

Pacific Journal of Mathematics

SIMPLE MODULES AND HEREDITARY RINGS

ABRAHAM ZAKS

SIMPLE MODULES AND HEREDITARY RINGS

ABRAHAM ZAKS

The purpose of this note is to prove that if in a semi-primary ring A , every simple module that is not a projective A -module is an injective A -module, then A is a semi-primary hereditary ring with radical of square zero. In particular, if A is a commutative ring, then A is a finite direct sum of fields. If A is a commutative Noetherian ring then if every simple module that is not a projective module, is an injective module, then for every maximal ideal M in A we obtain $\text{Ext}^1(A/M, A/M) = 0$. The technique of localization now implies that $\text{gl.dim } A = 0$.

1. We say that A is a semi-primary ring if its Jacobson radical N is a nilpotent ideal, and $\Gamma = A/N$ is a semi-simple Artinian ring.

Throughout this note all modules (ideals) are presumed to be left modules (ideals) unless otherwise stated. For any idempotent e in A we denote by Ne the ideal $N \cap Ae$.

We discuss first semi-primary rings A with radical N of square zero for which every simple module that is not a projective module is an injective module. We shall study the nonsemi-simple case, i.e., $N \neq 0$.

Under this assumption N becomes naturally a Γ -module.

Let e, e' be primitive idempotents in A for which $eNe' \neq 0$. In particular $Ne' \neq 0$. From the exact sequence $0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0$, it follows that S' is not a projective module since Ae' is indecomposable. Since S' is a simple module it follows that S' is an injective module.

Next consider the simple module $Ae/Ne = S$. Since $eNe' \neq 0$, since Ne' is a Γ -module, and since on N the Γ -module structure and the A -module structure coincide, Ne' contains a direct summand isomorphic with S . This gives rise to an exact sequence $0 \rightarrow S \rightarrow Ae' \rightarrow K \rightarrow 0$ with $K \neq 0$. If S were injective this sequence would split, and this contradicts the indecomposability of Ae' . Therefore S is a projective module.

Hence Ne' is a direct sum of projective modules, therefore Ne' is a projective module. The exact sequence $0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0$ now implies $\text{l.p.dim } S' \leq 1$, and since S' is not a projective module, then $\text{l.p.dim } S' = 1$.

Hence $\text{l.p.dim}_A \Gamma = 1$, and this implies that A is an hereditary ring (i.e., $\text{l.gl.dim } A = 1$) [1].

Conversely, assume that $\text{l.gl.dim } A = 1$. Every ideal in A is the direct sum of N_1, \dots, N_t where N_1 is contained in the radical, and

the others (if any) are components of \mathcal{A} , i.e., $N_i = \mathcal{A}e_i$ where e_2, \dots, e_t are primitive orthogonal idempotents in \mathcal{A} [4].

Let $\Gamma e'$ be any simple \mathcal{A} -module. Since $N_1 \subset N$, N_1 is a Γ -module. Since on N the Γ -module structure coincides with the \mathcal{A} -module structure, it easily follows that there exists a nonzero map of N_1 onto $\Gamma e'$ if and only if $\Gamma e'$ (up to isomorphism) is a direct summand of N_1 . This in particular implies that $\Gamma e'$ is a projective \mathcal{A} -module, since then $\Gamma e'$ is isomorphic to an ideal. If $\Gamma e'$ is not a projective \mathcal{A} -module, it follows that $\text{Hom}_{\mathcal{A}}(N_1, \Gamma e') = 0$. As a consequence, every map from an ideal in \mathcal{A} into $\Gamma e'$, extends to a map of \mathcal{A} into $\Gamma e'$, hence $\Gamma e'$ is an injective \mathcal{A} -module.

This proves:

THEOREM A. *Let \mathcal{A} be a semi-primary ring with radical of square zero. Then every simple \mathcal{A} -module that is not a projective \mathcal{A} -module is an injective \mathcal{A} -module if and only if \mathcal{A} is a hereditary ring.*

If \mathcal{A} is a semi-primary ring with radical N and $N^2 \neq 0$, then a simple module is projective if and only if it is isomorphic to a component, hence if $\mathcal{A}e/Ne$ is a projective module $Ne = 0$, and the idempotent e , when reduced mod N^2 (i.e., in \mathcal{A}/N^2) will still give rise to a projective module. If $\mathcal{A}e/Ne$ is an injective module e will give rise to an injective \mathcal{A}/N^2 -module. This will follow from the following two lemmas:

LEMMA 1. *Let e, e' be primitive idempotents in \mathcal{A} . Then $\mathcal{A}e$ is isomorphic to $\mathcal{A}e'$ if and only if $\text{Hom}_{\mathcal{A}}(\mathcal{A}e', \mathcal{A}e/Ne) \neq 0$.*

Proof. If $\mathcal{A}e$ is isomorphic to $\mathcal{A}e'$ then obviously

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}e', \mathcal{A}e/Ne) \neq 0.$$

Conversely, let $f: \mathcal{A}e' \rightarrow \mathcal{A}e/Ne$ be a nonzero map. Since $\mathcal{A}e/Ne$ is a simple module f is an epimorphism. Denote by π the canonical projection $\pi: \mathcal{A}e \rightarrow \mathcal{A}e/Ne$ then since $\mathcal{A}e'$ is a projective module there exists a map $g: \mathcal{A}e' \rightarrow \mathcal{A}e$ such that $f = \pi \circ g$. Since $\pi(Ne) = 0$, it follows that g is an epimorphism. Since $\mathcal{A}e$ is a projective module and $\mathcal{A}e'$ an indecomposable module g is an isomorphism.

LEMMA 2. *Let S be an injective simple \mathcal{A} -module and I an ideal that is contained in the radical. Then $\text{Hom}_{\mathcal{A}}(I, S) = 0$.*

Proof. Let f be a nonzero map of I into S . Since S is an

injective A module it follows that f extends to a map of A onto S , $f: A \rightarrow S$, but this implies that $f(N) = 0$. Since $f(I) \subset f(N)$ this is a contradiction. Therefore every map of I into S is the zero map.

THEOREM B. *Let A be a semi-primary ring then the following are equivalent:*

- (i) *A is an hereditary ring with radical of square zero.*
- (ii) *Every simple module that is not a projective A -module is an injective A -module.*

Proof. That (i) implies (ii) follows from Theorem A.

(ii) \Rightarrow (i): Let e_1, \dots, e_t be a complete set of orthogonal idempotents, i.e., each e_i is a primitive idempotent, and

$$A = Ae_1 + \dots + Ae_t.$$

Set $S_i = Ae_i/Ne_i$. We denote by $\bar{e}_1, \dots, \bar{e}_t$ the images of e_1, \dots, e_t in A/N^2 under the canonical epimorphism $A \rightarrow A/N^2$. Then S_1, \dots, S_t may be viewed as simple A/N^2 -modules, and every simple A/N^2 -module is necessarily isomorphic with some S_i . If S_j is A -projective then $Ne_j = 0$, and necessarily S_j is A/N^2 -projective. If S_j is A -injective then we claim that S_j is A/N^2 -injective. It suffices to prove that for any ideal I' in A/N^2 , and any A/N^2 -map f from I' to S_j , f extends to a map of A/N^2 into S_j . Since I' is a direct sum of ideals I_1, \dots, I'_r , $I'_1 \subset N/N^2$ and the others (if any) are components of A/N^2 , we will be done if we prove that $\text{Hom}_{A/N^2}(I'', S_j) = 0$ whenever $I'' \subset N/N^2$. Let I be the inverse image of I'' under the homomorphism $A \rightarrow A/N^2$, then $\text{Hom}_A(I, S_j) = 0$ since $I \subset N$ (Lemma 2). If we denote by h the map $I \rightarrow I''$ (restriction of the canonical projection) and if f is any map of I'' into S_j then if f is not the zero map, $f \circ h$ from I into S_j is a nonzero A -map of I into S_j . This contradiction implies that S_j is an injective A/N^2 -module.

By Theorem A it now follows, since A/N^2 is a semi-primary ring with radical of square zero, that $\text{l.gl.dim } A/N^2 \leq 1$. This necessarily implies that $N^2 = 0$ [2].

Remark that if all simple modules are projective modules, or if all simple modules are injective modules, then A is a semi-simple ring [1].

Finally, if $N \neq 0$ then there exist a simple projective (injective) module that is not an injective (projective) module.

REFERENCES

1. M. Auslander, *Global dimension*, Nagoya Math. J. **9** (1955), 67-77.
2. S. Eilenberg and T. Nakayama, *Dimension of residue rings*, Nagoya Math. J. **11** (1957), 9-12.
3. B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645-650.
4. A. Zaks, *Global dimension of Artinian rings*, Proc. Amer. Math. Soc. **18** (1967), 1102-1106.

Received December 5, 1967.

TECHNION, HAIFA
ISRAEL

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Leonard Asimow, <i>Universally well-capped cones</i>	421
Lawrence Peter Belluce, William A. Kirk and Eugene Francis Steiner, <i>Normal structure in Banach spaces</i>	433
William Jay Davis, <i>Bases in Hilbert space</i>	441
Larry Lee Dornhoff, <i>p-automorphic p-groups and homogeneous algebras</i>	447
William Grady Dotson, Jr. and W. R. Mann, <i>A generalized corollary of the Browder-Kirk fixed point theorem</i>	455
John Brady Garnett, <i>On a theorem of Mergelyan</i>	461
Matthew Gould, <i>Multiplicity type and subalgebra structure in universal algebras</i>	469
Marvin D. Green, <i>A locally convex topology on a preordered space</i>	487
Pierre A. Grillet and Mario Petrich, <i>Ideal extensions of semigroups</i>	493
Kyong Taik Hahn, <i>A remark on integral functions of several complex variables</i>	509
Choo Whan Kim, <i>Uniform approximation of doubly stochastic operators</i>	515
Charles Alan McCarthy and L. Tzafriri, <i>Projections in \mathcal{L}_1 and \mathcal{L}_∞-spaces</i>	529
Alfred Berry Manaster, <i>Full co-ordinals of RETs</i>	547
Donald Steven Passman, <i>p-solvable doubly transitive permutation groups</i>	555
Neal Jules Rothman, <i>An L^1 algebra for linearly quasi-ordered compact semigroups</i>	579
James DeWitt Stein, <i>Homomorphisms of semi-simple algebras</i>	589
Jacques Tits and Lucien Waelbroeck, <i>The integration of a Lie algebra representation</i>	595
David Vere-Jones, <i>Ergodic properties of nonnegative matrices. II</i>	601
Donald Rayl Wilken, <i>The support of representing measures for $R(X)$</i>	621
Abraham Zaks, <i>Simple modules and hereditary rings</i>	627