SIMPLE MODULES AND HEREDITARY RINGS

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The purpose of this note is to prove that if in a semi-primary ring \( A \), every simple module that is not a projective \( A \)-module is an injective \( A \)-module, then \( A \) is a semi-primary hereditary ring with radical of square zero. In particular, if \( A \) is a commutative ring, then \( A \) is a finite direct sum of fields. If \( A \) is a commutative Noetherian ring then if every simple module that is not a projective module, is an injective module, then for every maximal ideal \( M \) in \( A \) we obtain \( \text{Ext}^1(\frac{A}{M}, \frac{A}{M}) = 0 \). The technique of localization now implies that \( \text{gl.dim} A = 0 \).

1. We say that \( A \) is a semi-primary ring if its Jacobson radical \( N \) is a nilpotent ideal, and \( \Gamma = A/N \) is a semi-simple Artinian ring.

Throughout this note all modules (ideals) are presumed to be left modules (ideals) unless otherwise stated. For any idempotent \( e \) in \( A \) we denote by \( Ne \) the ideal \( N \cap Ae \).

We discuss first semi-primary rings \( A \) with radical \( N \) of square zero for which every simple module that is not a projective module is an injective module. We shall study the nonsemi-simple case, i.e., \( N \neq 0 \).

Under this assumption \( N \) becomes naturally a \( \Gamma \)-module.

Let \( e, e' \) be primitive idempotents in \( A \) for which \( eNe' \neq 0 \). In particular \( Ne' \neq 0 \). From the exact sequence \( 0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0 \), it follows that \( S' \) is not a projective module since \( Ae' \) is indecomposable. Since \( S' \) is a simple module it follows that \( S' \) is an injective module.

Next consider the simple module \( Ae/Ne = S \). Since \( eNe' \neq 0 \), since \( Ne' \) is a \( \Gamma \)-module, and since on \( N \) the \( \Gamma \)-module structure and the \( A \)-module structure coincide, \( Ne' \) contains a direct summand isomorphic with \( S \). This gives rise to an exact sequence \( 0 \rightarrow S \rightarrow Ae' \rightarrow K \rightarrow 0 \) with \( K \neq 0 \). If \( S \) were injective this sequence would split, and this contradicts the indecomposability of \( Ae' \). Therefore \( S \) is a projective module.

Hence \( Ne' \) is a direct sum of projective modules, therefore \( Ne' \) is a projective module. The exact sequence \( 0 \rightarrow Ne' \rightarrow Ae' \rightarrow S' \rightarrow 0 \) now implies \( l.p.d. S' \leq 1 \), and since \( S' \) is not a projective module, then \( l.p.d. S' = 1 \).

Hence \( l.p.d. \Gamma = 1 \), and this implies that \( A \) is an hereditary ring (i.e., \( l.gl.\dim A = 1 \) [1].

Conversely, assume that \( l.gl.\dim A = 1 \). Every ideal in \( A \) is the direct sum of \( N_i, \cdots, N_i \) where \( N_i \) is contained in the radical, and
the others (if any) are components of $A$, i.e., $N_i = Ae_i$ where $e_2, \ldots, e_t$ are primitive orthogonal idempotents in $A$ \cite{4}.

Let $\Gamma e'$ be any simple $A$-module. Since $N_i \subseteq N$, $N_i$ is a $\Gamma$-module. Since on $N$ the $\Gamma$-module structure coincides with the $A$-module structure, it easily follows that there exists a nonzero map of $N_i$ onto $\Gamma e'$ if and only if $\Gamma e'$ (up to isomorphism) is a direct summand of $N_i$. This in particular implies that $\Gamma e'$ is a projective $A$-module, since then $\Gamma e'$ is isomorphic to an ideal. If $\Gamma e'$ is not a projective $A$-module, it follows that $\text{Hom}_A(N_i, \Gamma e') = 0$. As a consequence, every map from an ideal in $A$ into $\Gamma e'$, extends to a map of $A$ into $\Gamma e'$, hence $\Gamma e'$ is an injective $A$-module.

This proves:

**Theorem A.** Let $A$ be a semi-primary ring with radical of square zero. Then every simple $A$-module that is not a projective $A$-module is an injective $A$-module if and only if $A$ is a hereditary ring.

If $A$ is a semi-primary ring with radical $N$ and $N^2 \neq 0$, then a simple module is projective if and only if it is isomorphic to a component, hence if $Ae/Ne$ is a projective module $Ne = 0$, and the idempotent $e$, when reduced mod $N^2$ (i.e., in $A/N^2$) will still give rise to a projective module. If $Ae/Ne$ is an injective module $e$ will give rise to an injective $A/N^2$-module. This will follow from the following two lemmas:

**Lemma 1.** Let $e, e'$ be primitive idempotents in $A$. Then $Ae$ is isomorphic to $Ae'$ if and only if $\text{Hom}_A(Ae', Ae/Ne) \neq 0$.

**Proof.** If $Ae$ is isomorphic to $Ae'$ then obviously

$$\text{Hom}_A(Ae', Ae/Ne) \neq 0.$$  

Conversely, let $f : Ae' \to Ae/Ne$ be a nonzero map. Since $Ae/Ne$ is a simple module $f$ is an epimorphism. Denote by $\pi$ the canonical projection $\pi : Ae \to Ae/Ne$ then since $Ae'$ is a projective module there exists a map $g : Ae' \to Ae$ such that $f = \pi \circ g$. Since $\pi(Ne) = 0$, it follows that $g$ is an epimorphism. Since $Ae$ is a projective module and $Ae'$ an indecomposable module $g$ is an isomorphism.

**Lemma 2.** Let $S$ be an injective simple $A$-module and $I$ an ideal that is contained in the radical. Then $\text{Hom}_A(I, S) = 0$.

**Proof.** Let $f$ be a nonzero map of $I$ into $S$. Since $S$ is an
injective $\Lambda$ module it follows that $f$ extends to a map of $\Lambda$ onto $S$, $f: \Lambda \to S$, but this implies that $f(N) = 0$. Since $f(I) \subset f(N)$ this is a contradiction. Therefore every map of $I$ into $S$ is the zero map.

**Theorem B.** Let $\Lambda$ be a semi-primary ring then the following are equivalent:

(i) $\Lambda$ is an hereditary ring with radical of square zero.

(ii) Every simple module that is not a projective $\Lambda$-module is an injective $\Lambda$-module.

**Proof.** That (i) implies (ii) follows from Theorem A.

(ii) $\Rightarrow$ (i): Let $e_1, \ldots, e_i$ be a complete set of orthogonal idempotents, i.e., each $e_i$ is a primitive idempotent, and

$$A = \Lambda e_i + \cdots + \Lambda e_i.$$

Set $S_i = \Lambda e_i / N e_i$. We denote by $\bar{e}_i, \ldots, \bar{e}_i$ the images of $e_i, \ldots, e_i$ in $A/N^2$ under the canonical epimorphism $A \to A/N^2$. Then $S_i, \ldots, S_i$ may be viewed as simple $A/N^2$-modules, and every simple $A/N^2$-module is necessarily isomorphic with some $S_i$. If $S_i$ is $\Lambda$-projective then $N e_i = 0$, and necessarily $S_i$ is $A/N^2$-projective. If $S_i$ is $\Lambda$-injective then we claim that $S_i$ is $A/N^2$-injective. It suffices to prove that for any ideal $I'$ in $A/N^2$, and any $A/N^2$-map $f$ from $I'$ to $S_i$, $f$ extends to a map of $A/N^2$ into $S_i$. Since $I'$ is a direct sum of ideals $I_1', \ldots, I'_l, I_1' \subset N/N^2$ and the others (if any) are components of $A/N^2$, we will be done if we prove that $\text{Hom}_{A/N^2}(I'', S_i) = 0$ whenever $I'' \subset N/N^2$.

Let $I$ be the inverse image of $I''$ under the homomorphism $A \to A/N^2$, then $\text{Hom}_{A}(I, S_i) = 0$ since $I \subset N$ (Lemma 2). If we denote by $h$ the map $I \to I''$ (restriction of the canonical projection) and if $f$ is any map of $I''$ into $S_i$ then if $f$ is not the zero map, $f \circ h$ from $I$ into $S_i$ is a nonzero $\Lambda$-map of $I$ into $S_i$. This contradiction implies that $S_i$ is an injective $A/N^2$-module.

By Theorem A it now follows, since $A/N^2$ is a semi-primary ring with radical of square zero, that $l.\text{gl.dim } A/N^2 \leq 1$. This necessarily implies that $N^2 = 0$ [2].

Remark that if all simple modules are projective modules, or if all simple modules are injective modules, then $\Lambda$ is a semi-simple ring [1].

Finally, if $N \neq 0$ then there exist a simple projective (injective) module that is not an injective (projective) module.
REFERENCES


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