THE INTEGRATION OF A LIE ALGEBRA REPRESENTATION

JACQUES TITS AND LUCIEN WAELEBROECK
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Let \( u: G \rightarrow A \) be a differentiable representation of a Lie group into a \( b \)-algebra. The differential \( u_0 = du_e \) of \( u \) at the neutral element \( e \) of \( G \) is a representation of the Lie algebra \( \mathfrak{g} \) of \( G \) into \( A \). Because a Lie group is locally the union of one-parameter subgroups and since the infinitesimal generator of a differentiable (multiplicative) sub-semi-group of \( A \) determines this sub-semi-group, the representation \( u_0 \) determines \( u \) if \( G \) is connected.

We shall be concerned with the converse: given a representation \( u_0 \) of \( \mathfrak{g} \), when can it be obtained by differentiating a representation \( u \) of \( G \)? We shall assume \( G \) connected and simply connected, which means that we are only interested in the local aspect of the problem.

Call \( a \in A \) integrable if a differentiable \( r: \mathbb{R} \rightarrow A \) can be found such that \( r(s + t) = r(s)r(t) \) and \( r'(0) = a \). We can only hope to integrate \( u_0: \mathfrak{g} \rightarrow A \) to a differentiable \( u: G \rightarrow A \) if \( u_0x \) is integrable for all \( x \in \mathfrak{g} \). We shall prove the

**Theorem.** The set \( \mathfrak{h} \) of all elements \( x \in \mathfrak{g} \) such that \( u_0x \) is integrable, is a Lie subalgebra of \( \mathfrak{g} \); the representation \( u_0 \) can be integrated to a representation \( u: G \rightarrow A \) of the simply connected group \( G \) if and only if \( \mathfrak{h} = \mathfrak{g} \).

This result is "best possible" in the following sense:

**Proposition 1.** Given a real Lie algebra \( \mathfrak{g} \) and a subalgebra \( \mathfrak{h} \), there exists a representation \( u_0: \mathfrak{g} \rightarrow A \) of \( \mathfrak{g} \) in a \( b \)-algebra \( A \), so that

\[
\mathfrak{h} = \{x \in \mathfrak{g} \mid u_0x \text{ is integrable}\}.
\]

As a consequence of the theorem, we have the following result: Let \( x, y \) be two integrable elements of a \( b \)-algebra, and assume that the Lie algebra \( \mathfrak{g} \) they generate is finite-dimensional. Then all elements of \( \mathfrak{g} \) are integrable.

We cannot drop the assumption that \( \mathfrak{g} \) is finite-dimensional. There exists a \( b \)-algebra which contains integrable elements \( x, y \) such that neither \( x + y \) nor \( xy - yx \) is integrable.

Elementary properties of \( b \)-spaces and \( b \)-algebras can be found in [2] or [3]. Differentiable mappings into such spaces are investigated.
The results we need about differentiable semi-groups are established in [5], [6]. Our results are related to, but different from, those of R. T. Moore [1].

2. We first prove Proposition 1. Let \( G \) be a Lie group having \( \mathfrak{g} \) as Lie algebra and let \( H \) be the subgroup of \( G \) "generated" by \( \mathfrak{h} \). Call \( A \) the ring of distributions on \( G \) whose support is compact and contained in \( H \). The product in \( A \) is the convolution. A subset \( B \) of \( A \) is bounded if \( B \) is a bounded set of distributions with compact support, the union of the supports being relatively compact in \( H \). Then, it is easily seen that the elements of \( \mathfrak{g} \) whose image by the natural inclusion \( u_{\mathfrak{c}}: \mathfrak{g} \to A \) are integrable, are precisely the elements of \( \mathfrak{h} \). This completes the proof.

Remark. If \( H \) is simply connected, the algebra \( A \) described above is the solution of a universal problem: every representation \( u: \mathfrak{g} \to A' \) of \( \mathfrak{g} \) in a \( b \)-algebra \( A' \) such that \( u_{\mathfrak{h}} \) is integrable can be factorized in a unique way as \( u = v \circ u_{\mathfrak{c}} \), where \( v: A \to A' \) is a morphism of \( b \)-algebras. An easy but somewhat technical modification of our definition of \( A \) would provide a solution of this problem in general (for an arbitrary \( H \)); the reader will have no difficulty to figure it out.

3. Let \( u \) be a differentiable mapping of a manifold \( D \) into another manifold \( D' \) or into a \( b \)-space \( E \). We denote by \( du(x; \cdot) \) the derivative of \( u \) at \( x \), so that \( du(x, \xi) \) is a tangent vector to \( D' \) at \( ux \) or an element of \( E \) when \( \xi \) is a tangent vector at \( x \in D \). The chain rule says that if \( D, D', D'' \) are manifolds, if \( E \) is a \( b \)-space and if \( u: D \to D' \), \( v: D' \to D'' \) or \( D' \to E \) are differentiable mappings, then

\[
(1) \quad d(v \circ u)(x; \xi) = dv(ux; du(x; \xi)) .
\]

Let \( G \) be a Lie group whose neutral element will be denoted by \( e \) and let \( g \) be its Lie algebra. If \( x, y \in G \) and if \( \xi \) is a tangent vector at \( x \), then \( y \xi \) and \( \xi y \) will be the tangent vectors at \( yx, xy \) respectively obtained by translating \( \xi \) to the left or to the right. We shall denote by \( \pi: G \times G \to G \) the product mapping \( (\pi(x, y) = xy) \), by \( i: G \to G \) the inverse mapping \( (i(x) = x^{-1}) \), by \( Ad: G \to \text{Aut} \, \mathfrak{g} \) the adjoint representation \( (Adx \cdot \xi = x \xi x^{-1}) \) and by \( ad \) the derivative of \( Ad \) at \( e \) \((ad \xi \cdot \eta = [\xi, \eta])\). We have

\[
(2) \quad d\pi(x, y; \xi, \eta) = x\eta + \xi y ;
\]

\[
(3) \quad di(x; \xi) = -x^{-1} \cdot \xi \cdot x^{-1} .
\]

Let \( H \) be a Lie group, let \( A \) be a \( b \)-algebra and let \( u \) denote
either a Lie group homomorphism \( G \to H \) or a differentiable mapping \( G \to A \) which is a homomorphism of \( G \) in the multiplicative group of \( A \). Finally, set \( u_0 = du(e; \cdot) : g \to \mathfrak{h} = \text{Lie } H \) or \( g \to A \) accordingly. Then

\[
(4) \quad du(x; \xi) = u(x)u_0(x^{-1}\xi) = u_0(\xi x^{-1})u(x) .
\]

In particular

\[
(5) \quad d\text{Ad}(x; \xi) = \text{Ad } x \cdot ad(x^{-1}\xi) = ad(\xi x^{-1}) \cdot \text{Ad } x .
\]

4. Let \( A \) be a \( b \)-algebra and \( A^\ast \) be the set of its invertible elements. A mapping \( u : D \to A^\ast \) will be called differentiable if both \( x \to u(x) \) and \( x \to u(x)^{-1} \) are differentiable mappings.

It is not difficult to construct differentiable \( A \)-valued mappings which are \( A^\ast \)-valued but are not differentiable \( A^\ast \)-valued mappings.

Consideration of the resolvent identity

\[
a^{-1} - b^{-1} = -a^{-1}(a - b)b^{-1}
\]

and standard proofs show that a differentiable mapping \( u : D \to A^\ast \) with values in \( A^\ast \) is a differentiable \( A^\ast \)-valued mapping in the above sense if and only if \( u^{-1} : D \to A \) is locally bounded. It turns out that

\[
(6) \quad du^{-1}(x; \xi) = -u^{-1}(x) \cdot du(x; \xi) \cdot u^{-1}(x) .
\]

5. From now on, \( G \) will be a connected, simply connected Lie group, \( \mathfrak{g} \) will be its Lie algebra, \( A \) a \( b \)-algebra and \( u_0 : \mathfrak{g} \to A \) a representation. A differentiable submanifold \( D \) of \( G \) is called right (resp. left) integrable for \( u_0 \) if a differentiable \( u : D \to A^\ast \) exists such that the equation (7) (resp. (8)) holds:

\[
(7) \quad du(x; \xi) = u_0(\xi \cdot x^{-1})u(x) ;
\]

\[
(8) \quad du(x; \xi) = u(x)u_0(x^{-1} \cdot \xi) .
\]

It will follow from Proposition 2 that the representation \( u_0 \) is integrable in the sense of §1 if and only if the manifold \( G \) itself is right or left integrable; therefore the terminology. We note that, if \( u \) satisfies (7), then

\[
(9) \quad du^{-1}(x; \xi) = -u^{-1}(x)u_0(\xi \cdot x^{-1}) .
\]

A right translate of a right integrable manifold is right integrable. If \( u \) satisfies (7), so does \( au \) for every \( a \in A^\ast \).

**Lemma 1.** Let \( D \) be connected, right integrable, containing \( e \), and let \( u \) be a solution of (7) such that \( u(e) = 1 \). Then
(10) \[ u_0(x\xi x^{-1}) = u(x)u_0(\xi)u(x)^{-1} \]
for all \( x \in D \) and \( \xi \in g \).

It suffices to show that if \( \varphi: D \to A \) is defined by
\[
\varphi(x) = u(x)^{-1}u_0(x\xi x^{-1})u(x) ,
\]
then \( d\varphi = 0 \), and this follows from a straightforward computation using (7), (9), (5) and the fact that \( u_0: g \to A \) is a homomorphism of Lie algebras.

**Lemma 2.** If \( D \) is connected, right integrable and contains \( e \), it is also left integrable. Furthermore, the solution \( u \) of (7) such that \( u(e) = 1 \) is also a solution of (8).

This is clear since, by (10),
\[ u(x)u_0(x^{-1}\xi) = u_0(x^{-1}\xi x^{-1})u(x) = u_0(\xi x^{-1})u(x) . \]

In view of Lemma 2, it is now meaningful to say that a manifold containing \( e \) is integrable.

6. Let \( D, D' \) be two differentiable manifolds. The rank \( r_x \) of a differentiable mapping \( u: D \to D' \) at a point \( x \in D \) is the dimension of the image of the derivative \( du(x; \cdot) \). We recall that \( r_x \) is upper semi-continuous as a function of \( x \). The mapping \( u \) is said to be regular at \( x \) if \( r_x \) is constant in a neighborhood of \( x \); in that case, there exists a neighborhood \( U \) of \( x \), a submanifold \( D'' \) of \( D' \), a manifold \( E \) and a diffeomorphism \( w': U \to D'' \times E \), so that \( u|_U = p_{D''} \circ w' \) where \( p_{D''} \) denotes the projection of \( D'' \times E \) of its first factor.

**Lemma 3.** For \( i = 1, 2 \), let \( D_i \) be an integrable submanifold of \( G \) containing \( e \), and let \( u_i: D_i \to A \) be a solution of (7) mapping \( e \) on 1. Assume that the product mapping \( D'_1 \times D'_2 \to G \) is regular at \( (e, e) \). Then, one can find neighborhoods \( D'_1, D'_2 \) of \( e \) in \( D_1, D_2 \) respectively, so that \( D = D'_1 \cdot D'_2 \) is an integrable manifold and the relation

\[ u(x_i, x_2) = u_i(x_1)u_2(x_2) \quad (x_i \in D'_i) \]

defines a mapping \( u: D \to A \) which is a solution of (7).

Put \( v(x_1, x_2) = u_1(x_1)u_2(x_2) \), differentiate and apply (7), (10) and (2). This yields
\[ dv(x_1, x_2; \xi_1, \xi_2) = u_0(d\pi(x_1, x_2; \xi_1, \xi_2)x_1^{-1}x_2^{-1})v(x_1, x_2) . \]
In particular, \( dv = 0 \) whenever \( d\pi = 0 \). This, the regularity assump-
tion and the implicit function theorem imply the existence of a function \( u \) satisfying (11) locally. In view of (12), this function is locally a solution of (7).

7. Our main theorem is an immediate consequence of the

**Proposition 2.** Let \( D \) be an integrable submanifold of \( G \) of maximum dimension containing \( e \) and let \( u: D \to A \) be the solution of (7) with \( u(e) = 1 \). Then \( D \) is a local subgroup, \( u \) is a local homomorphism of \( D \) into \( A^* \) and \( D \) contains locally every integrable sub-

manifold of \( G \) containing \( e \).

We first show that

\( (*) \) if \( D' \) is any integrable submanifold of \( G \) containing \( e \), the tangent space to \( D' \) at \( e \) is contained in that of \( D \).

Assume the contrary. Then there exists a neighborhood \( U \) of \((e, e)\) in \( D \times D' \) such that, for every \((x, x') \in U \), the tangent space to \( x^{-1}D \) at \( e \) does not contain that to \( D'x'^{-1} \). Let \((f, f') \in U \) be a point where the product mapping \( D \times D' \to D \cdot D' \) is regular (one knows that the set of those points is dense). Then, by Lemma 3, there exist neighborhoods \( E \) of \( f \) in \( D \) and \( E' \) of \( f' \) in \( D' \) such that \( f^{-1}EE'f'^{-1} \) is an integrable manifold, which is obviously of dimension greater than that of \( D \), in contradiction to the maximality assumption.

It follows from \( (*) \) that the tangent space to \( D \) at any one of its points, say \( x \), is a translate of its tangent space at \( e \) (take \( D' = x^{-1}D \)). This ensures that \( D \) is a local group.

Since \( D \) is a local group, the product mapping \( D \times D \to D \) is regular in \((e, e)\). It then follows from Lemma 3 that there exist a neighborhood \( U \) of \((e, e)\) in \( D \times D \) and a function \( v \) defined in a neighborhood of \( e \) in \( D \) so that

\[
v(x_1, x_2) = u(x_1)u(x_2)\]

for \((x_1, x_2) \in U \). But then, for points \( x_1, x_2 \) close enough to \( e \), we have

\[
u(x_1)u(x_2) = v(x_1, x_2 \cdot e) = u(x_1, x_2) \cdot u(e) = u(x_1, x_2),\]

and \( u \) is a local representation.

Finally, if \( D' \) is integrable (right or left), it follows from (8) that the tangent space to \( D' \) at any one of its points is contained in a translate of the tangent space to \( D \) at \( e \). If \( e \in D' \), this implies that \( D' \) is locally (at \( e \)) contained in \( D \).
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