H\textsc{ENSOR PRODUCTS OF W*-ALGEBRAS}

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TENSOR PRODUCTS OF W*-ALGEBRAS

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This paper deals primarily with a characterization of the tensor products of a family of W*-algebras (abstract von Neumann algebras). It is especially concerned with infinite tensor products; the results, however, apply and have interest in the finite case.

A tensor product for a family \((\mathcal{A}_i)\) of W*-algebras is defined to be a W*-algebra \(\mathcal{A}\) together with injections \(\alpha_i\) of \(\mathcal{A}_i\) into \(\mathcal{A}\) satisfying four conditions: the first two are that the \(\alpha_i(\mathcal{A}_i)\) commute and generate \(\mathcal{A}\); the last two are conditions on the set of positive normal functionals of \(\mathcal{A}\) which are products with respect to the \(\alpha_i(\mathcal{A}_i)\). A local tensor product is defined to be a tensor product satisfying a fifth condition—that its tail reduce to the scalars. It is shown that the local tensor products of \((\mathcal{A}_i)\) are precisely the incomplete direct products \(\bigotimes(\mathcal{A}_i, \mu_i)\), and that every tensor product is a direct sum of local tensor products which are not product isomorphic.

Suppose that \((\mathcal{A}_i)_{i \in I}\) is a family of W*-algebras. We call \((\mathcal{A}, (\alpha_i)_{i \in I})\) a product for the family \((\mathcal{A}_i)_{i \in I}\) if \(\mathcal{A}\) is a W*-algebra if, for each \(i \in I\), \(\alpha_i\) is an injection of \(\mathcal{A}_i\) into \(\mathcal{A}\) with \(\alpha_i(1) = 1\) and if the following conditions hold:

(I). \(\alpha_i(\mathcal{A}_i)\) commutes with \(\alpha_j(\mathcal{A}_j)\) for all \(i, j \in I\) with \(i \neq j\).

(II). \(\mathcal{B}(\alpha_i(\mathcal{A}_i); i \in I) = \mathcal{A}\): that is, \(\mathcal{A}\) is the smallest W* subalgebra of \(\mathcal{A}\) which contains all \(\mathcal{A}_i\) for \(i \in I\).

By a product functional for \((\mathcal{A}, (\alpha_i))\) we mean a nonzero norm positive functional \(\mu\) on \(\mathcal{A}\) for which there exist normal positive functionals \(\mu_i\) on \(\mathcal{A}_i\) for each \(i \in I\) such that:

\[ \mu\left(\prod_{i \in I} \alpha_i(A_i)\right) = \prod_{i \in I} \mu_i(A_i) \]

whenever each \(A_i \in \mathcal{A}_i\) and \(A_i = 1\) for a.a. \(i \in I\). (a.a. \(i \in I\) means—here and throughout the paper—all but a finite number of \(i \in I\))

Because of (II), it is evident that the \(\mu_i\) determine \(\mu\) uniquely, and we write \(\mu = \bigotimes_{i \in I} \mu_i\). We will denote the set of product functional for \((\mathcal{A}, (\mathcal{A}_i))\) by \(\Sigma_p\).

We call \((\mathcal{A}, (\alpha_i))\) a tensor product for \((\mathcal{A}_i)\) if it is a product for \((\mathcal{A}_i)\) (i.e., if (I) and (II) hold) and if the following conditions hold

(III). \(\Sigma_p\) is separating: i.e., if \(A \in \mathcal{A}^+\) and \(\mu(A) = 0\) for all \(\mu \in \Sigma_p\), then \(A = 0\).
(IV). For all $\mu \in \Sigma_p$, (IV-$\mu$) holds.

(IV-$\mu$). $\mu = \otimes_{i \in I} \mu_i \in \Sigma_p$, and, if $\nu_i$ is a nonzero normal positive functional on $\mathcal{A}_i$ with $\nu_i = \mu_i$ for a.a. $i \in I$, then $\otimes_{i \in I} \nu_i$ exists in $\Sigma_p$.

We define the tail $\mathcal{I}$ of a product $(\mathcal{A}_i (\alpha_i))$ to be the intersection over all finite subsets $F$ of $I$ of the algebras

$$\mathcal{A}_{I-F} = \mathcal{B} \{ \alpha_i (\mathcal{A}_i) : i \in I - F \}.$$ 

We call $(\mathcal{A}_i (\alpha_i))$ a local tensor product if it is a tensor product and the following condition holds:

(V). The tail $\mathcal{I}$ of the product $(\mathcal{A}_i (\alpha_i))$ consists of the scalars only.

A local tensor product will be called a ($\mu_i$)-local tensor product if $\otimes \mu_i \in \Sigma_p$.

We show (Theorem 4.7) that, for every family $(\mathcal{A}_i, \mu_i)_{i \in I}$ with $\mu_i$ a normal positive functional on the $W^*$-algebra $\mathcal{A}_i$ and

$$0 < \prod_{i \in I} \mu_i(1) < \infty,$$

a ($\mu_i$)-local tensor product exists and is unique up to isomorphism. (An isomorphism of a product $(\mathcal{A}_i (\alpha_i))$ with a product $(\mathcal{A}_i (\beta_i))$ is an isomorphism $\psi$ of $\mathcal{A}_i$ onto $\mathcal{B}_i$ such that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$.)

In fact, a ($\mu_i$)-local tensor product for $(\mathcal{A}_i)$ can be constructed as follows. For each $i \in I$ let $\phi_i$ be an isomorphism of $\mathcal{A}_i$ onto a von Neumann algebra on the Hilbert space $H_i$ and let $x_i \in H_i$ induce $\mu_i$:

$$\mu_i(A_i) = (\phi_i(A_i)x_i | x_i) \quad \text{for all } A_i \in \mathcal{A}_i.$$ 

Let $\mathcal{A}$ be $\otimes_{i \in I} (\phi_i (\mathcal{A}_i), x_i)$, i.e., von Neumann's incomplete direct product of $(\phi_i (\mathcal{A}_i))_{i \in I}$ with respect to the $C^*$-sequence $(x_i)$ (see [7], [1], [2], or §2 below); and for each $i \in I$ let $\alpha_i = \gamma_i \circ \phi_i$, where $\gamma_i$ is the natural injection of $\phi_i (\mathcal{A}_i)$ into $\mathcal{A}$. Then $(\mathcal{A}_i (\alpha_i))$ is a ($\mu_i$)-local tensor product for $(\mathcal{A}_i)_{i \in I}$. A special consequence of the uniqueness of ($\mu_i$)-local tensor products is, then, roughly that the tensor product of a family of von Neumann algebras depends on their algebraic structure only (see Corollary 3.5, below, for a proper statement). This is an easy result which can be proved also from [9] or directly (see remark in [2, §3]). For finite $I$, it is a result due to Misonou [4].

If $I$ is finite, all tensor products of $(\mathcal{A}_i)_{i \in I}$ are local and all are isomorphic. Thus properties (I), (II), (III), and (IV) characterize the finite tensor product. A special case of this result was proved by Nakamura [6]: he showed that (I) and (II) characterize the finite tensor product of finite factors. A stronger result of this kind was proved by Takesaki [10]: he showed that (I), (II) and the existence
of a nonzero ultraweakly continuous (not necessarily positive) product functional characterize the finite tensor product of factors (c.f. Lemma 6.2, below).

In §5, we determine all possible tensor products for \((\mathcal{A}_i)_{i \in I}\). Let \(A\) be the set of all families \((\mu_i)_{i \in I}\) where each \(\mu_i\) is a normal positive functional on \(\mathcal{A}_i\) and \(0 < \prod_{i \in I} \mu_i(1) < \infty\). Define an equivalence relation \(R\) on \(A\) by writing \((\mu_i) \sim (\nu_i)\) when a \((\mu_i)\)-local tensor product is necessarily a \((\nu_i)\)-local tensor product. Denote \(A/R\) by \(\Delta\) and the natural quotient map \(A \to A/R\) by \(\varphi\). If \(\Gamma\) is a subset of \(\Delta\), we call \((\mathcal{A}_i, (\mu_i))\) a \(\Gamma\)-tensor product for \((\mathcal{A}_i)_{i \in I}\) if \((\mathcal{A}_i, (\mu_i))\) is a tensor product for \((\mathcal{A}_i)_{i \in I}\) and if

\[
\{(\mu_i) \in A : \bigotimes \mu_i \text{ exists on } \mathcal{A} \} = \varphi^{-1}(\Gamma).
\]

Then:

1. Every tensor product for \((\mathcal{A}_i)_{i \in I}\) is a \(\Gamma\)-tensor product for some subset \(\Gamma\) of \(\Delta\).
2. For every nonempty subset \(\Gamma\) of \(\Delta\) a \(\Gamma\)-tensor product exists for \((\mathcal{A}_i)_{i \in I}\).
3. A \(\Gamma_1\)-tensor product is isomorphic (as a product) to a \(\Gamma_2\)-tensor product if and only if \(\Gamma_1 = \Gamma_2\).
4. A \(\Gamma\)-tensor product is a local tensor product if and only if \(\Gamma\) consists of only one point.
5. A \(\Gamma\)-tensor product is the direct sum of \(\{\alpha\}\)-tensor products as \(\alpha\) runs through \(\Gamma\).

In case each \(\mathcal{A}_i\) is semi-finite, the equivalence relation \(R\) may be defined explicitly by using the Kakutani product theorem for \(W^*\)-algebras [2]. We obtain \((\mu_i) \sim (\nu_i)\) if and only if

\[
\sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty,
\]

where \(d(\mu, \nu)\) is roughly the infimum of \(\|x - y\|\) over all representations of \(\mathcal{A}\) as a von Neumann algebra and all \(x, y\) inducing \(\mu\) and \(\nu\) respectively.

It is not difficult to see that Takeda's infinite direct product of \((\mathcal{A}_i)_{i \in I}\) (see [9]) is a \(\Delta\)-tensor product for \((\mathcal{A}_i)_{i \in I}\).

Section 6 contains some special results on tensor products of factors. Section 7 contains a few simple counterexamples which demonstrate that conditions (III) and (IV) are necessary.

1. Products and factorizations. If \(\mu\) is a normal positive functional on a \(W^*\)-algebra \(\mathcal{A}\), we denote the support of \(\mu\) by \(S(\mu)\), and the central support (the smallest projection of the center of \(\mathcal{A}\) larger than \(S(\mu)\)) by \(Z(\mu)\).

Throughout this section \((\mathcal{A}_i)_{i \in I}\) will be a factorization of the
$W^*$-algebra $\mathcal{A}$. By this we mean that each $\mathcal{A}_i$ is a $W^*$-subalgebra of $\mathcal{A}$ and that, if $\lambda_i$ denotes the inclusion mapping of $\mathcal{A}_i$ into $\mathcal{A}$, $(\mathcal{A}, (\lambda_i))$ is a product for $(\mathcal{A}_i)_{i \in I}$. $\mathcal{Z}$ will denote the center of $\mathcal{A}$ and $\mathcal{Z}_i$ the center of $\mathcal{A}_i$. For $J$ a subset of $I$ we let $\mathcal{A}_J = R(\mathcal{A}; i \in J)$. We call an element of $\mathcal{A}$ tail if it is in $\mathcal{I} = \bigcap_F \mathcal{A}_{F^{-1}}$. For $\mu \in \Sigma_p$, $T(\mu)$ will denote the smallest tail projection larger than $S(\mu)$.

**Lemma 1.1.** (i). If $\mu \in \Sigma_p$ and $x > 0$, then $x\mu \in \Sigma_p$, where $(x\mu)(A) = x(\mu(A))$ for all $A \in \mathcal{A}$.

(ii). Suppose that $\mu$ is a normal positive functional on $\mathcal{A}$ with $\mu(1) = 1$. Then $\mu \in \Sigma_p$ if and only if the family $(\mathcal{A}_i)_{i \in I}$ is independent with respect to $\mu$: i.e., if and only if

$$u\left(\prod_{i \in F} A_i\right) = \prod_{i \in F} \mu(A_i)$$

for all $A_i \in \mathcal{A}_i$ and all finite subsets $F$ of $I$.

**Proof.** (i) is obvious. Suppose that $\mu$ is a normal positive functional on $\mathcal{A}$ with $\mu(1) = 1$. If (1.1) holds let $\mu_i$ be the restriction of $\mu$ to $\mathcal{A}_i$; then $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$. Suppose, on the other hand, that $\mu \in \Sigma_p$. Then $\mu = \bigotimes_{i \in I} \mu_i$ for normal positive functionals $\mu_i$ on $\mathcal{A}_i$. We have $\mu(1) = \prod_{i \in I} \mu_i(1)$, so that $\mu = \bigotimes_{i \in I} \mu_i'$ where $\mu_i' = (\mu_i(1))^{-1} \mu_i$ and $\mu_i'(1) = 1$. Evidently $\mu_i'$ is the restriction of $\mu$ to $\mathcal{A}_i$, and (1.1) follows.

**Lemma 1.2.** (i) $\mathcal{I} \subseteq \mathcal{Z}$.

(ii) $Z(\mu) \leq T(\mu)$ for all $\mu \in \Sigma_p$.

**Proof.** $\mathcal{I}$ commutes with each $\mathcal{A}_i$ because $\mathcal{I} \subseteq \mathcal{A}_{F^{-1}}$; therefore $\mathcal{I}$ commutes with $\mathcal{A} = R\{\mathcal{A}_i; i \in I\}$.

**Lemma 1.3.** (i) $\mathcal{Z} \supseteq \mathcal{Z}_i$ for each $i \in I$.

(ii) If $\mathcal{A}$ is a factor then each $\mathcal{A}_i$ is a factor.

**Lemma 1.4.** For all $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$:

$$S(\mu) \leq \prod_{i \in I} S(\mu_i)$$

and

$$Z(\mu) \leq \prod_{i \in I} Z(\mu_i)$$

**Proof.**
\[ \mu \left( \prod_{i \in I} S(\mu_i) \right) = \prod_{i \in I} \mu_i(S(\mu_i)) = \prod_{i \in I} \mu_i(1) = \mu(1) . \]

Therefore (1.2) holds. (1.3) holds because \( \prod_{i \in I} Z(\mu_i) \) is a projection of \( \mathcal{A} \) larger than \( \prod_{i \in I} S(\mu_i) \) and, hence, by (1.2), larger than \( S(\mu) \).

**Remark.** The two propositions which follow are stated now for convenience in referring to them later. For the moment, we need only parts (i) and (ii) of Proposition 1.6.

**Proposition 1.5.** Suppose that \( J \) is a subset of \( I \). Then:

(i). \((\mathcal{A}_i)_{i \in I}\) is a factorization of \( \mathcal{A}_J \).

(ii). If \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_\mu \), then the restriction \( \mu' \) of \( \mu \) to \( \mathcal{A}_J \) is a product functional on \( \mathcal{A}_J \) for the factorization \((\mathcal{A}_i)_{i \in J}\), and \( \mu' \) is a scalar multiple of \( \mu_J = \bigotimes_{i \in J} \mu_i \).

(iii). If \( \Sigma \) is a separating subset of \( \Sigma_\mu \), then \( \{\mu_J: \bigotimes_{i \in I} \mu_i \in \Sigma\} \) is separating on \( \mathcal{A}_J \).

(iv). If (III) holds for \((\mathcal{A}_i)_{i \in I}\) then (III) holds for \((\mathcal{A}_i)_{i \in J}\).

(v). If (III) and (IV) hold for \((\mathcal{A}_i)_{i \in I}\), then (IV-\(\mu\)) holds for \((\mathcal{A}_i)_{i \in J}\) for \( \mu \) in a separating subset of product functionals on \( \mathcal{A}_J \).

(vi). If (V) holds for \((\mathcal{A}_i)_{i \in I}\) then (V) holds for \((\mathcal{A}_i)_{i \in J}\).

**Proof.** (i) and (ii) are obvious, (iii) follows from (ii) and (iv) from (iii). To prove (v) observe that (IV-\(\mu_j\)) clearly holds for all \( \mu \in \Sigma_\mu \). To prove (vi) let \( T_J \) be the tail of the factorization \((\mathcal{A}_i)_{i \in J}\). For every finite subset \( F \) of \( I \):

\[ \mathcal{A}_{I-F} \cap \mathcal{A}_J = \mathcal{A}_{I-F} \]

Taking the intersection as \( F \) runs over all finite subsets of \( I \), since \( F \cap J \) runs over all finite subsets of \( J \), we obtain \( T_J \subseteq T \).

**Proposition 1.6.** Suppose that \((I(j))_{j \in J}\) is a mutually disjoint family of subsets of \( I \) whose union is \( I \). Then:

(i). \((\mathcal{A}_{I(j)})_{j \in J}\) is a factorization of \( \mathcal{A}_J \).

(ii). If \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_\mu \) then \( \mu \) is a product functional for the factorization \((\mathcal{A}_{I(j)})_{j \in J}\) and \( \mu' = \bigotimes_{j \in J} (\bigotimes_{i \in I(j)} \mu_i) \).

(iii). If (III) holds for the factorization \((\mathcal{A}_i)_{i \in I}\) then (III) holds for the factorization \((\mathcal{A}_{I(j)})_{j \in J}\).

(iv). If (V) holds for the factorization \((\mathcal{A}_i)_{i \in I}\) then (V) holds for the factorization \((\mathcal{A}_{I(j)})_{j \in J}\).

**Remark.** (IV) holding for \((\mathcal{A}_i)_{i \in J}\) does not necessarily mean that (IV) holds for \((\mathcal{A}_{I(j)})_{j \in J}\); see Example 7.3.
Proposition 1.7. (Zero-one law). For all \( \mu \in \Sigma_p \) with \( \mu(1) = 1 \) and all tail projections \( T \):

\[
\mu(T) = 0 \text{ or } 1.
\]

Proof. Let \( F \) be a finite subset of \( I \). Then \( \mu \) is a product functional for the factorization \( \{ \mathcal{A}_F, \mathcal{A}_{\sim F} \} \) of \( \mathcal{A} \) (Proposition 1.6. (i)) and \( T \in \mathcal{A}_{\sim F} \); therefore (Lemma 1.1), for all \( A \in \mathcal{A}_F \):

\[
(1.4) \quad \mu(AT) = \mu(A)\mu(T).
\]

Now \( \bigcup_F \mathcal{A}_F \) is ultraweakly dense in \( \mathcal{A} \), so (1.4) holds for all \( A \in \mathcal{A} \). Putting \( A = T \in \mathcal{A} \), we obtain:

\[
\mu(T) = (\mu(T))^2.
\]

Corollary 1.8. If \( \mu \in \Sigma_p \) and \( T \) is a tail projection:

\[
\mu(T) \neq 0 \text{ implies } S(\mu) \leq T.
\]

Proposition 1.9. For every \( \mu \in \Sigma_p \), \( T(\mu) \) is an atomic projection of \( \mathcal{T} \).

Proof. Suppose that \( T \) is a projection of \( \mathcal{T} \) with \( 0 \leq T \leq T(\mu) \). Then either \( \mu(T) = 0 \) or \( S(\mu) \leq T \), by Corollary 1.8. If \( S(\mu) \leq T \) then \( T = T(\mu) \) by definition. If \( \mu(T) = 0 \) then \( T \leq 1 - S(\mu) \) and \( T(\mu) - T \geq S(\mu) \); that implies \( T = 0 \).

Corollary 1.10. For all \( \mu, \nu \in \Sigma_p \):

\[
either T(\mu) = T(\nu) \text{ or } [T(\mu)][T(\nu)] = 0.
\]

Corollary 1.11. If condition (III) holds, then \( \mathcal{T} \) is an atomic \( W^* \)-algebra.

Lemma 1.12. Suppose that conditions (III) and (IV) hold and that \( i \in I \). For all \( A_i \in \mathcal{A}_i^+ \) and all \( T \in \mathcal{T}^+ \):

\[
A_i T = 0 \text{ implies } A_i = 0 \text{ or } T = 0.
\]

Proof. Suppose that \( T \neq 0 \). Then because of (III), there exists \( \mu \in \Sigma_p \) with \( \mu(T) \neq 0 \). By Proposition 1.6, \( \{ \mathcal{A}_i, \mathcal{A}_{\sim i} \} \) is a factorization for \( \mathcal{A} \) and \( \mu = \mu_i \otimes \mu' \) is a product functional for this factorization. We have \( T \in \mathcal{A}_{\sim i} \) and \( \mu'(T) \neq 0 \). Now for every nonzero normal positive functional \( \nu_i \) on \( \mathcal{A}_i \), \( \nu_i \otimes \mu' \) exists on \( \mathcal{A} \) by (IV). Hence \( A_i T = 0 \) implies that \( (\nu_i \otimes \mu')(A_i T) = 0 \) or that \( \nu_i(A_i) = 0 \) for each \( \nu_i \). Therefore \( A_i T = 0 \) implies \( A_i = 0 \).
**Definition 1.13.** Suppose that \((\mathcal{A}_i, (\alpha_i))\) is a product for \((\mathcal{A}_i)_{i \in I}\) and that \(\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p\). Let \(E(\mu) = \sup \{S(\nu): \nu \in \Sigma_p\text{ and } \nu = \bigotimes_{i \in I} \nu_i\text{ with } \nu_i = \mu_i \text{ for a.a. } i \in I\}\).

**Remark.** It is clear that \(E(\mu)\) is well defined: i.e., \(E(\mu)\) does not depend on how \(\mu\) is expressed as \(\bigotimes \mu_i\).

**Definition 1.14.** A product \((\mathcal{A}_i, (\alpha_i))\) for \((\mathcal{A}_i)_{i \in I}\) will be said to satisfy \((VI-(\mu_i))\), where each \(\mu_i\) is a normal positive functional on \(\mathcal{A}_i\), if the following conditions hold:

(i). \(\mu = \bigotimes_{i \in I} \mu_i\) exists on \(\mathcal{A}\).

(ii). \((IV-\mu)\) holds.

(iii). \(E(\mu) = 1\).

**Proposition 1.15.** For all \(\mu \in \Sigma_p\):

\[E(\mu) \leq T(\mu).\]

**Proof.** Suppose that \(\nu = \bigotimes_{i \in I} \nu_i \in \Sigma_p\) with \(\nu_i = \mu_i\) for a.a. \(i \in I\). Let \(F = \{i \in I: \nu_i = \mu_i\}\). Then \(F\) is finite so that \(T(\mu) \in \mathcal{A}_{I-F}\). By Proposition 1.6, \(\{\mathcal{A}_F, \mathcal{A}_{I-F}\}\) is a factorization of \(\mathcal{A}\) for which \(\mu\) and \(\nu\) are product functionals: \(\mu = \mu_F \bigotimes \mu'\) and \(\nu = \nu_F \bigotimes \nu'\). Clearly \(\mu' = \nu'\). We have \(0 \neq \nu(T(\mu)) = \mu'(T(\mu))\), so that \(\nu'(T(\mu)) \neq 0\). Hence \(\nu(T(\mu)) = \nu_F(1)\nu'(T(\mu)) = 0\) and by Corollary 1.8, \(S(\nu) \leq T(\mu)\).

Since \(E(\mu)\) is the supremum of such \(S(\nu)\), \(E(\mu) \leq T(\mu)\).

**Proposition 1.16.** Condition \((VI-(\mu_i))\) implies conditions (III), (V) and (IV-\(\nu\)) for \(\nu\) in a separating subset of \(\Sigma_p\).

**Proof.** Evidently \((VI-(\mu_i))\) implies (IV-\(\nu\)) for \(\nu\) in a separating subset of \(\Sigma_p\), and hence (III). That it implies (V) is a consequence of Proposition 1.9 and Proposition 1.15.

**Lemma 1.17.** Suppose that \(Z\) is a projection of \(\mathcal{A}\). Let \(\alpha_i: \mathcal{A}_i \to Z\mathcal{A}\) be defined, for each \(i \in I\), by:

\[\alpha_i(A_i) = ZA_i \quad \text{for all } A_i \in \mathcal{A}_i.\]

Let \(Z_i\) be the support of \(\alpha_i\). Then \((Z, \mathcal{A}_i, (\alpha_i'))\) is a product for \((Z, \mathcal{A}_i)\), where \(\alpha_i'\) denotes the restriction of \(\alpha_i\) to \(Z_i\mathcal{A}_i\). Suppose that \(\mu' = \bigotimes \mu_i'\) is a product functional for \((Z, \mathcal{A}_i, (\alpha_i'))\). Define \(\mu\) on \(\mathcal{A}\) and \(\mu_i\) on \(\mathcal{A}_i\) by:

\[\mu(A) = \mu'(ZA) \quad \text{for all } A \in \mathcal{A};\]

\[\mu_i(A_i) = \mu_i'(Z_iA_i) \quad \text{for all } A_i \in \mathcal{A}_i.\]
Then \( \mu \) is in \( \Sigma_p \), \( S(\mu) = S(\mu') \), and \( \mu = \bigotimes \mu_i \).

**Proof.** Obviously, since \( \alpha'_i(Z_i, \mathcal{A}_i) = \alpha_i(\mathcal{A}_i) = Z_i, (Z, \mathcal{A}, (\alpha_i)) \) is a product for \( (Z_i, \mathcal{A}_i) \). Suppose \( \mu', \mu_i', \mu_i \), and \( \mu_i \) are as in the lemma. Then whenever each \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for a.a. \( i \in I \):

\[
\mu\left( \prod_{i \in I} A_i \right) = \mu'\left( Z \prod_{i \in I} A_i \right) = \mu'\left( \prod_{i \in I} \alpha_i(A_i) \right) \\
= \mu'\left( \prod_{i \in I} \alpha'_i(Z_i A_i) \right) = \prod_{i \in I} \mu'_i(Z_i A_i) = \prod_{i \in I} \mu_i(A_i).
\]

**Proposition 1.18.** Suppose that the factorization \((\mathcal{A}_i)_{i \in I} \) satisfies (III) and (IV), and suppose that \( T \) is a nonzero tail projection. Let \( \alpha_i : \mathcal{A}_i \to T_\mathcal{A}_i \) be defined, for each \( i \in I \), by:

\[
\alpha_i(A_i) = TA_i \quad \text{for all } A_i \in \mathcal{A}_i.
\]

Then:

(i). Each \( \alpha_i \) is an isomorphism and \((T_\mathcal{A}_i, (\alpha_i)) \) is a tensor product for \((\mathcal{A}_i) \) i.e., \((T_\mathcal{A}_i, (\alpha_i)) \) is a product for \((\mathcal{A}_i) \) satisfying (III) and (IV).

(ii). \((T_\mathcal{A}_i, (\alpha_i)) \) is a local tensor product if and only if \( T \) is atomic in \( \mathcal{F} \).

(iii). There is a one-to-one correspondence \( \mu' \leftrightarrow \mu \) between product functionals \( \mu' \) for \((T_\mathcal{A}_i, (\alpha_i)) \) and product functionals \( \mu \) on \( \mathcal{A} \) for \((\mathcal{A}_i) \) with \( S(\mu) \leq T \), where \( \mu' \) is the restriction of \( \mu \) to \( T_\mathcal{A}_i \) and \( \mu(A) = \mu'(TA) \) for all \( A \in \mathcal{A}_i \). We have \( S(\mu) = S(\mu') \) and \( \mu = \bigotimes \mu_i \) if and only if \( \mu' = \bigotimes \mu_i \).

**Proof.** Lemma 1.12 shows that each \( \alpha_i \) is an isomorphism. Then Lemma 1.17 shows both that \((T_\mathcal{A}_i, (\alpha_i)) \) is a product for \((\mathcal{A}_i) \), and also that, if \( \mu' = \bigotimes \mu_i \) is a product functional for \((T_\mathcal{A}_i, (\alpha_i)) \), then the \( \mu \) corresponding to \( \mu' \) is in \( \Sigma_p \), \( \mu = \bigotimes \mu_i \), and \( S(\mu) = S(\mu') \). Suppose that \( \mu = \bigotimes \mu_i \) is in \( \Sigma_p \) with \( S(\mu) \leq T \), and let \( \mu' \) be the restriction of \( \mu \) to \( T_\mathcal{A}_i \). Suppose \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for all \( i \in I - F \) for a finite subset \( F \) of \( I \). Then:

\[
\mu' \left( \prod_{i \in I} \alpha_i(A_i) \right) = \mu \left( \prod_{i \in I} TA_i \right) = \left[ \prod_{i \in F} \mu_i(A_i) \right] \left[ \mu_{I-F}(T) \right]
\]

because \( T \in \mathcal{A}_{I-F} \). Now \( \mu(T) = \mu_F(1)\mu_{I-F}(T) \), and, since \( S(\mu) \leq T \), \( \mu(T) = \mu(1) = \mu_F(1)\mu_{I-F}(1) \). Therefore:

\[
\mu_{I-F}(T) = \mu_{I-F}(1) = \prod_{i \in I-F} \mu_i(1).
\]

Combining (1.5) and (1.6), we conclude that \( \mu = \bigotimes \mu_i \in \Sigma_p \). That completes the proof of (iii).
Since (III) holds for the factorization \((\mathcal{A}_i)\), evidently Corollary 1.8 and (iii) demonstrate that (III) holds for \((T, \mathcal{A}, (\alpha_i))\). To prove (IV) for \((T, \mathcal{A}, (\alpha_i))\), let us assume that \(\mu' = \bigotimes \mu_i\) is a product functional for \((T, \mathcal{A}, (\alpha_i))\) and that \(\nu_i\) is a non-zero normal positive functional on \(\mathcal{A}_i\) with \(\nu_i = \mu_i\) for a.a. \(i \in I\). Let \(\mu\) correspond to \(\mu'\) as in (iii) so that \(\mu = \bigotimes \mu_i\) for \((\mathcal{A}_i)\) and \(S(\mu) \leq T\). Now (IV) holds for \((\mathcal{A}_i)\), so that \(\nu = \bigotimes \nu_i\) exists on \(\mathcal{A}\). We have \(S(\nu) \leq E(\mu) \leq T(\mu) \leq T\) by Propositions 1.15 and 1.9. \(\nu' = \bigotimes \nu_i\) exists as a product functional for \((T, \mathcal{A}, (\alpha_i))\) by (iii). That demonstrates (IV) and thus (i).

Since \(T\) is in \(\mathcal{K}\) and each \(\mathcal{A}_i\), a direct calculation shows that the tail of the product \((T, \mathcal{A}, (\alpha_i))\) is precisely \(T\mathcal{F}\). Hence (V) holds for \((T, \mathcal{A}, (\alpha_i))\) if and only if \(T\) is atomic in \(\mathcal{F}\). That proves (ii).

2. Direct products of von Neumann algebras. We summarize here the definition and same basic properties of the direct product of a family of von Neumann algebras. For details and omitted proofs, see [7] or [1].

Let \(I\) be an arbitrary indexing set. Suppose that \((H_i)_{i \in I}\) is a family of Hilbert spaces and that, for each \(i \in I\), \(x_i\) is in \(H_i\) with \(0 < \prod_{i \in I} \|x_i\| < \infty\). Then we denote by \(\bigotimes_{i \in I} (H_i, x_i)\) von Neumann's incomplete direct product of the family \((H_i)\) with respect to the \(C_0\)-sequence \((x_i)\), (see [7]). Let \(A = \{(y_i)\} : \) each \(y_i \in H_i, \sum |1 - (x_i|y_i)| < \infty\) and \(\sum |1 - \|y_i\|| < \infty\}\). Then there is a natural multilinear mapping \((y_i) \mapsto \bigotimes y_i\) from \(A\) into a dense subset of \(H\) with:

\[(\bigotimes y_i | \bigotimes z_i) = \prod (y_i, z_i)\quad \text{for all} \quad (y_i), (z_i) \in A.
\]

**Lemma 2.1.** Suppose that \(x_i, y_i \in H_i\) with \(0 < \prod \|x_i\|, \prod \|y_i\| < \infty\) and that \(\sum |1 - (x_i|y_i)| < \infty\). Then \(\bigotimes (H_i, x_i) = \bigotimes (H_i, y_i)\).

**Lemma 2.2.** Suppose that, for each \(i \in I\), \(L_i\) is a dense linear subset of \(H_i\) with \(x_i \in L_i\), and suppose that

\[0 < \prod \|x_i\| < \infty.
\]

Let \(L = \{\bigotimes_{i \in I} y_i : y_i \in L_i\ \text{for all} \ i \in I\ \text{and} \ y_i = x_i \ \text{for a.a.} \ i \in I\}\). Then \(L\) is dense in \(\bigotimes_{i \in I} (H_i, x_i)\).

**Lemma 2.3.** Let \(H = \bigotimes_{i \in I} (H_i, x_i)\). Then, for each \(j \in I\), there exists a normal isomorphism \(\alpha_j\) of \(\mathcal{L}(H_j)\) into \(\mathcal{L}(H)\) such that, for all \(A_j \in \mathcal{N}_j\) and all \((y_i) \in A\):

\[(\alpha_j(A_j))(\bigotimes y_i) = \bigotimes y'_i\]

where \(y'_i = y_i\) for \(i \neq j\) and \(y'_j = A_j y_j\). We call \(\alpha_j\) the natural injection of \(\mathcal{L}(H_j)\) into \(\mathcal{L}(H)\).
DEFINITION 2.4. Suppose that, for each \( i \in I \), \( \mathcal{A} \) is a von Neumann algebra on \( H_i \) and \( x_i \in H_i \), and suppose that \( 0 < \prod ||x_i|| < \infty \). Then by \( \bigotimes_{i \in I} (\mathcal{A}, x_i) \) we will mean \( \mathcal{B}(\alpha_i(\mathcal{A}); i \in I) \), where \( \alpha_i \) is the natural injection of \( \mathcal{F}(H_i) \) into \( \mathcal{L}(\bigotimes (H_i, x_i)) \).

LEMMA 2.5. (i). \( \bigotimes_{i \in I} (\mathcal{A}(H_i), x_i) = \mathcal{L}(\bigotimes_{i \in I} (H_i, x_i)) \).

(ii). \( \bigotimes_{i \in I} (\mathcal{A}, x_i) \) is a factor if and only if each \( \mathcal{A} \) is a factor.

PROPOSITION 2.6. Suppose that, for each \( i \in I \), \( \mathcal{A} \) is a von Neumann algebra on \( H_i \) and \( x_i \in H_i \), and suppose that \( 0 < \prod ||x_i|| < \infty \). Let \( \mu_i(A_i) = \langle A_i x_i | x_i \rangle \). Let \( \alpha_i \) be \( \bigotimes_{i \in I} (\mathcal{A}, x_i) \), and let \( \alpha_i \) be the natural injection of \( \mathcal{A} \) into \( \mathcal{A} \) for each \( i \in I \). Then \( (\mathcal{A}, \alpha_i) \) is a product for \( (\mathcal{A}, \alpha_i) \) which satisfies (VI-(\( \mu_i \)). Furthermore, if \( \mu = \bigotimes_{i \in I} \mu_i \), then

\[
S(\mu) = \prod_{i \in I} \alpha_i(S(\mu_i)) .
\]

REMARK. (IV) also holds, of course, and is easily proved directly. See Proposition 4.2.

Proof. Obviously \((\mathcal{A}, \alpha_i)\) is a product for \((\mathcal{A})\) and \( \mu = \bigotimes_{i \in I} \mu_i \) exists in \( \Sigma \); in fact, if \( x = \bigotimes_{i \in I} x_i \) then \( \mu(A) = \langle A x | x \rangle \) for all \( A \in \mathcal{A} \). By Lemma 1.4,

\[
S(\mu) \leq \prod_{i \in I} \alpha_i(S(\mu_i)) .
\]

Now \( S(\mu) = \text{pr} [\mathcal{A}' x] \) (By \([L]\) we mean the closure of \( L \); by \( \text{pr} [L] \) we mean the orthogonal projection onto \([L]\)). Because \( \mathcal{A}' \) contains each \( \alpha_i(\mathcal{A}') \), \([\mathcal{A}' x] \) contains the closure of

\[
\{ \bigotimes A'_i x_i; \text{each } A'_i \in \mathcal{A}' \text{ and } A'_i = 1 \text{ for a.a. } i \in I \} .
\]

Thus (Lemma 2.2) \([\mathcal{A}' x] \) contains \( \bigotimes ([\mathcal{A}' x_i], x_i) \). The projection onto this last subspace of \( H = \bigotimes (H_i, x_i) = \prod \alpha_i(S(\mu_i)) \). Hence \( S(\mu) \geq \prod \alpha_i(S(\mu_i)) \) and (2.1) follows from (2.2).

To prove (VI-(\( \mu_i \)), let us assume first that every normal positive functional on \( \mathcal{A} \) is induced by a vector of \( H_i \). Let

\[
L = \{ \bigotimes y_i; y_i \in H_i, y_i = x_i \text{ for a.a. } i \in I \} .
\]

Then \( L \) is dense in \( H \) by Lemma 2.2. For each nonzero \( y \in L \), let \( \nu_y \) be the functional induced by \( y 
\]

\[
\nu_y(A) = \langle A y | y \rangle \text{ for all } A \in \mathcal{A} .
\]

Then a direct calculation shows that \( \nu_y = \bigotimes \nu_i \) where \( \nu_i \) is induced by \( y_i \) and \( \nu_i = \mu_i \) for a.a. \( i \in I \). We have \( (S(\nu_y))y = y \). Since every
normal positive functional on $\mathcal{A}$ is induced by a vector, as $y$ runs through $L, y_i$ runs through $\Sigma = \{\otimes y_i: y_i = \mu_i \text{ for a.a. } i \in I\}$.

Thus (IV-$\mu$) holds, and

$$E(\mu) = \sup \{S(y): y \in \Sigma\} \geq \text{pr } [L] = 1.$$  

To prove (VI-$\mu_i$) in the general case we will show that there exist von Neumann algebras $\mathcal{B}_i$ on $G_i$ and vectors $z_i \in G_i$, and that there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\otimes (\mathcal{B}_i, z_i)$ such that:

1. Every normal positive functional on $\mathcal{B}_i$ is induced by a vector.

2. $\psi(\alpha_i(\mathcal{A}_i)) = \beta_i(\mathcal{B}_i)$ where $\beta_i$ is the natural injection of $\mathcal{A}(G_i)$ into $\mathcal{L}(G)$ and $G = \otimes (G_i, z_i)$.

3. If $z = \otimes z_i$ then

$$\langle \psi(A)z | z \rangle = \mu(A) \text{ for all } A \in \mathcal{A}.$$

Then by the preceding paragraph (VI-$\mu_i$) will hold for the product $(\mathcal{B}, (\beta_i))$ and thus for the product $(\mathcal{A}, (\alpha_i))$.

For each $i \in I$, let $H'_i$ be a Hilbert space of infinite dimension, let $x'_i \in H'_i$ with $\|x'_i\| = 1$, and let $\mathcal{C}_i$ be the algebra of scalars on $H'_i$. Let $\mathcal{B}_i = \mathcal{A}_i \otimes \mathcal{C}_i$ on $G_i = H_i \otimes H'_i$ and let $z_i = x_i \otimes x'_i$. Let $G = \otimes (G_i, z_i) = \otimes (H_i \otimes H'_i, x_i \otimes x'_i)$ and let $H' = \otimes (H'_i, x'_i)$. Then [7] it is easy to construct a natural isometry $\phi$ from $H \otimes H'$ onto $G$ such that:

$$\phi(\alpha_i(T_i) \otimes 1_{H'})\phi^{-1} = \beta_i(T_i \otimes 1_{H'})$$

for all $T_i \in \mathcal{A}(H_i)$ and all $i \in I$. Define $\psi: \mathcal{A} \rightarrow \mathcal{L}(G)$ by:

$$\psi(A) = \phi(A \otimes 1_{H'})\phi^{-1} \text{ for all } A \in \mathcal{A}.$$

Then (2.4), (2.5), and $\psi(\mathcal{A}) = \mathcal{B}$ follow immediately.

**Corollary 2.7.** Suppose that $(\mathcal{A}_i)_{i \in F}$ is a finite family of von Neumann algebras. Let $\mathcal{A} = \bigotimes_{i \in F} \mathcal{A}_i$ and let $\alpha_i$ be the natural injection of $\mathcal{A}_i$ into $\mathcal{A}$. Then $(\mathcal{A}_i, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)_{i \in F}$ which satisfies (V). In particular $\otimes \mu_i$ exists in $\Sigma$, for every nonzero normal positive functional $\mu_i$ of $\mathcal{A}_i$.

**Lemma 2.8.** Suppose that $(H_i)_{i \in I}$ is a family of Hilbert spaces and that, for each $i \in I, H_i = \bigoplus_{j \in J(i)} H_i^j$ where $0 \in J(i)$ (by $H_i = \bigoplus_{j \in J(i)} H_i^j$ we mean that the $H_i^j$ are mutually orthogonal subspaces of $H_i$ which span $H_i$). Suppose that, for each $i \in I$ and $j \in J(i), x_i^j$
is a nonzero vector of \( H_i \), and suppose that \( 0 < \prod_{i \in I} ||x_i^*|| < \infty \). Denote by \( J \) the set of families \((j(i))_{i \in I}\) with each \( j(i) \in J(i) \) and \( j(i) = 0 \) for a.a. \( i \in I \). If \( j = (j(i)) \in J \) let \( H_j = \bigotimes_{i \in I} (H_i^{j(i)}, x_i^{j(i)}). \) Then each \( H_j \) is a subspace of \( H = \bigoplus_{j \in J} H_j. \) Furthermore, if \( \alpha_j \) denotes, for each \( j = (j(i)) \in J \), the natural injection of \( \mathcal{L}(H_j^{j(i)}) \) into \( \mathcal{L}(H_j) \), then:

\[
(2.6) \quad \alpha_j(\bigoplus_{i \in J(i)} T_i^j) = \bigoplus_{i \in J(j(i))} [\alpha_i^j(T_i^j)] \quad \text{for all } (T_i^j)_{j \in J}, \quad \text{with each } T_i^j \in \mathcal{L}(H_j^i). \quad \text{(Here } \bigoplus \text{ maps } T_i^j : \otimes x_i^j \rightarrow \bigoplus T_i^j x_i^j).\]

**Proof.** The \( H_j \) are clearly mutually orthogonal, and \( [H_j ; j \in J] \) is \( H \) by Lemma 2.2. Formula (2.6) can be confirmed by a direct calculation.

3. The basic isomorphism theorems. By a representation \( \psi \) of a \( W^* \)-algebra \( \mathcal{A} \) on a Hilbert space \( H \) we mean a normal homomorphism \( \mathcal{A} \rightarrow \mathcal{B}(H) \) onto a von Neumann algebra on \( H \) (Notice that \( \psi(1) \) is the identity on \( H \)). If \( \psi \) is a representation of \( \mathcal{A} \) on \( H \) and \( \mu \) is a normal positive functional on \( \mathcal{A} \), a vector \( x \in H \) will be called a \( \mu \)-cyclic vector for \( \psi \) if \([\psi(\mathcal{A})x] = H \) and

\[
\mu(A) = \langle \psi(A)x | x \rangle \quad \text{for all } A \in \mathcal{A}. \]

Given \( \mathcal{A} \) and \( \mu \) it is well known (see [3, p. 51], for example) that a representation \( \psi \) with a \( \mu \)-cyclic vector exists (and is essentially unique), and that such a \( \psi \) acts isomorphically on \( (Z(\mu))_{\mathcal{A}} \) and takes \((1 - Z(\mu))_{\mathcal{A}} \) into 0.

**PROPOSITION 3.1.** Suppose that \((\mathcal{A}_i)_{i \in I}\) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \) and that \( \mu = \bigotimes_{i \in I} \mu_i \) is a product functional for this factorization. Suppose that \( \psi \) is a representation of \( \mathcal{A} \) on \( H \) with \( \mu \)-cyclic vector \( x \). Suppose that, for each \( i \in I \), \( \psi_i \) is a representation of \( \mathcal{A}_i \) on \( H_i \) with \( \mu_i \)-cyclic vector \( x_i \). Then there exists an isometry \( \phi \) of \( H \) onto \( \bigotimes_{i \in I} (H_i, x_i) \) such that:

(i). \( \phi(x) = \bigotimes_{i \in I} x_i. \)

(ii). \( \phi(\psi(\mathcal{A}))\phi^{-1} = \bigotimes_{i \in I} (\psi_i(\mathcal{A}_i), x_i). \)

(iii). For all \( A_i \in \mathcal{A}_i \) and each \( i \in I \):

\[
\phi(\psi(A_i))\phi^{-1} = \alpha_i(\psi_\mathcal{A}(A_i)) \]

where \( \alpha_i \) denotes the natural injection of \( \mathcal{L}(H_i) \) into \( \mathcal{L}(\bigotimes_{i \in I} (H_i, x_i)). \)

**Proof.** Let \( \mathcal{K} \) denote the set of families \((A_i)_{i \in I}\) with each \( A_i \in \mathcal{A}_i \) and \( A_i = 1 \) for a.a. \( i \in I \). Let

\[
M = \left\{ \left[ \psi\left( \prod_{i \in I} A_i \right) \right] : (A_i) \in \mathcal{K} \right\}
\]
First we claim that $M$ is a dense subset of $H$. For $\mathcal{A}$ the $*$-algebra $\{\prod_{i \in I} A_i : (A_i) \in \mathcal{A}\}$, is ultrastrongly dense in $\mathcal{A}$ (a corollary of the double-commutant theorem); hence $\psi(\mathcal{A})$ is strongly dense in $\psi(\mathcal{A})$ and $[\psi(\mathcal{A})] = [\psi(\mathcal{A})] = H$ because $x$ is a cyclic vector for $\psi(\mathcal{A})$.

Secondly, $N$ is a dense subset of $\bigotimes_{i \in I} (H_i, x_i)$ by Lemma 2.2, since $x_i \in [\psi_i(\mathcal{A}_i)x_i] = H_i$ for each $i \in I$.

Fix $(A_i) \in \mathcal{A}$. Then:

$$\left\| \left( \psi \left( \prod_{i \in I} A_i \right) \right) x \right\| = \mu \left( \prod_{i \in I} A_i^* A_i \right) = \prod_{i \in I} \mu_i (A_i^* A_i) \left\| \bigotimes_{i \in I} \left( \psi_i (A_i) x_i \right) \right\| = \prod_{i \in I} \left\| \psi_i (A_i) x_i \right\|^2 = \prod_{i \in I} \mu_i (A_i^* A_i).$$

Therefore, since $M$ is dense in $H$ and $N$ is dense in $\bigotimes_{i \in I} (H_i, x_i)$, there exists a (unique) isometry $\phi$ of $H$ onto $\bigotimes_{i \in I} (H_i, x_i)$ such that, for all $(A_i) \in \mathcal{A}$:

$$\phi \left[ \left( \prod_{i \in I} A_i \right) x \right] = \bigotimes_{i \in I} \left( \psi_i (A_i) x_i \right).$$

Now (i) follows immediately, (iii) by a direct calculation, and (ii) from (iii).

**Theorem 3.2.** Suppose that $(\mathcal{A}_i)_{i \in I}$ is a factorization of the $W^*$-algebra $\mathcal{A}$. Suppose that $\mu = \bigotimes_{i \in I} \mu_i$ is a product functional for this factorization, and suppose that (IV-$\mu$) holds. Then there exist, for each $i \in I$, a faithful representation $\mathcal{A}_i$ of $\mathcal{A}_i$ on $H_i$ and a vector $x_i \in H_i$, and there exists a representation $\psi$ of $\mathcal{A}$ on $H = \bigotimes_{i \in I} (H_i, x_i)$ such that:

(i) $\psi$ maps $(1 - E(\mu))\mathcal{A}$ into 0 and maps $(E(\mu))\mathcal{A}$ isomorphically onto $\psi(\mathcal{A}) = \bigotimes_{i \in I} (\mathcal{A}_i, x_i)$.

(ii) For each $i \in I$ and all $A_i \in \mathcal{A}_i$:

$$\psi(A_i) = \alpha_i(A_i),$$

where $\alpha_i$ denotes the natural injection of $\mathcal{A}(H_i)$ into $\mathcal{L}(H)$.

(iii). For each $i \in I$ and all $A_i \in \mathcal{A}_i$:

$$((\mathcal{A}_i(A_i)) x_i | x_i) = \mu_i (A_i).$$

(iv). If $x$ denotes $\bigotimes_{i \in I} x_i$, then, for all $A \in \mathcal{A}$:

$$((\psi(A)) x | x) = \mu(A).$$
Proof. For each $i \in I$, select (by Zorn’s lemma) a family $(\mu^i)_{j \in J(i)}$ of normal nonzero positive functionals on $\mathcal{A}_i$ such that $\sum_{j \in J(i)} Z(\mu^i) = 1$ and $0 \in J(i)$ with $\mu^i_0 = \mu_i$. Let $J$ be the subset of $\Pi_{i \in I} J(i)$ consisting of $(j(i))$ with $j(i) = 0$ for a.a. $i \in I$. Since $(IV-\mu)$ holds, each $j = (j(i)) \in J$ the product functional $\mu^i = \bigotimes_{i \in I} \mu^i_{j(i)}$ exists on $\mathcal{A}_i$. We have $Z(\mu^i) \leq \Pi_{i \in I} Z(\mu^i_{j(i)})$ by Lemma 1.4, so that $(Z(\mu^i))_{j \in J}$ is a mutually orthogonal family of central projections of $\mathcal{A}$. Let $Z = \sum_{j \in J} Z(\mu^i)$.

For each $j \in J$ let $\Gamma^j$ be a representation of $\mathcal{A}$ on $G^j$ with a $\mu^j$-cyclic vector $y^j$. Let $\Gamma$ be the direct sum representation $\bigoplus_{j \in J} \Gamma^j$ of $\mathcal{A}$ on $G = \bigoplus_{j \in J} G^j$.

\[(3.1) \quad \Gamma^j(A) = \bigoplus_{j \in J} \Gamma^j(A) \quad \text{for all } A \in \mathcal{A}_i.\]

Then $\Gamma$ maps $(1 - Z)\mathcal{A}$ into 0 and maps $Z\mathcal{A}$ isomorphically onto $\Gamma(\mathcal{A})$.

For each $i \in I$ and each $j \in J(i)$, let $\Delta^i_j$ be a representation of $\mathcal{A}_i$ on $H_i^j$ with $\mu_i^j$-cyclic vector $x_i^j$. Let $\Delta_i$ be the direct sum representation $\bigoplus_{j \in J(i)} \Delta^i_j$ of $\mathcal{A}_i$ on $H_i = \bigoplus_{j \in J(i)} H_i^j$.

\[(3.2) \quad \Delta_i(A_i) = \bigoplus_{j \in J(i)} \Delta^i_j(A_i) \quad \text{for all } A_i \in \mathcal{A}_i.\]

Then each $\Delta_i$ is faithful.

Fix $j = (j(i))$ in $J$. We know that $\mu^j = \bigotimes_{i \in I} \mu^i_{j(i)}$, that $\Gamma^j$ is a representation of $\mathcal{A}$ on $G^j$ with $\mu^j$-cyclic vector $y^j$, and that $\Delta^i_{j(i)}$, for each $i \in I$, is a representation of $\mathcal{A}_i$ on $H^j_{i(i)}$-cyclic vector $x^j_{i(i)}$. Therefore Proposition 3.1 demonstrates the existence of an isometry $\phi^j$ from $G^j$ onto $H^j = \bigotimes_{i \in I} (H^j_{i(i)}, x^j_{i(i)})$ such that:

\[(3.3) \quad \phi^j(y^j) = \bigotimes_{i \in I} x^j_{i(i)}\]

and

\[(3.4) \quad \phi^j(\Gamma^j(A_i))(\phi^j)^{-1} = \alpha_i(\Delta^j_{i(i)}(A_i)) \quad \text{for all } A_i \in \mathcal{A}_i,\]

where $\alpha_i$ denotes the natural injection of $L(H^j_{i(i)})$ into $L(H^j)$.

Let $x_i^j$ denote $x^j_{i(i)}$ for each $i \in I$. Let $H = \bigoplus_{i \in I} (H_i, x_i)$, and denote by $\alpha_i$ the natural injection of $L(H_i)$ into $L(H)$. Then (Lemma 2.8), $H = \bigoplus_{j \in J} H^j$, and, for each $i \in I$ and all operators $T_i \in L(H_i)$ with $T_i = \bigoplus_{j \in J(i)} T_i^j$ and with each $T_i^j \in L(H_i^j)$:

\[(3.5) \quad \alpha_i(T_i) = \bigoplus_{j = (j(i)) \in J} (\alpha_i(T^j_{i(i)})).\]

Define the isometry $\phi$ of $G$ onto $H$ by:

\[\phi\left(\bigoplus_{j \in J} f^j\right) = \bigoplus_{j \in J} \phi^j(f^j) \quad \text{for all } f^j \in G^j.\]

Let $\psi$ be the representation of $\mathcal{A}$ on $H$ defined by:
\[ \psi(A) = \phi(\Gamma(A))\varphi^{-1} \quad \text{for all } A \in \mathcal{A}. \]

Evidently \( \psi \) has the same kernel as \( \Gamma \): \( \psi \) maps \((1 - Z)\mathcal{A}\) into 0 and \( Z\mathcal{A} \) isomorphically onto \( \psi(\mathcal{A}) \).

Now fix \( i \in I \) and \( A_i \in \mathcal{A}_i \). In view of (3.2), applying (3.5) to \( A_i \) we obtain:

\[ \alpha_i(\mathcal{A}_i) = \bigoplus_{j = (j_i)} [\alpha_i(\mathcal{A}_j)] . \]

Using (3.1), the definitions of \( \psi \) and \( \varphi \), and (3.4), we get:

\[ \psi(A_i) = \phi\left[ \bigoplus_{j \in J} \Gamma^i(A_i) \right] \varphi^{-1} = \bigoplus_{j \in J} \phi^i(\Gamma^i(A_i))\varphi^{-1} \]

\[ = \bigoplus_{j = (j_i)} [\alpha_i(\mathcal{A}_j)] . \]

We conclude, from (3.6) and (3.7), that:

\[ \psi(A_i) = \alpha_i(\mathcal{A}_i) \quad \text{for all } A_i \in \mathcal{A}_i \text{ and all } i \in I. \]

Hence \( \psi \) maps \( \mathcal{A} = \mathcal{B}(\mathcal{A}_i; i \in I) \) onto

\[ \mathcal{B}(\alpha_i(\mathcal{A}_i); i \in I) = \bigotimes_{i \in I} (\mathcal{A}_i, x_i). \]

Assertion (ii) of the theorem is precisely (3.8). (iii) holds because \( x_i = x_i^i \) is a \( \mu \)-cyclic vector for \( \mathcal{A}_i \). (iv) holds because of (3.4) and the choice of \( \psi \) to be a \( \mu \)-cyclic vector for \( \Gamma^o \). To complete the proof of the theorem, then, we need to show only that \( Z = E(\mu) \).

Evidently \( Z \leq E(\mu) \). Let \( \beta_i : \mathcal{A}_i \to Z\mathcal{A}_i \) be defined by \( \beta_i(A_i) = ZA_i \) for all \( A_i \in \mathcal{A}_i \). Then we have just proved that \( (Z, \mathcal{A}_i, (\beta_i)) \) is isomorphic to the product \( (\bigotimes (\mathcal{A}_i, x_i), (\alpha_i, \cdot, \cdot)) \), which satisfies (VI-\( \mu \)) by Proposition 2.6. Hence (\( Z, \mathcal{A}_i, (\beta_i) \)) is a product for \( (\mathcal{A}_i) \) which satisfies (VI-\( \mu \)). Now suppose that each \( \nu_i \) is a nonzero normal positive functional on \( \mathcal{A}_i \) and that \( \nu_i = \mu_i \) for a.a. \( i \in I \). Then \( \nu' = \bigotimes \nu_i \) exists as a product functional for \( (Z, \mathcal{A}_i, (\beta_i)) \). Hence, by Lemma 1.17, \( \nu = \bigotimes \nu_i \) exists in \( \Sigma_\mu \) with \( S(\nu) = S(\nu') \leq Z \). Since \( E(\mu) \) is the supremum of such \( S(\nu) \), \( E(\mu) \leq Z \). This completes the proof.

**Corollary 3.3.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a factorization of the \( W^* \)-algebra \( \mathcal{A} = \bigotimes_{i \in I} \mathcal{A}_i \), that \( \mu = \bigotimes_{i \in I} \mu_i \) is a product functional for this factorization, and that (IV-\( \mu \)) holds. Then

\[ S(\mu) = [E(\mu)] \prod_{i \in I} S(\mu_i). \]

**Proof.** Use Theorem 3.2 and (2.1) of Proposition 2.6.

**Corollary 3.4.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a family of \( W^* \)-algebras,
and that, for each \( i \in I \), \( \mu_i \) is a normal positive functional of \( \mathcal{A} \). Suppose that \( (\mathcal{A}, (\alpha_i)) \) and \( (\mathcal{B}, (\beta_i)) \) are products for \( (\mathcal{A}) \) which satisfy (VI-(\( \mu_i \))). Then \( (\mathcal{A}, (\alpha_i)) \) and \( (\mathcal{B}, (\beta_i)) \) are isomorphic: i.e., there exists an isomorphism \( \psi \) of \( \mathcal{A} \) onto \( \mathcal{B} \) such that \( \psi \circ \alpha_i = \beta_i \) for all \( i \in I \).

**Corollary 3.5.** Suppose that, for each \( i \in I \), \( \mathcal{A}_i \) and \( \mathcal{B}_i \) are von Neumann algebras on \( H_i \) and \( G_i \) respectively, that \( x_i \in H_i \) and \( y_i \in G_i \) with

\[
0 < \prod \| x_i \|, \quad \prod \| y_i \| < \infty,
\]

and that \( \psi_i \) is an isomorphism of \( \mathcal{A}_i \) onto \( \mathcal{B}_i \) such that:

\[
(\psi_i(A_i))y_i | y_i = (A_i x_i | x_i) \quad \text{for all } A_i \in \mathcal{A}_i.
\]

Then there exists an isomorphism \( \psi \) of \( \mathcal{A} = \bigotimes (\mathcal{A}_i, x_i) \) onto \( \mathcal{B} = \bigotimes (\mathcal{B}_i, y_i) \) such that \( \psi \circ \alpha_i = \beta_i \circ \psi_i \) for each \( i \in I \), where \( \alpha_i \) is the natural injection of \( \mathcal{A}_i \) into \( \mathcal{A} \) and \( \beta_i \) is the natural injection of \( \mathcal{B}_i \) into \( \mathcal{B} \).

**Proof.** Use Corollary 3.4 and Proposition 2.6.

**Theorem 3.6.** Suppose that \( (\mathcal{A}_i)_{i \in F} \) is a finite family of \( W^* \)-algebras. Suppose that \( (\mathcal{A}, (\alpha_i)) \) is a product for \( (\mathcal{A}_i)_{i \in F} \) satisfying (III) and (IV-\( \mu \)) for some product functional \( \mu \). Then there exists an isomorphism \( \psi \) of \( \mathcal{A} \) onto \( \bigotimes_{i \in F} \mathcal{A}_i \) such that:

\[
\psi \left( \prod_{i \in F} \alpha_i(A_i) \right) = \bigotimes_{i \in F} A_i \quad \text{for all } A_i \in \mathcal{A}_i.
\]

(We write \( \bigotimes_{i \in F} A_i \) for \( \prod_{i \in F} \lambda_i(A_i) \), where \( \lambda_i \) is the natural injection of \( \mathcal{A}_i \) into \( \bigotimes_{i \in F} \mathcal{A}_i \).) Furthermore, for every product functional \( \nu = \bigotimes \nu_i \) for \( (\mathcal{A}_i, (\alpha_i)) \):

\[
S(\nu) = \prod_{i \in F} \alpha_i(S(\nu_i)).
\]

**Proof.** If \( \mu = \bigotimes_{i \in F} \mu_i \), \( E(\mu) = 1 \) because (III) holds and \( F \) is finite. Hence (VI-(\( \mu \))) holds, and Corollary 3.4 and Proposition 2.6 complete the proof.

4. Local tensor products.

**Lemma 4.1.** Suppose that \( (\mathcal{A}_i)_{i=1}^N \) is a factorization of the \( W^* \)-algebra \( \mathcal{A} \), and that \( \mu_i \) is a normal positive functional on \( \mathcal{A}_i \). Let \( \Sigma_i \) be the set of normal positive functionals \( \mu_i \) on \( \mathcal{A}_i \) such that
\( \mu_1 \otimes \mu_2 \) exists as a product functional on \( \mathcal{A} \) for the factorization \( (\mathcal{A}_i) \).

Then:

(i). If \( \mu_i \in \Sigma_i \) and \( x > 0 \), then \( x \mu_i \in \Sigma_i \).

(ii). If \( \mu_1^\ast \in \Sigma_1 \) and \( \sum_i \mu_i^\ast(1) < \infty \), then \( \sum_i \mu_i^\ast \in \Sigma_i \).

(iii). If \( \mu_i \in \Sigma_i \) and \( A_i \in \mathcal{A}_i \), with \( \mu_i(A_i^\ast A_i) \neq 0 \), then \( (\mu_i)_{A_i} \in \Sigma_i \)

(iv). If \( \nu_i \) is a nonzero normal positive functional on \( \mathcal{A}_i \) with \( S(\nu_i) \leq Z(\mu_i) \) and \( \mu_i \in \Sigma_i \), then \( \nu_i \in \Sigma_i \).

(v). If \( \Sigma_i \) is separating then \( \Sigma_i \) is the set of all nonzero normal positive functionals on \( \mathcal{A}_i \).

**Proof.** (i), (ii), and (iii) are obvious by direct calculation. To prove (iv) suppose that \( \nu_i \) is a normal positive functional on \( \mathcal{A}_i \) and that \( S(\nu_i) \leq Z(\mu_i) \) with \( \mu_i \in \Sigma_i \). Then, by Proposition 3.1, there exists a normal homomorphism \( \psi \) from \( \mathcal{A} \) onto \((Z(\mu_i))_i \otimes (Z(\mu_2))_{\mathcal{A}_2}\) such that:

\[
\psi(A_1 A_2) = (\nu_i(A_1)) \otimes (Z(\mu_2)A_2)
\]

for all \( A_1 \in \mathcal{A}_1 \) and \( A_2 \in \mathcal{A}_2 \).

Now, since \( S(\nu_i) \leq Z(\mu_i) \), by Corollary 2.7 there exists a normal positive functional \( \omega = \nu_i \otimes \mu_2 \) on \((Z(\mu_1))_i \otimes (Z(\mu_2))_{\mathcal{A}_2}\) such that:

\[
\omega((Z(\mu_i)A_1) \otimes (Z(\mu_2)A_2)) = (\nu_i(A_1))(\mu_2(A_2))
\]

for all \( A_1 \in \mathcal{A}_1 \) and \( A_2 \in \mathcal{A}_2 \).

Evidently \( \omega \circ \psi \) equals \( \nu_i \otimes \mu_2 \in \Sigma_p \) and \( \nu_i \in \Sigma_i \).

To prove (v) assume that \( \Sigma_i \) is separating. Then, using (iii) and Zorn's lemma, we can choose a family \( (\mu_i^\ast)_{j \in J} \) with each \( \mu_i^\ast \in \Sigma_i \) and \( \sum_{j \in J} Z(\mu_i^\ast) = 1 \). Suppose that \( \nu_i \) is a normal state of \( \mathcal{A}_i \). Then \( \sum \nu_j(Z(\mu_i^\ast)) < \infty \) and therefore \( \nu_j(Z(\mu_i^\ast)) = 0 \) for all but a countable number of \( j \in J \). Hence a suitable countable linear combination \( \mu_i \) of the \( \mu_i^\ast \) satisfies \( S(\nu_i) \leq Z(\mu_i) \) and \( \mu_i \in \Sigma_i \) by (ii). Then \( \nu_i \in \Sigma_i \) by (iv).

**Remark.** Lemma 4.1 may be proved directly (without using Proposition 3.1 or properties of the tensor product) by using Sakai's Radon-Nikodým theorem [8] and the weak Radon-Nikodým type result of [5, p. 211].

**Proposition 4.2.** Suppose that \( (\mathcal{A}_i)_{i \in I} \) is a factorization of the \( W^\ast \)-algebra \( \mathcal{A} \). Let \( \Sigma_{IV} \) be the set of product functionals on \( \mathcal{A} \) for which (IV-\( \mu \)) holds, and suppose that \( \Sigma_{IV} \) is separating. Suppose that \( F \) is a finite subset of \( I \). Then there exists an isomorphism \( \psi \) of \( \mathcal{A} \) onto \( \mathcal{A}_F \otimes \mathcal{A}_{-F} \) such that:

\[
\psi(AB) = A \otimes B \quad \text{for all } A \in \mathcal{A}_F \text{ and } B \in \mathcal{A}_{-F}.
\]
Proof. By Proposition 1.6 \( \{\mathcal{A}_f, \mathcal{A}_{-f}\} \) is a factorization of \( \mathcal{A} \) and each \( \mu \in \Sigma_{Y} \) is a product functional for this factorization: \( \mu = \mu_f \otimes \mu_{i-f} \). Let \( \Sigma_1 = \{\mu_{i-f}: \mu \in \Sigma_Y\} \) and let \( \Sigma_i \) be the set of product functionals on \( \mathcal{A}_f \) for the factorization \( (\mathcal{A}_i)_{i \in I} \). By Proposition 1.5. (iii), \( \Sigma_1 \) is separating on \( \mathcal{A}_f \) and \( \Sigma_2 \) is separating on \( \mathcal{A}_{-f} \). Because (IV-\( \mu \)) holds for all \( \mu \in \Sigma_Y, \nu \otimes \mu_{i-f} \) exists on \( \mathcal{A} \) for all \( \nu \in \Sigma_1 \) and all \( \mu_{i-f} \in \Sigma_2 \). Hence, by Lemma 4.1 (\( \nu \)), \( \nu \otimes \omega \) exists on \( \mathcal{A} \) for all \( \nu \in \Sigma_1 \) and all nonzero normal positive functionals \( \omega \) on \( \mathcal{A}_{-f} \); and, from there, by the same lemma, \( \nu \otimes \omega \) exists on \( \mathcal{A} \) for all nonzero normal positive functionals \( \nu \) of \( \mathcal{A}_f \) and \( \omega \) of \( \mathcal{A}_{-f} \). Thus (IV) holds for the factorization \( \{\mathcal{A}_f, \mathcal{A}_{-f}\} \). (III) obviously holds for \( (\mathcal{A}_i) \) and thus for \( \{\mathcal{A}_f, \mathcal{A}_{-f}\} \) (Proposition 1.6. (iii)). Now Theorem 3.6 completes the proof.

Remark. Proposition 4.2 is false if the hypothesis that \( F \) be finite is omitted (see Example 7.3).

Corollary 4.3. If \( \Sigma_Y \) is separating then (III) and (IV) hold.

Proof. That (III) holds is obvious. To prove (IV) use Proposition 4.2 and Corollary 2.7.

Corollary 4.4. If a product \( (\mathcal{A}_i, (\alpha_i)) \) satisfies (VI-\( \mu_i \)), then it satisfies (III), (IV), and (V): i.e., it is a \( (\mu_i) \)-local tensor product.

Proof. Use Proposition 1.16 and Proposition 4.2.

Proposition 4.5. Suppose that \( (\mathcal{A}_i, (\alpha_i)) \) is a tensor product for \( (\mathcal{A}_i)_{i \in I} \); i.e., that (III) and (IV) hold. Then, for all \( \mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p \):

\[
E(\mu) = T(\mu)
\]

and

\[
S(\mu) = [T(\mu)] \prod_{i \in I} \alpha_i(S(\mu_i)).
\]

Proof. \( E(\mu) \leq T(\mu) \) by Proposition 1.15. To prove (4.1), then, it suffices to prove that \( E(\mu) \) is tail. Let \( \Sigma = \{\nu \in \Sigma_p: \nu = \bigotimes \nu_i \text{ for a.a. } i \in I\} \). Then \( E(\mu) = \sup \{S(\nu): \nu \in \Sigma\} \). Suppose that \( F \) is a finite subset of \( I \). Then \( \{\mathcal{A}_f, \mathcal{A}_{-f}\} \) is a factorization of \( \mathcal{A} \) for which each \( \nu \in \Sigma \) is a product functional: \( \nu = \nu_f \otimes \nu_{i-f} \). By Proposition 4.2 and (2.1) of Proposition 2.6, for all \( \nu \in \Sigma \):

\[
S(\nu) = [S(\nu_f)][S(\nu_{i-f})].
\]

Thus:
Because of (IV), for fixed $\nu_{I-F}$, $\nu_F$ runs through all product functionals for $(\mathcal{A}_F, (\alpha_i)_{i \in F})$ and therefore (Proposition 1.5. (iv)):

$$\sup \{S(\nu_F): \nu \in \Sigma, \nu_{I-F} \text{ fixed} \} = 1.$$  

Using (4.3), we obtain:

$$E(\mu) \geq \sup \{S(\nu_{I-F}): \nu \in \Sigma \}. \quad (4.5)$$

Now (4.4) and (4.5) show that $E(\mu) \in \mathcal{A}_{I-F}$. Since $F$ was an arbitrary finite subset of $I$, we have shown that $E(\mu)$ is tail. That proves (4.1).

(4.2) follows from (4.1) and Corollary 3.3.

**Corollary 4.6.** A product $(\mathcal{A}, (\alpha_i))$ for $(\mathcal{A})$ is a $(\mu_i)$-local tensor product for $(\mathcal{A})$ if and only if (VI-(\mu_i)) holds.

**Proof.** Corollary 4.4 shows that (VI-(\mu_i)) is sufficient. Suppose that $(\mathcal{A}, (\alpha_i))$ is a $(\mu_i)$-local tensor product for $(\mathcal{A})$. Let $\mu = \bigotimes \mu_i$. Then (IV-\mu) holds because (IV) does, and, using Proposition 4.5 and (V), we see that $E(\mu) = T(\mu) = 1$.

**Theorem 4.7.** Suppose that, for each $i \in I$, $\mathcal{A}_i$ is a $W^{*}$-algebra and $\mu_i$ is a normal positive functional on $\mathcal{A}_i$. Suppose that $0 < \prod_{i \in I} \mu_i(1) < \infty$. Then a $(\mu_i)$-local tensor product exists and is unique up to isomorphism.

**Proof.** Proposition 2.6, Corollary 3.4, and Corollary 4.6.

5. Tensor products. Throughout this section we suppose that $(\mathcal{A}_i)_{i \in I}$ is a family of $W^{*}$-algebras. Let $A$ be the set of families $(\mu_i)_{i \in I}$, each $\mu_i$ a normal positive functional on $\mathcal{A}_i$ and

$$0 < \prod_{i \in I} \mu_i(1) < \infty.$$ 

Let the relation $R$ on $A$ be defined by writing $(\mu_i) \sim (\nu_i) \pmod{R}$ to mean that a $(\mu_i)$-local tensor product for $(\mathcal{A}_i)_{i \in I}$ is necessarily a $(\nu_i)$-local tensor product for $(\mathcal{A}_i)_{i \in I}$. $R$ is a well defined equivalence relation because a $(\mu_i)$-local tensor product exists and is unique up to isomorphism. The following lemma is a trivial consequence of the definition of $(\mu_i)$-local tensor product.

**Lemma 5.1.** Let $(\mathcal{A}, (\alpha_i))$ be a $(\mu_i)$-local tensor product and let $(\nu_i) \in A$. Then $(\mu_i) \sim (\nu_i) \pmod{R}$ if and only if $\bigotimes \nu_i$ exists on $(\mathcal{A}, (\alpha_i))$. 
REMARK. If \( \sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty \) then \((\mu_i) \sim (\nu_i)\), and the converse holds provided each \( \mathcal{A}_i \) is semi-finite [2].

**Lemma 5.2.** If \((\mu_i)\) and \((c_i\mu_i)\) are in \( A \) (where the \( c_i \) are positive real numbers), then \((\mu_i) \sim (c_i\mu_i)\).

**Proof.** Since \( \prod \mu_i(1) \) and \( \prod c_i\mu_i(1) \) both converge to a nonzero number, so must \( \prod c_i \) converge to \( c \neq 0 \). If \( \mu = \bigotimes \mu_i \) exists as a product for \((\mathcal{A}_i)\), \( c\mu \) is a product state equal to \( \bigotimes (c_i\mu_i) \) by direct calculation.

**Remark.** This lemma shows that we could, without loss of generality, confine ourselves to \((\mu_i)\) with each \( \mu_i(1) = 1 \).

Define \( A \) to be the quotient set \( \Lambda/R \) and let \( \varphi \) be the quotient map \( A \to A/R = \Delta \).

**Definition 5.3.** A tensor product \((\mathcal{A}, (\alpha_i))\) for \((\mathcal{A}_i)\) will be called a \( \Gamma \)-tensor product for \((\mathcal{A}_i)\) when:

\[ \{ (\mu_i) \in A : \bigotimes \mu_i \text{ exists on } (\mathcal{A}, (\alpha_i)) \} = \varphi^{-1}(\Gamma) \]

**Lemma 5.4.** Let \( \gamma = \varphi((\mu_i)) \). Then a \((\mu_i)\)-local tensor product is a \( \{\gamma\} \)-tensor product.

**Theorem 5.5.** Suppose that \((\mathcal{A}, (\alpha_i))\) is a tensor product for \((\mathcal{A}_i)_{i \in I}\). Let

\[ \Gamma = \varphi((\mu_i) \in A : \bigotimes \mu_i \text{ exists on } (\mathcal{A}, (\alpha_i))) \]

Then:

(i) \((\mathcal{A}, (\alpha_i))\) is a \( \Gamma \)-tensor product for \((\mathcal{A}_i)\).

(ii) If \( \mu = \bigotimes \mu_i \) and \( \nu = \bigotimes \nu_i \) are product functionals for \((\mathcal{A}, (\alpha_i))\) then:

\[ T(\mu) = T(\nu) \text{ if and only if } (\mu_i) \sim (\nu_i) \]

and

\[ [T(\mu)][T(\nu)] = 0 \]

otherwise.

(iii) If \( \mu = \bigotimes \mu_i \) is a product functional for \((\mathcal{A}, (\alpha_i))\) then:

\[ S(\mu) = [T(\mu)] \prod_{i \in I} \alpha_i(S(\mu_i)) \tag{5.1} \]

\[ T(\mu) = \sup \{ S(\bigotimes \nu_i) : (\nu_i) \sim (\mu_i) \} \tag{5.2} \]
(iv). \((N, (\alpha_i))\) is a \((\mu_i)\)-local tensor product if and only if \(\Gamma = \{\varphi((\mu_i))\}\).

(v). For each \(\gamma \in \Gamma\), define \(T(\gamma)\) to be \(T(\mu)\) for \(\mu = \bigotimes \mu_i\) and \(\varphi((\mu_i)) = \gamma\). Let \(N(\gamma) = [T(\gamma)]N\) and let \(\alpha_i(\gamma)\) be defined by:

\[
(\alpha_i(\gamma))(A_i) = \left[\Gamma(\gamma)\right][\alpha_i(A_i)]
\]

for all \(A_i \in N\) and all \(i \in I\).

Then, for each \(\gamma \in \Gamma\), \((N, (\alpha_i))\) is a \((\mu_i)\)-local tensor product for \((N)\) provided that \(\gamma = \varphi((\mu_i))\).

Furthermore:

\[N = \bigoplus_{\gamma \in \Gamma} N(\gamma)\]

and \(\alpha_i = \bigoplus_{\gamma \in \Gamma} \alpha_i(\gamma)\) for all \(i \in I\),

with respect to the same direct sum decomposition of \(N\).

Proof. Suppose that \(\mu = \bigotimes \mu_i\) is a product functional on \((N, (\alpha_i))\).

For each \(i \in I\), define \(\beta_i : N \to N\) by:

\[
\beta_i(A_i) = [T(\mu)][\alpha_i(A_i)]
\]

for all \(A_i \in N\).

Then, by Proposition 1.18:

\[([T(\mu)]N, (\alpha_i)) \text{ is a } (\mu_i)\text{-local tensor product for } (N)\]  

By Lemma 5.1, therefore, for all \((\nu_i) \in N, (\nu_i) \sim (\mu_i)\) if and only if \(\bigotimes \nu_i\) exists on \([T(\mu)]N, (\beta_i))\). According to Proposition 1.18. (iii), however, this happens precisely when \(\bigotimes \nu_i\) exists on \((N, (\alpha_i))\) and \(S(\bigotimes \nu_i) \leq T(\mu)\).

We have shown that, for all \((\nu_i) \in N\), and for all product functionals \(\mu = \bigotimes \mu_i\) for \((N, (\alpha_i))\):

\[\beta_i(A_i) = [T(\mu)][\alpha_i(A_i)] \text{ for all } A_i \in N\]  

(5.3) shows that, if \((\nu_i) \sim (\mu_i)\) and if \(\bigotimes \mu_i\) exists on \((N, (\alpha_i))\), then \(\bigotimes \nu_i\) exists on \((N, (\alpha_i))\); (i) follows. (ii) is an immediate consequence of (5.4) and the fact that \(I\) is atomic (Proposition 1.9). (5.1) of (iii) is just (4.2) of Proposition 4.5, and (5.2) is a consequence of (4.1) of Proposition 4.5 and (5.4). (iv) follows from (ii). (5.3), together with (ii), proves (v).

**Theorem 5.6.** Suppose that \((N)_{i \in I}\) is a family of \(W^\ast\)-algebras and that \(A\) is as defined above. Then:

(i). If \(A\) is a nonempty subset of \(A\), a \(\Gamma\)-tensor product for \((N)\) exists and is unique up to isomorphism.

(ii). Suppose that \((N, (\alpha_i))\) is a \(\Gamma\)-tensor product for \((N)\) and that \((P, (\beta_i))\) is a \(\Gamma\)-tensor product for \((N)\). Then \(\Gamma_1 = \Gamma_2\) if and only if \((N, (\alpha_i))\) and \((P, (\beta_i))\) are isomorphic: i.e., if and only if
there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$.

Proof. Everything but the existence of a $\Gamma$-tensor product for $(\mathcal{A})$ follows from Theorem 5.5. For each $\gamma \in \mathcal{A}$, a $\{\gamma\}$-tensor product exists by Theorem 4.7. Hence the existence is a result of the following proposition.

PROPOSITION 5.7. Suppose that $\Gamma$ is a subset of $\mathcal{A}$ and that, for each $\gamma \in \Gamma$, $(\mathcal{A}(\gamma), \alpha_i(\gamma))$ is a $\{\gamma\}$-tensor product for $(\mathcal{A})_i$. Let $\mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A}(\gamma)$ and $\alpha_i = \bigoplus_{\gamma \in \Gamma} \alpha_i(\gamma)$ for all $i \in I$. Then $(\mathcal{A}, (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)$.

Proof. Let $\mathcal{A}$ and $\alpha_i$ be defined as above and let $E(\gamma)$ be the projection of $\mathcal{A}$ with $\mathcal{A}(\gamma) = [E(\gamma)]\mathcal{A}$. Let $\mathcal{B} = \bigoplus_i \mathcal{A}(\alpha_i)$ and $(\mathcal{B}, (\alpha_i))$ is a product for $(\mathcal{A}_i)$. If $\varphi((\mu_i)) = \gamma \in \Gamma$, then $\mu' = \bigotimes \mu_i$ exists on $(\mathcal{A}(\gamma), (\alpha_i(\gamma)))$, and, if $\mu$ is defined by $\mu(B) = \mu'(E(\gamma)B)$ for all $B \in \mathcal{B}$, we can see by direct calculation that $\mu = \bigotimes \mu_i$ on $(\mathcal{B}, (\alpha_i))$ with

$$(5.5) \quad S(\mu) \leq E(\gamma) \quad \text{where} \quad \gamma = \varphi((\mu_i)).$$

It is clear that such $\mu$ form a separating subset $\Sigma$ of the normal positive functionals on $\mathcal{B}$, and that—since $\nu_i = \mu_i$ for a.a. $i \in I$ implies $(\nu_i) \sim (\mu_i)$—(IV-$\mu$) holds for each $\mu \in \Sigma$. Therefore (Corollary 4.3), $(\mathcal{B}, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)$. By (5.2) of Theorem 5.5 (iii), and by (5.5):

$$T(\gamma) = E(\gamma) \quad \text{for all} \quad \gamma \in \Gamma.$$

Hence each $E(\gamma) \in \mathcal{B}$ and $\mathcal{B} = \mathcal{A}$. Furthermore $\sum_{\gamma \in \Gamma} T(\gamma) = 1$, so that, if $\mu = \bigotimes \mu_i$ is a product functional for $(\mathcal{A}_i, (\alpha_i))$, then $T(\mu) = T(\gamma)$ for some $\gamma \in \Gamma$ (Proposition 1.9) and $\varphi((\mu_i)) = \gamma \in \Gamma$ by Theorem 5.5. (ii). Therefore $(\mathcal{A}, (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)$.

PROPOSITION 5.8. Suppose that, for each $i \in I$, $\mathcal{A}_i$ is a von Neumann algebra on $H_i$ and every normal positive functional on $\mathcal{A}_i$ is induced by a vector. Let $H$ denote von Neumann's complete direct product [7] of $(H_i)_{i \in I}$ and let $\alpha_i$ be the natural injection of $L(H_i)$ into $L(H)$ for each $i \in I$. Let $\mathcal{A}_i = R(\alpha_i, (\mathcal{A}_i))$ for each $i \in I$. Then $(\mathcal{A}_i, (\alpha_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)_{i \in I}$. Furthermore, for every nonempty subset $\Gamma$ of $\mathcal{A}$, there exists a projection $T(\Gamma)$ in the tail of $(\mathcal{A}_i, (\alpha_i))$ such that $(T(\Gamma), \mathcal{A}_i(\beta_i))$ is a $\Gamma$-tensor product for $(\mathcal{A}_i)_{i \in I}$, where $\beta_i(A_i) = [T(\Gamma)]A_i$ for all $A_i \in \mathcal{A}_i$. 


The proof is easy and is omitted.

6. Tensor products of factors.

**Lemma 6.1.** If $(\mathcal{A}_i, (\alpha_i))$ is a product for $(\mathcal{A}_i)$ and if each $\mathcal{A}_i$ is a factor, then (IV) holds for $(\mathcal{A}_i, (\alpha_i))$.

**Proof.** Use Lemma 4.1. (iv) and mathematical induction.

**Lemma 6.2.** Suppose that $(\mathcal{A}_i, (\alpha_i))$ is a product for $(\mathcal{A}_i)$ and that $\mathcal{A}$ is a factor. Then $(\mathcal{A}_i, (\alpha_i))$ is a local tensor product for $(\mathcal{A}_i)$ if and only if there exists a product state of $(\mathcal{A}_i, (\alpha_i))$.

**Proof.** Each $\mathcal{A}_i$ is a factor by Lemma 1.3, and therefore (IV) holds by Lemma 6.1. (V) holds because $\mathcal{A} \subset \mathcal{B}$. If $\mu$ is a product functional on $(\mathcal{A}_i, (\alpha_i))$, then $E(\mu) = 1$, for $E(\mu)$ is central by Theorem 3.2. Thus (III) holds if and only if a product state $\mu$ exists.

**Proposition 6.3.** Suppose that $(\mathcal{A}_i, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)$ and that each $\mathcal{A}_i$ is a factor. Then $\mathcal{A} = \mathcal{B}$: the tail of $(\mathcal{A}_i, (\alpha_i))$ equals the center of $\mathcal{A}$.

**Proof.** By Theorem 5.5, the family $(T(\gamma))_{\gamma \in I}$ of atomic projections of $\mathcal{A}$ is such that each $[T(\gamma), \mathcal{A}]$ is a local tensor product for $(\mathcal{A}_i)$. By Lemma 2.5, each $[T(\gamma), \mathcal{A}]$ is a factor. Hence the center of $\mathcal{A} = \bigoplus [T(\gamma), \mathcal{A}]$ is $\mathcal{B} (T(\gamma); \gamma \in I) = \mathcal{A}$.

**Corollary 6.4.** Suppose that $(\mathcal{A}_i, (\alpha_i))$ is a tensor product for $(\mathcal{A}_i)$ and that each $\mathcal{A}_i$ is a factor. Then $\mathcal{A}$ is a factor if and only if $(\mathcal{A}_i, (\alpha_i))$ is a local tensor product: i.e., if and only if (V) holds.

**Proposition 6.5.** Suppose that $\mathcal{A}$ is a finite factor and that $(\mathcal{A}_i)_{i \in I}$ is a factorization of $\mathcal{A}$. Let $\mu_i$ be the restriction of the normalized trace on $\mathcal{A}$ to $\mathcal{A}_i$. Let $(\mathcal{B}, (\alpha_i))$ be a $(\mu_i)$-local tensor product for $(\mathcal{A}_i)_{i \in I}$. Then there exists an isomorphism $\psi$ of $\mathcal{A}$ onto $\mathcal{B}$ such that, for each $i \in I$:

$$\psi(A_i) = \alpha_i(A_i) \quad \text{for all } A_i \in \mathcal{A}_i.$$

**Proof.** (c.f. [6]). If $\mu$ is the normalized trace on $\mathcal{A}$, a direct calculation (see the proof of Theorem 4.3 in [2]) demonstrates that $\mu = \bigotimes \mu_i$ for $(\mathcal{A}_i)$. From there Lemma 6.2, Theorem 4.7, Corollary 4.4 and Proposition 2.6 complete the proof.
7. Some simple counterexamples.

**EXAMPLE 7.1.** Let \( \mathcal{A} \) be a factor of Type II\(_1\) on the Hilbert space \( H \). Then \( \{\mathcal{A}, \mathcal{A}'\} \) is a factorization of \( \mathcal{L}(H) \) which satisfies (IV) and (V), and for which no product functional exists.

See Lemmas 6.1, 6.2 and Theorem 3.6.

**EXAMPLE 7.2.** For \( i = 1 \) and \( 2 \), let \( \mathcal{A}_i \) be a \( W^* \)-algebra with central projection \( Z_i \neq 0 \) or 1. Let \[
Z = (Z_1 \otimes Z_2) + (1 - Z_1) \otimes (1 - Z_2)
\]
in \( \mathcal{A}_1 \otimes \mathcal{A}_2 \), and let \( \mathcal{A} = Z(\mathcal{A}_1 \otimes \mathcal{A}_2) \). Let \( \alpha_i : \mathcal{A}_i \to \mathcal{A} \) be defined by
\[
\alpha_i(A_i) = Z(A_i \otimes 1) \quad \text{for all } A_i \in \mathcal{A}_i
\]
\[
\alpha_i(A_2) = Z(1 \otimes A_2) \quad \text{for all } A_2 \in \mathcal{A}_2.
\]
Then \( (\mathcal{A}, (\alpha_i)) \) is a product for \( (\mathcal{A}_i)_{i=1,2} \) which satisfies (III) and (V) but not (IV).

**EXAMPLE 7.3.** Let \( I = \{1, 2\} \times J \) where \( J \) is infinite, and, for each \( i \in I \), let \( \mathcal{A}_i \) be an abelian \( W^* \)-algebra generated by its two atomic projection \( E_i \) and \( 1 - E_i \). Let the states \( \mu_i \) and \( \nu_i \) of \( \mathcal{A}_i \) be defined by \( \mu_i(1) = \nu_i(1) = 1 \) and \( \mu_i(E_i) = 1 \) and \( \nu_i(E_i) = 1/2 \). Let \( I' = \varphi((\mu_i), (\nu_i)) \) and let \( (\mathcal{A}, (\alpha_i)) \) be a \( I' \)-tensor product for \( (\mathcal{A}_i) \). Let
\[
\mathcal{A}_\delta = \mathcal{R}(\alpha_i(\mathcal{A}_i) : i \in \{\delta\} \times J)
\]
and let \( \lambda_\delta : \mathcal{A}_\delta \to \mathcal{A} \) be the inclusion map, for \( \delta = 1 \) and \( 2 \). Then \( (\mathcal{A}, (\lambda_i)) \) is a product for \( (\mathcal{A}_i)_{i=1,2} \) which satisfies (III) and (V) but not (IV). In particular, \( (\mathcal{A}, (\lambda_i)) \) is not isomorphic to \( \mathcal{A}_1 \otimes \mathcal{A}_2 \).

To make this clearer, let \( \mathcal{B} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) with \( \lambda_\delta \) the natural injection of \( \mathcal{A}_\delta \) into \( \mathcal{B} \). Let \( \beta_i : \mathcal{A}_i \to \mathcal{B} \) be defined for each \( i = (\delta, j) \in I \) by \( \beta_i = \lambda_\delta \circ \alpha_i \). Then \( (\mathcal{B}, (\beta_i)) \) is a \( I'' \)-tensor product for \( (\mathcal{A}_i) \) where \( I'' \) contains four points. In fact
\[
I'' = \varphi((\mu_i), (\nu_i), (\omega_i), (\rho_i))
\]
where:
\[
\omega_i = \mu_i \quad \text{and} \quad \rho_i = \nu_i \quad \text{for } i \in \{1\} \times J
\]
and
\[
\omega_i = \nu_i \quad \text{and} \quad \rho_i = \mu_i \quad \text{for } i \in \{2\} \times J.
\]
EXAMPLE 7.4. Let $I = \{1, 2\} \times J$ with $J$ infinite. For each $i \in I$, let $H_i$ be a Hilbert space (of arbitrary dimension $\geq 2$) and let $\varphi_i$ and $\psi_i$ be orthogonal unit vectors in $H_i$. For each $j \in J$, let $H_j = H_{(1,j)} \otimes H_{(2,j)}$ and let $x_j = [\varphi_{(1,j)} \otimes \omega_{(2,j)} + \varphi_{(1,j)} \otimes \psi_{(2,j)}] \sqrt{2}$. Let $H = \bigotimes_{i \in I} (H_i, x_i)$ and let $\beta_j$ be the natural injection of $\mathcal{L}(H_j)$ into $\mathcal{L}(H)$. Let $\gamma_{(\delta,j)}$ be the natural injection of $\mathcal{L}(H_{(\delta,j)})$ into $\mathcal{L}(H_j)$. Let $\alpha_{(\delta,j)} = \beta_j \circ \gamma_{(\delta,j)}$ for all $(\delta,j) \in I$. Then $(\mathcal{L}(H), (\alpha_i))$ is a product for $(\mathcal{L}(H_i))_{i \in I}$, and there exist no product functionals for $(\mathcal{L}(H), (\alpha_i))$.

See [2] or the remark which follows Lemma 5.1.

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