THE INTEGRATION OF A LIE ALGEBRA REPRESENTATION

JACQUES TITS AND LUCIEN WAELBROECK
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Let \( u: G \to A \) be a differentiable representation of a Lie group into a \( b \)-algebra. The differential \( u_0 = du_e \) of \( u \) at the neutral element \( e \) of \( G \) is a representation of the Lie algebra \( \mathfrak{g} \) of \( G \) into \( A \). Because a Lie group is locally the union of one-parameter subgroups and since the infinitesimal generator of a differentiable (multiplicative) sub-semi-group of \( A \) determines this sub-semi-group, the representation \( u_0 \) determines \( u \) if \( G \) is connected.

We shall be concerned with the converse: given a representation \( u_0 \) of \( \mathfrak{g} \), when can it be obtained by differentiating a representation \( u \) of \( G \)? We shall assume \( G \) connected and simply connected, which means that we are only interested in the local aspect of the problem.

Call \( a \in A \) integrable if a differentiable \( r: \mathbb{R} \to A \) can be found such that \( r(s + t) = r(s)r(t) \) and \( r'(0) = a \). We can only hope to integrate \( u_0: \mathfrak{g} \to A \) to a differentiable \( u: G \to A \) if \( u_0x \) is integrable for all \( x \in \mathfrak{g} \). We shall prove the

**THEOREM.** The set \( \mathfrak{h} \) of all elements \( x \in \mathfrak{g} \) such that \( u_0x \) is integrable, is a Lie subalgebra of \( \mathfrak{g} \); the representation \( u_0 \) can be integrated to a representation \( u: G \to A \) of the simply connected group \( G \) if and only if \( \mathfrak{h} = \mathfrak{g} \).

This result is "best possible" in the following sense:

**PROPOSITION 1.** Given a real Lie algebra \( \mathfrak{g} \) and a subalgebra \( \mathfrak{h} \), there exists a representation \( u_0: \mathfrak{g} \to A \) of \( \mathfrak{g} \) in a \( b \)-algebra \( A \), so that

\[
\mathfrak{h} = \{ x \in \mathfrak{g} \mid u_0x \text{ is integrable} \} .
\]

As a consequence of the theorem, we have the following result: Let \( x, y \) be two integrable elements of a \( b \)-algebra, and assume that the Lie algebra \( \mathfrak{g} \) they generate is finite-dimensional. Then all elements of \( \mathfrak{g} \) are integrable.

We cannot drop the assumption that \( \mathfrak{g} \) is finite-dimensional. There exists a \( b \)-algebra which contains integrable elements \( x, y \) such that neither \( x + y \) nor \( xy - yx \) is integrable.

Elementary properties of \( b \)-spaces and \( b \)-algebras can be found in [2] or [3]. Differentiable mappings into such spaces are investigated.
The results we need about differentiable semi-groups are established in [5], [6]. Our results are related to, but different from, those of R. T. Moore [1].

2. We first prove Proposition 1. Let $G$ be a Lie group having $\mathfrak{g}$ as Lie algebra and let $H$ be the subgroup of $G$ "generated" by $\mathfrak{h}$. Call $A$ the ring of distributions on $G$ whose support is compact and contained in $H$. The product in $A$ is the convolution. A subset $B$ of $A$ is bounded if $B$ is a bounded set of distributions with compact support, the union of the supports being relatively compact in $H$. Then, it is easily seen that the elements of $\mathfrak{g}$ whose image by the natural inclusion $\iota_0: \mathfrak{g} \rightarrow A$ are integrable, are precisely the elements of $\mathfrak{h}$. This completes the proof.

REMARK. If $H$ is simply connected, the algebra $A$ described above is the solution of a universal problem: every representation $u: \mathfrak{g} \rightarrow A'$ of $\mathfrak{g}$ in a $\mathfrak{b}$-algebra $A'$ such that $u_0$ is integrable can be factorized in a unique way as $u = v \circ u_0$, where $v: A \rightarrow A'$ is a morphism of $\mathfrak{b}$-algebras. An easy but somewhat technical modification of our definition of $A$ would provide a solution of this problem in general (for an arbitrary $H$); the reader will have no difficulty to figure it out.

3. Let $u$ be a differentiable mapping of a manifold $D$ into another manifold $D'$ or into a $b$-space $E$. We denote by $du(x; \xi)$ the derivative of $u$ at $x$, so that $du(x; \xi)$ is a tangent vector to $D'$ at $ux$ or an element of $E$ when $\xi$ is a tangent vector at $x \in D$. The chain rule says that if $D, D', D''$ are manifolds, if $E$ is a $b$-space and if $u: D \rightarrow D'$, $v: D' \rightarrow D''$ or $D' \rightarrow E$ are differentiable mappings, then

$$d(v \circ u)(x; \xi) = dv(ux; du(x; \xi)).$$

Let $G$ be a Lie group whose neutral element will be denoted by $e$ and let $\mathfrak{g}$ be its Lie algebra. If $x, y \in G$ and if $\xi$ is a tangent vector at $x$, then $y\xi$ and $\xi y$ will be the tangent vectors at $yx, xy$ respectively obtained by translating $\xi$ to the left or to the right. We shall denote by $\pi: G \times G \rightarrow G$ the product mapping ($\pi(x, y) = xy$), by $i: G \rightarrow G$ the inverse mapping ($i(x) = x^{-1}$), by $Ad: G \rightarrow \text{Aut} \mathfrak{g}$ the adjoint representation ($Ad x \cdot \xi = x\xi x^{-1}$) and by $\text{ad}$ the derivative of $Ad$ at $e$ ($\text{ad} \xi \cdot \eta = [\xi, \eta]$). We have

$$d\pi(x, y; \xi, \eta) = x\eta + \xi y;$$

$$di(x; \xi) = -x^{-1} \cdot \xi \cdot x^{-1}.$$
either a Lie group homomorphism \( G \to H \) or a differentiable mapping \( G \to A \) which is a homomorphism of \( G \) in the multiplicative group of \( A \). Finally, set \( u_0 = du(e; \cdot) : \mathfrak{g} \to \mathfrak{h} = \text{Lie} H \) or \( \mathfrak{g} \to A \) accordingly. Then

\[
(4) \quad du(x; \xi) = u(x)u_0(x^{-1}\xi) = u_0(\xi x^{-1})u(x) .
\]

In particular

\[
(5) \quad d\text{Ad}(x; \xi) = \text{Ad} x \cdot ad(x^{-1}\xi) = ad(\xi x^{-1}) \cdot \text{Ad} x .
\]

4. Let \( A \) be a \( b \)-algebra and \( A^* \) be the set of its invertible elements. A mapping \( u : D \to A^* \) will be called differentiable if both \( x \to u(x) \) and \( x \to u(x)^{-1} \) are differentiable mappings.

It is not difficult to construct differentiable \( A \)-valued mappings which are \( A^* \)-valued but are not differentiable \( A^* \)-valued mappings.

Consideration of the resolvent identity

\[
a^{-1} - b^{-1} = -a^{-1}(a - b)b^{-1}
\]

and standard proofs show that a differentiable mapping \( u : D \to A^* \) with values in \( A^* \) is a differentiable \( A^* \)-valued mapping in the above sense if and only if \( u^{-1} : D \to A \) is locally bounded. It turns out that

\[
(6) \quad du^{-1}(x; \xi) = -u^{-1}(x) \cdot du(x; \xi) \cdot u^{-1}(x) .
\]

5. From now on, \( G \) will be a connected, simply connected Lie group, \( \mathfrak{g} \) will be its Lie algebra, \( A \) a \( b \)-algebra and \( u_0 : \mathfrak{g} \to A \) a representation. A differentiable submanifold \( D \) of \( G \) is called right (resp. left) integrable for \( u_0 \) if a differentiable \( u : D \to A^* \) exists such that the equation (7) (resp. (8)) holds:

\[
(7) \quad du(x; \xi) = u_0(\xi x^{-1})u(x) ;
\]

\[
(8) \quad du(x; \xi) = u(x)u_0(x^{-1}\xi) .
\]

It will follow from Proposition 2 that the representation \( u_0 \) is integrable in the sense of §1 if and only if the manifold \( G \) itself is right or left integrable; therefore the terminology. We note that, if \( u \) satisfies (7), then

\[
(9) \quad du^{-1}(x; \xi) = -u^{-1}(x)u_0(\xi x^{-1}) .
\]

A right translate of a right integrable manifold is right integrable. If \( u \) satisfies (7), so does \( au \) for every \( a \in A^* \).

**Lemma 1.** Let \( D \) be connected, right integrable, containing \( e \), and let \( u \) be a solution of (7) such that \( u(e) = 1 \). Then
for all \( x \in D \) and \( \xi \in \mathfrak{g} \).

It suffices to show that if \( \varphi: D \to A \) is defined by
\[
\varphi(x) = u(x)^{-1}u_0(x\xi x^{-1})u(x),
\]
then \( d\varphi = 0 \), and this follows from a straightforward computation using (7), (9), (5) and the fact that \( u_0: \mathfrak{g} \to A \) is a homomorphism of Lie algebras.

**Lemma 2.** If \( D \) is connected, right integrable and contains \( e \), it is also left integrable. Furthermore, the solution \( u \) of (7) such that \( u(e) = 1 \) is also a solution of (8).

This is clear since, by (10),
\[
\varphi(x) = u(x)^{-1}u_0(x\xi x^{-1})u(x) = u_0(x\xi x^{-1})u(x).
\]

In view of Lemma 2, it is now meaningful to say that a manifold containing \( e \) is integrable.

6. Let \( D, D' \) be two differentiable manifolds. The rank \( r_x \) of a differentiable mapping \( u: D \to D' \) at a point \( x \in D \) is the dimension of the image of the derivative \( du(x; \cdot) \). We recall that \( r_x \) is upper semi-continuous as a function of \( x \). The mapping \( u \) is said to be regular at \( x \) if \( r_x \) is constant in a neighborhood of \( x \); in that case, there exists a neighborhood \( U \) of \( x \), a submanifold \( D'' \) of \( D' \), a manifold \( E \) and a diffeomorphism \( w': U \to D'' \times E \), so that \( u|_U = p_{D''} \circ w \) where \( p_{D''} \) denotes the projection of \( D'' \times E \) of its first factor.

**Lemma 3.** For \( i = 1, 2 \), let \( D_i \) be an integrable submanifold of \( G \) containing \( e \), and let \( u_i: D_i \to A \) be a solution of (7) mapping \( e \) on 1. Assume that the product mapping \( D_1 \times D_2 \to G \) is regular at \( (e, e) \). Then, one can find neighborhoods \( D'_1, D'_2 \) of \( e \) in \( D_1, D_2 \) respectively, so that \( D = D'_1 \cdot D'_2 \) is an integrable manifold and the relation
\[
(11) \quad u(x_1, x_2) = u_1(x_1) \cdot u_2(x_2) \quad (x_i \in D'_i)
\]
defines a mapping \( u: D \to A \) which is a solution of (7).

Put \( v(x_1, x_2) = u_1(x_1)u_2(x_2) \), differentiate and apply (7), (10) and (2). This yields
\[
(12) \quad dv(x_1, x_2; \xi_1, \xi_2) = u_0(d\pi(x_1, x_2; \xi_1, \xi_2)x_2^{-1}x_1^{-1})v(x_1, x_2).
\]

In particular, \( dv = 0 \) whenever \( d\pi = 0 \). This, the regularity assump-
tion and the implicit function theorem imply the existence of a function $u$ satisfying (11) locally. In view of (12), this function is locally a solution of (7).

7. Our main theorem is an immediate consequence of the

**Proposition 2.** Let $D$ be an integrable submanifold of $G$ of maximum dimension containing $e$ and let $u: D \rightarrow A$ be the solution of (7) with $u(e) = 1$. Then $D$ is a local subgroup, $u$ is a local homomorphism of $D$ into $A^*$ and $D$ contains locally every integrable submanifold of $G$ containing $e$.

We first show that

(*) if $D'$ is any integrable submanifold of $G$ containing $e$, the tangent space to $D'$ at $e$ is contained in that of $D$.

Assume the contrary. Then there exists a neighborhood $U$ of $(e, e)$ in $D \times D'$ such that, for every $(x, x') \in U$, the tangent space to $x^{-1}D$ at $e$ does not contain that to $D'x'^{-1}$. Let $(f, f') \in U$ be a point where the product mapping $D \times D' \rightarrow D \cdot D'$ is regular (one knows that the set of those points is dense). Then, by Lemma 3, there exist neighborhoods $E$ of $f$ in $D$ and $E'$ of $f'$ in $D'$ such that $f^{-1}EE'f'^{-1}$ is an integrable manifold, which is obviously of dimension greater than that of $D$, in contradiction to the maximality assumption.

It follows from (*) that the tangent space to $D$ at any one of its points, say $x$, is a translate of its tangent space at $e$ (take $D' = x^{-1}D$). This ensures that $D$ is a local group.

Since $D$ is a local group, the product mapping $D \times D \rightarrow D$ is regular in $(e, e)$. It then follows from Lemma 3 that there exist a neighborhood $U$ of $(e, e)$ in $D \times D$ and a function $v$ defined in a neighborhood of $e$ in $D$ so that

$$v(x, x_2) = u(x_1)u(x_2)$$

for $(x_1, x_2) \in U$. But then, for points $x_1, x_2$ close enough to $e$, we have

$$u(x_1)u(x_2) = v(x_1, x_2 \cdot e) = u(x_1x_2) \cdot u(e) = u(x_1x_2),$$

and $u$ is a local representation.

Finally, if $D'$ is integrable (right or left), it follows from (8) that the tangent space to $D'$ at any one of its points is contained in a translate of the tangent space to $D$ at $e$. If $e \in D'$, this implies that $D'$ is locally (at $e$) contained in $D$. 


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