1 of [4], the two notions of  $\alpha$ -order indecomposability are similar enough that most of the results of [4] are also correct for this definition of  $\alpha$ -order indecomposability. If Definition 0.1 of [4] is modified by defining  $S_\alpha = \{ X: X = 0 \}$ , then the two definitions of  $P_\alpha$  and  $I_\alpha$  are the same. Replacing some occurrences of  $P_\alpha \cap S_\alpha (P_\alpha - S_\alpha)$  with occurrences of  $\bigcup_{\beta < \alpha} I_\beta (P_\alpha - \bigcup_{\beta < \alpha} I_\beta)$  respectively, all results of § 1 of [4] remain valid except Lemma 1.1 and Theorem 1.4. In particular  $P_\alpha$  is closed under predecessor (Lemma 1.5) so that  $I_\alpha$  is the ideal generated by  $P_\alpha$ . Moreover the arguments used in Construction I of § 2 and the first part of § 3 (through Theorem 3.2) are still valid under the present interpretation and show the existence of  $c$  strictly  $\alpha$ -order indecomposable isols for each countable ordinal  $\alpha$ .

The main result of this paper is the following theorem.

**THEOREM.** *Let  $\alpha$  be a positive countable ordinal. If  $X$  is a sum of  $n$  strictly  $\alpha$ -order indecomposables,  $\| \mathcal{F}(X) \| = [\omega, \omega^\alpha(n + 1)]$ . If  $X \notin \bigcup_{\alpha < \omega_1} I_\alpha$ , then  $\| \mathcal{F}(X) \| = [\omega, \omega_1]$ .*

**LEMMA 1.** *Let  $0 < \alpha < \omega_1$ . If  $X$  is a sum of  $n$   $\alpha$ -order indecomposables and  $Y \in \mathcal{F}(X)$ , then  $|Y| < \omega^\alpha(n + 1)$ .*

*Proof.* The proof is an induction on  $\alpha$ . Assume inductively that for  $\beta < \alpha$  if  $Z$  is a sum of  $m$   $\beta$ -order indecomposables and  $T \in \mathcal{F}(Z)$  then  $|T| < \omega^\beta(m + 1)$ . (Note that the remainder of the proof applies for all  $\alpha \geq 1$ .) In particular, if  $Z$  is  $\alpha$ -small and  $T \in \mathcal{F}(Z)$  then  $|T| < \omega^\alpha$ . Suppose  $X = X_1 + \dots + X_n$  where  $n \geq 1$  and each  $X_i \in P_\alpha$ . Suppose  $Y \in \mathcal{F}(X)$  and  $|Y| \geq \omega^\alpha(n + 1)$ . Since  $Y$  is full,  $Y = Y_1 + \dots + Y_{n+1}$  where  $|Y_i| = \omega^\alpha$  for  $1 \leq i \leq n$  and  $|Y_{n+1}| \geq \omega^\alpha$ . Each  $Y_i$  is full since initial segments and tails of full co-ordinals are full.  $X = Z_1 + \dots + Z_{n+1}$  where each  $Z_i$  is the field of  $Y_i$ . By the inductive hypotheses each  $Z_i$  is not  $\alpha$ -small since  $Y_i \in \mathcal{F}(Z_i)$  and  $|Y_i| \geq \omega^\alpha$ . By the refinement property (Theorem 15 (l) of Dekker-Myhill [3]) there exist RETs  $X_{i,j}$  satisfying the following system of equations.

$$\begin{array}{r}
 X_{1,1} + \cdots + X_{1,n+1} = X_1 \\
 + \qquad \qquad \qquad + \\
 X_{2,1} + \cdots + X_{2,n+1} = X_2 \\
 + \qquad \qquad \qquad + \\
 \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdot \\
 + \qquad \qquad \qquad + \\
 X_{n,1} + \cdots + X_{n,n+1} = X_n \\
 \parallel \quad \cdots \quad \parallel \\
 Z_1 \quad \cdots \quad Z_{n+1} .
 \end{array}$$

Since each  $Z_j$  is not  $\alpha$ -small, for each  $j$  there is at least one  $i$  such that  $X_{i,j}$  is not  $\alpha$ -small. Since there are  $n + 1$  columns but only  $n$  rows there must be a row, say row  $i$ , in which there are at least two terms,  $X_{i,j}$  and  $X_{i,k}$ , neither of which is  $\alpha$ -small. Thus  $X_i \notin P_\alpha$ . Contradiction.

Lemma 1 shows that if  $X$  is a sum of  $n$   $\alpha$ -order indecomposables, then  $\|\mathcal{F}(X)\| \subseteq [\omega, \omega^\alpha(n + 1))$ . The next sequence of lemmas lead to the converse inclusion.

**LEMMA 2.** *If  $X$  is not  $\alpha$ -small and  $\beta$  is any ordinal less than  $\alpha$ , then there exist  $X_1$  and  $X_2$  such that  $X = X_1 + X_2$ ,  $X_1$  is not  $\beta$ -small, and  $X_2$  is not  $\alpha$ -small.*

*Proof.* Suppose  $X$  is not  $\alpha$ -small. Since, in particular,  $X \notin P_\beta$ , there exist  $X_1$  and  $X_2$  such that  $X = X_1 + X_2$  and neither  $X_1$  nor  $X_2$  is  $\beta$ -small. Not both  $X_1$  and  $X_2$  can be  $\alpha$ -small.

**LEMMA 3.** *If  $X$  is not  $\alpha$ -small, then there is a  $Y \in \mathcal{F}(X)$  of order type  $\omega^\alpha$ .*

*Proof.* The proof is an induction on  $\alpha$ . The base step,  $\alpha = 1$ , is easy since every infinite RET is the field of full co-ordinals of type  $\omega$ . Let  $1 < \alpha < \omega_1$ . Let  $\{\beta_i\}_{i < \omega}$  be a sequence of ordinals such that  $1 \leq \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots < \alpha$  and such that for every  $\beta < \alpha$  there is an  $i$  such that  $\beta \leq \beta_i$ . (If  $\alpha = \gamma + 1$ , let each  $\beta_i = \gamma$ .)

Since  $X$  is not  $\alpha$ -small and  $\beta_0 < \alpha$ , by Lemma 2 there exist  $X_0$  and  $Z_0$  such that  $X = X_0 + Z_0$ ,  $X_0$  is not  $\beta_0$ -small, and  $Z_0$  is not  $\alpha$ -small. Inductively for each  $n$ ,  $X = X_0 + \cdots + X_n + Z_n$  where  $Z_n$  is not  $\alpha$ -small. Since  $\beta_{n+1} < \alpha$  by Lemma 2 there exist  $X_{n+1}$  and  $Z_{n+1}$  such that  $Z_n = X_{n+1} + Z_{n+1}$ ,  $X_{n+1}$  is not  $\beta_{n+1}$ -small and  $Z_{n+1}$  is not  $\alpha$ -small. Thus

$$X = X_0 + \cdots + X_n + X_{n+1} + Z_{n+1}$$

where each  $X_i$  is not  $\beta_i$ -small and  $Z_{n+1}$  is not  $\alpha$ -small.

Since  $\beta_n < \alpha$  the inductive hypothesis asserts the existence of a full co-ordinal  $Y_n$  of order type  $\omega^{\beta_n}$  with field  $X_n$ . Unfortunately the ordinal sum of the  $Y_n$  is not well defined and even if it were it would not, in general, be a co-ordinal with field  $X$ . However, it would be a full co-ordinal of type  $\omega^\alpha$ . To remove these difficulties, it seems necessary to work with a representative of  $X$ .

Let  $\langle \xi \rangle = X$ . For each  $n$  let  $\xi_n$  be a representative of  $X$  such that  $\bigcup_n \xi_n \subseteq \xi$  and  $\bigcup_{i=1}^n \xi_i$  is RE separable from  $\xi - \bigcup_{i=1}^n \xi_i$ . Define

$$\xi'_n = \xi_n \cup \left( \{n\} \cap \xi \cap \overline{\bigcup_{m < n} \xi_m} \right)$$

and  $X'_n = \langle \xi'_n \rangle$ . For any RET  $Z$  both  $Z$  and  $Z + 1$  have the same order of indecomposability, if any. Thus  $X'_n$  is not  $\beta_n$ -small. Now apply the inductive hypothesis to obtain full co-ordinals  $Y'_n$  of order type  $\omega^{\beta_n}$  with field  $X'_n$ . Let  $Y'_n = \langle \xi'_n, <_n \rangle$  and define

$$Y = \left\langle \bigcup_{n < \omega} \xi'_n, \bigcup_{n < \omega} <_n \cup \bigcup_{m < n} (\xi'_m \times \xi'_n) \right\rangle.$$

$Y$  is a full co-ordinal of type  $\sum \omega^{\beta_n} = \omega^\alpha$  with field  $X$ .

LEMMA 4. *If  $Y \in \mathcal{F}(X)$ ,  $|Y| = \alpha$ , and  $\omega \leq \beta < \alpha$ , then there is a  $Y' \in \mathcal{F}(X)$  such that  $|Y'| = \beta$ .*

*Proof.* First consider the case in which  $\alpha - \beta$  is finite.  $Y = Y_1 + Y_2$  where  $|Y_1| = \beta$  and  $|Y_2| = \alpha - \beta < \omega$ .  $Y' = Y_2 + Y_1$  is then a full co-ordinal of type  $(\alpha - \beta) + \beta = \beta$  with field  $X$ .

Next consider the case in which  $\alpha - \beta \geq \omega$  and  $\omega^2 \leq \beta$ .  $Y = Y_1 + Y_2$  where  $|Y_1| = \beta$  and  $|Y_2| = \alpha - \beta \geq \omega$ . Let  $Y'_2$  be a (necessarily full) ordering of type  $\omega$  of the field of  $Y_2$ .  $Y' = Y'_2 + Y_1$  satisfies the conclusion of the lemma.

Finally consider the case in which  $\alpha - \beta \geq \omega$  and  $\omega^2 > \beta$ . Observe that for every  $k < \omega$ , any co-ordinal of type  $\omega + k$  is full. Let  $\beta = \omega n + k$  where  $n$  and  $k$  are finite. If  $n = 1$  the field of  $Y$  has full co-ordinals of type  $\beta$  by the observation just made. Otherwise let  $n = m + 1$  where  $m \geq 1$ .  $Y = Y_1 + Y_2$  where  $|Y_1| = \omega m$  and

$$|Y_2| = \alpha - \omega m \geq \alpha - (\omega n + k) = \alpha - \beta \geq \omega.$$

Let  $Y'_2$  be a full co-ordinal of type  $\omega + k$  with the same field as the field of  $Y_2$ .  $Y = Y_1 + Y'_2$  is a full co-ordinal of type  $\omega m + \omega + k = \beta$  with field  $X$ .

LEMMA 5. *If  $X$  is not  $\alpha$ -small then  $[\omega, \omega^\alpha] \subseteq \|\mathcal{F}(X)\|$ .*

*Proof.* Apply Lemmas 3 and 4.

*Proof of theorem.* First suppose  $X$  is not  $\alpha$ -small for any countable ordinal  $\alpha$ .  $\omega^\alpha \in \|\mathcal{F}(X)\|$  for each countable  $\alpha$  by Lemma 3. Thus  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$  by Lemma 4.

To prove that if  $X$  is a sum of  $n$  strictly  $\alpha$ -order indecomposables than  $\|\mathcal{F}(X)\| = [\omega, \omega^\alpha(n + 1))$ , consider first the case  $n = 1$ . Since  $X$  is not  $\alpha$ -small there is, by Lemma 3, a full co-ordinal  $Y$  with field  $X$  of order type  $\omega^\alpha$ . Let  $\omega^\alpha \leq \beta < \omega^\alpha \cdot 2$ . For some  $\gamma < \omega^\alpha$ ,  $\beta = \omega^\alpha + \gamma$ . Let  $Y = Y_1 + Y_2$  where  $|Y_1| = \gamma$  and  $|Y_2| = \omega^\alpha - \gamma = \omega^\alpha$ .  $Y' = Y_2 + Y_1$  is a full co-ordinal with field  $X$  of type  $|Y_2| + |Y_1| = \omega^\alpha + \gamma = \beta$ . This result, Lemma 5, and Lemma 1 show  $\|\mathcal{F}(X)\| = [\omega, \omega^\alpha \cdot 2)$ .

Finally consider the case in which  $X = X_1 + \dots + X_n$ ,  $n \geq 2$ , and each  $X_i$  is strictly  $\alpha$ -order indecomposable. Let

$$\beta \in [\omega^\alpha \cdot n, \omega^\alpha(n + 1)) .$$

For some  $\gamma < \omega^\alpha$ ,  $\beta = \omega^\alpha \cdot n + \gamma$ . Let  $Y_i$  be a full co-ordinal with field  $X_i$  of order type  $\omega^\alpha$  for each  $i < n$ . Let  $Y_n$  be a full co-ordinal of order type  $\omega^\alpha + \gamma$  with field  $X_n$ .  $Y = Y_1 + \dots + Y_n$  is a full co-ordinal of type  $\beta$  with field  $X$ . This result, Lemma 4, and Lemma 1 show  $\|\mathcal{F}(X)\| = [\omega, \omega^\alpha(n + 1))$ .

It remains to show that if the RET  $X$  is not an isol then  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$  and show the existence of  $c$  isols such that  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$ . For the former result it suffices to show that if  $X$  is not an isol then  $X \notin \bigcup_{\alpha < \omega_1} I_\alpha$  and hence it suffices to show that  $X \notin \bigcup_{\alpha < \omega_1} P_\alpha$ . Let  $\alpha$  be the least ordinal  $\beta$  such that some non-isol  $X$  is  $\beta$ -order indecomposable.  $X = X_1 + X_2$  where neither  $X_1$  nor  $X_2$  is an isol but one of them, say  $X_1$ , is a finite sum of  $\beta$ -order indecomposables for some  $\beta < \alpha$ . Since  $\beta < \alpha$  every  $\beta$ -order indecomposable is an isol. Since every finite sum of isols is an isol,  $X_1$  must be an isol. This contradiction shows that every nonisol is not in  $\bigcup_{\alpha < \omega_1} I_\alpha$ .

Examples of isols  $X$  such that  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$  are provided by first-order highly decomposable isols in the sense of Manaster [4]. It will be shown, as in the preceding paragraph, that if  $X$  is first-order highly decomposable then  $X \notin \bigcup_{\alpha < \omega_1} P_\alpha$ . Let  $\alpha$  be the least ordinal  $\beta$  such that some first-order highly decomposable  $X$  is  $\beta$ -order indecomposable.  $X = X_1 + X_2$  where both  $X_1$  and  $X_2$  are infinite but one, say  $X_1$ , is a finite sum of  $\beta$ -order indecomposables for some  $\beta < \alpha$ . Since  $X_1$  is infinite, there is an infinite  $X_3 \leq X_1$  such that  $X_3$  is  $\beta$ -order indecomposable. Since  $X_3 \leq X_1 \leq X$ ,  $X_3$  is highly decomposable contradicting the minimality of  $\alpha$ . The existence of  $c$  first-order highly decomposable isols is shown in Dekker-Myhill [3, pp. 112-113]

and Manaster [4]. Thus there are  $c$  isols  $X$  such that  $\|\mathcal{F}(X)\| = [\omega, \omega_1)$ .

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Leonard Asimow, <i>Universally well-capped cones</i> .....	421
Lawrence Peter Belluce, William A. Kirk and Eugene Francis Steiner, <i>Normal structure in Banach spaces</i> .....	433
William Jay Davis, <i>Bases in Hilbert space</i> .....	441
Larry Lee Dornhoff, <i>p-automorphic p-groups and homogeneous algebras</i> .....	447
William Grady Dotson, Jr. and W. R. Mann, <i>A generalized corollary of the Browder-Kirk fixed point theorem</i> .....	455
John Brady Garnett, <i>On a theorem of Mergelyan</i> .....	461
Matthew Gould, <i>Multiplicity type and subalgebra structure in universal algebras</i> .....	469
Marvin D. Green, <i>A locally convex topology on a preordered space</i> .....	487
Pierre A. Grillet and Mario Petrich, <i>Ideal extensions of semigroups</i> .....	493
Kyong Taik Hahn, <i>A remark on integral functions of several complex variables</i> .....	509
Choo Whan Kim, <i>Uniform approximation of doubly stochastic operators</i> .....	515
Charles Alan McCarthy and L. Tzafriri, <i>Projections in <math>\mathcal{L}_1</math> and <math>\mathcal{L}_\infty</math>-spaces</i> .....	529
Alfred Berry Manaster, <i>Full co-ordinals of RETs</i> .....	547
Donald Steven Passman, <i>p-solvable doubly transitive permutation groups</i> .....	555
Neal Jules Rothman, <i>An <math>L^1</math> algebra for linearly quasi-ordered compact semigroups</i> .....	579
James DeWitt Stein, <i>Homomorphisms of semi-simple algebras</i> .....	589
Jacques Tits and Lucien Waelbroeck, <i>The integration of a Lie algebra representation</i> .....	595
David Vere-Jones, <i>Ergodic properties of nonnegative matrices. II</i> .....	601
Donald Rayl Wilken, <i>The support of representing measures for <math>R(X)</math></i> .....	621
Abraham Zaks, <i>Simple modules and hereditary rings</i> .....	627