

# Pacific Journal of Mathematics

**SIMPLE MODULES AND HEREDITARY RINGS**

ABRAHAM ZAKS

## SIMPLE MODULES AND HEREDITARY RINGS

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The purpose of this note is to prove that if in a semi-primary ring  $A$ , every simple module that is not a projective  $A$ -module is an injective  $A$ -module, then  $A$  is a semi-primary hereditary ring with radical of square zero. In particular, if  $A$  is a commutative ring, then  $A$  is a finite direct sum of fields. If  $A$  is a commutative Noetherian ring then if every simple module that is not a projective module, is an injective module, then for every maximal ideal  $M$  in  $A$  we obtain  $\text{Ext}^t(A/M, A/M) = 0$ . The technique of localization now implies that  $\text{gl.dim } A = 0$ .

1. We say that  $A$  is a semi-primary ring if its Jacobson radical  $N$  is a nilpotent ideal, and  $\Gamma = A/N$  is a semi-simple Artinian ring.

Throughout this note all modules (ideals) are presumed to be left modules (ideals) unless otherwise stated. For any idempotent  $e$  in  $A$  we denote by  $Ne$  the ideal  $N \cap Ae$ .

We discuss first semi-primary rings  $A$  with radical  $N$  of square zero for which every simple module that is not a projective module is an injective module. We shall study the nonsemi-simple case, i.e.,  $N \neq 0$ .

Under this assumption  $N$  becomes naturally a  $\Gamma$ -module.

Let  $e, e'$  be primitive idempotents in  $A$  for which  $eNe' \neq 0$ . In particular  $Ne' \neq 0$ . From the exact sequence  $0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0$ , it follows that  $S'$  is not a projective module since  $Ae'$  is indecomposable. Since  $S'$  is a simple module it follows that  $S'$  is an injective module.

Next consider the simple module  $Ae/Ne = S$ . Since  $eNe' \neq 0$ , since  $Ne'$  is a  $\Gamma$ -module, and since on  $N$  the  $\Gamma$ -module structure and the  $A$ -module structure coincide,  $Ne'$  contains a direct summand isomorphic with  $S$ . This gives rise to an exact sequence  $0 \rightarrow S \rightarrow Ae' \rightarrow K \rightarrow 0$  with  $K \neq 0$ . If  $S$  were injective this sequence would split, and this contradicts the indecomposability of  $Ae'$ . Therefore  $S$  is a projective module.

Hence  $Ne'$  is a direct sum of projective modules, therefore  $Ne'$  is a projective module. The exact sequence  $0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0$  now implies  $\text{l.p.dim } S' \leq 1$ , and since  $S'$  is not a projective module, then  $\text{l.p.dim } S' = 1$ .

Hence  $\text{l.p.dim}_A \Gamma = 1$ , and this implies that  $A$  is an hereditary ring (i.e.,  $\text{l.gl.dim } A = 1$ ) [1].

Conversely, assume that  $\text{l.gl.dim } A = 1$ . Every ideal in  $A$  is the direct sum of  $N_1, \dots, N_t$  where  $N_1$  is contained in the radical, and

the others (if any) are components of  $A$ , i.e.,  $N_i = Ae_i$  where  $e_2, \dots, e_t$  are primitive orthogonal idempotents in  $A$  [4].

Let  $\Gamma e'$  be any simple  $A$ -module. Since  $N_1 \subset N$ ,  $N_1$  is a  $\Gamma$ -module. Since on  $N$  the  $\Gamma$ -module structure coincides with the  $A$ -module structure, it easily follows that there exists a nonzero map of  $N_1$  onto  $\Gamma e'$  if and only if  $\Gamma e'$  (up to isomorphism) is a direct summand of  $N_1$ . This in particular implies that  $\Gamma e'$  is a projective  $A$ -module, since then  $\Gamma e'$  is isomorphic to an ideal. If  $\Gamma e'$  is not a projective  $A$ -module, it follows that  $\text{Hom}_A(N_1, \Gamma e') = 0$ . As a consequence, every map from an ideal in  $A$  into  $\Gamma e'$ , extends to a map of  $A$  into  $\Gamma e'$ , hence  $\Gamma e'$  is an injective  $A$ -module.

This proves:

**THEOREM A.** *Let  $A$  be a semi-primary ring with radical of square zero. Then every simple  $A$ -module that is not a projective  $A$ -module is an injective  $A$ -module if and only if  $A$  is a hereditary ring.*

If  $A$  is a semi-primary ring with radical  $N$  and  $N^2 \neq 0$ , then a simple module is projective if and only if it is isomorphic to a component, hence if  $Ae/Ne$  is a projective module  $Ne = 0$ , and the idempotent  $e$ , when reduced mod  $N^2$  (i.e., in  $A/N^2$ ) will still give rise to a projective module. If  $Ae/Ne$  is an injective module  $e$  will give rise to an injective  $A/N^2$ -module. This will follow from the following two lemmas:

**LEMMA 1.** *Let  $e, e'$  be primitive idempotents in  $A$ . Then  $Ae$  is isomorphic to  $Ae'$  if and only if  $\text{Hom}_A(Ae', Ae/Ne) \neq 0$ .*

*Proof.* If  $Ae$  is isomorphic to  $Ae'$  then obviously

$$\text{Hom}_A(Ae', Ae/Ne) \neq 0.$$

Conversely, let  $f: Ae' \rightarrow Ae/Ne$  be a nonzero map. Since  $Ae/Ne$  is a simple module  $f$  is an epimorphism. Denote by  $\pi$  the canonical projection  $\pi: Ae \rightarrow Ae/Ne$  then since  $Ae'$  is a projective module there exists a map  $g: Ae' \rightarrow Ae$  such that  $f = \pi \circ g$ . Since  $\pi(Ne) = 0$ , it follows that  $g$  is an epimorphism. Since  $Ae$  is a projective module and  $Ae'$  an indecomposable module  $g$  is an isomorphism.

**LEMMA 2.** *Let  $S$  be an injective simple  $A$ -module and  $I$  an ideal that is contained in the radical. Then  $\text{Hom}_A(I, S) = 0$ .*

*Proof.* Let  $f$  be a nonzero map of  $I$  into  $S$ . Since  $S$  is an

injective  $A$  module it follows that  $f$  extends to a map of  $A$  onto  $S$ ,  $f: A \rightarrow S$ , but this implies that  $f(N) = 0$ . Since  $f(I) \subset f(N)$  this is a contradiction. Therefore every map of  $I$  into  $S$  is the zero map.

**THEOREM B.** *Let  $A$  be a semi-primary ring then the following are equivalent:*

- (i)  *$A$  is an hereditary ring with radical of square zero.*
- (ii) *Every simple module that is not a projective  $A$ -module is an injective  $A$ -module.*

*Proof.* That (i) implies (ii) follows from Theorem A.

(ii)  $\Rightarrow$  (i): Let  $e_1, \dots, e_t$  be a complete set of orthogonal idempotents, i.e., each  $e_i$  is a primitive idempotent, and

$$A = Ae_1 + \dots + Ae_t.$$

Set  $S_i = Ae_i/Ne_i$ . We denote by  $\bar{e}_1, \dots, \bar{e}_t$  the images of  $e_1, \dots, e_t$  in  $A/N^2$  under the canonical epimorphism  $A \rightarrow A/N^2$ . Then  $S_1, \dots, S_t$  may be viewed as simple  $A/N^2$ -modules, and every simple  $A/N^2$ -module is necessarily isomorphic with some  $S_i$ . If  $S_j$  is  $A$ -projective then  $Ne_j = 0$ , and necessarily  $S_j$  is  $A/N^2$ -projective. If  $S_j$  is  $A$ -injective then we claim that  $S_j$  is  $A/N^2$ -injective. It suffices to prove that for any ideal  $I'$  in  $A/N^2$ , and any  $A/N^2$ -map  $f$  from  $I'$  to  $S_j$ ,  $f$  extends to a map of  $A/N^2$  into  $S_j$ . Since  $I'$  is a direct sum of ideals  $I_1, \dots, I_r$ ,  $I'_1 \subset N/N^2$  and the others (if any) are components of  $A/N^2$ , we will be done if we prove that  $\text{Hom}_{A/N^2}(I'', S_j) = 0$  whenever  $I'' \subset N/N^2$ . Let  $I$  be the inverse image of  $I''$  under the homomorphism  $A \rightarrow A/N^2$ , then  $\text{Hom}_A(I, S_j) = 0$  since  $I \subset N$  (Lemma 2). If we denote by  $h$  the map  $I \rightarrow I''$  (restriction of the canonical projection) and if  $f$  is any map of  $I''$  into  $S_j$  then if  $f$  is not the zero map,  $f \circ h$  from  $I$  into  $S_j$  is a nonzero  $A$ -map of  $I$  into  $S_j$ . This contradiction implies that  $S_j$  is an injective  $A/N^2$ -module.

By Theorem A it now follows, since  $A/N^2$  is a semi-primary ring with radical of square zero, that  $\text{l.gl.dim } A/N^2 \leq 1$ . This necessarily implies that  $N^2 = 0$  [2].

Remark that if all simple modules are projective modules, or if all simple modules are injective modules, then  $A$  is a semi-simple ring [1].

Finally, if  $N \neq 0$  then there exist a simple projective (injective) module that is not an injective (projective) module.

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