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**INTEGRAL INEQUALITIES INVOLVING SECOND ORDER
DERIVATIVES**

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An integral inequality involving second order derivatives is derived. A most important consequence of this inequality is that the Dirichlet form

$$D(u, u) = \int_{D^{i,k}} \sum a_{ik} D_i^2 u D_k^2 \bar{u} = q |u|^2 dx \geq 0,$$

for functions $q(x)$ which are positive and "not too large" in a sense which will be made precise later and for functions $u(x)$ with compact support contained in D . Some examples are given and an application is made to an existence theorem for a fourth order uniformly elliptic P.D.E.

An earlier paper by the author [1] contains some similar results for inequalities involving first derivatives. The following definitions and notations will be used throughout the paper. Let

$$x = (x_1, x_2, \dots, x_n) \in R^n.$$

Let D be an open domain in R^n which may be unbounded. Let $C^\infty(D)$ denote the set of infinitely differentiable complex valued functions on D and let $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ consisting of functions with compact support contained in D . Let

$$\|u\|_q = \left(\int_D \sum_{i=1}^n |D_i^2 u|^2 + q |u|^2 dx \right)^{1/2}, \text{ where } D_i^2 u = \frac{\partial^2 u}{\partial x_i^2}$$

and q is either equal to 1 or to one of the positive functions to be defined later. Let $H_q(D)$ be the completion of $\{u \in C^\infty(D) : \|u\|_q < \infty\}$ with respect to $\|u\|_q$ and let $\dot{H}_q(D)$ be the completion of $C_0^\infty(D)$ with respect to $\|u\|_q$. The functions u in $H_q(D)$ or $\dot{H}_q(D)$ have strong L_2 second derivatives which we will denote by the same symbol as for the ordinary derivative. So that

$$\lim_{n \rightarrow \infty} \int_D |D_i^2 u - D_i^2 u_n|^2 dx = 0$$

where $\{u_n\}$ is any sequence of elements in $C^\infty(D)$ such that $\|u - u_n\|_q \rightarrow 0$. All coefficient functions considered will be real valued. The variable functions u may be complex valued. There do not seem to be any analogues of the basic results with complex valued coefficients.

THEOREM 1. *Suppose that the boundary of D is smooth enough*

to apply Gauss' Theorem. Let $a_{ik} \in C^1(D)$ and (a_{ik}) be symmetric positive definite. Let $f_1, f_2, \dots, f_n \in C^1(D)$, $q_k = (f_k + D_k)f_k$ and suppose that $\sum_i a_{ik}q_i \leq 0$, for every $k = 1, 2, \dots, n$. Then, for any $u \in C^1(D)$,

$$\begin{aligned} & \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - (f_i + D_i)^2 (a_{ik} q_k) |u|^2 dx \\ & \geq \int_{\bar{D}} \sum_{i,k} [a_{ik} q_i D_k |u|^2 - (D_k(a_{ik} q_i) + 2a_{ik} q_i f_k) |u|^2] \nu_k ds \end{aligned}$$

where ν_k is the k^{th} component of the normal and the integral on the right is assumed to exist. Equality holds if and only if $D_i^2 u = q_i u$ and $D_i u = f_i u$, for every i .

Proof. We shall require two integrations by parts.

$$\begin{aligned} & \int_D a_{ik} q_i (u D_k^2 \bar{u} + \bar{u} D_k^2 u) dx \\ & = - \int_D [a_{ik} q_i D_k u + u D_k (a_{ik} q_i)] D_k \bar{u} \\ & \quad + [a_{ik} q_i D_k \bar{u} + \bar{u} D_k (a_{ik} q_i)] D_k u dx \\ & \quad + \int_{\bar{D}} a_{ik} q_i (\bar{u} D_k u + u D_k \bar{u}) \nu_k ds \\ & = \int_D D_k^2 (a_{ik} q_i) |u|^2 - 2a_{ik} q_i |D_k u|^2 dx \\ & \quad + \int_{\bar{D}} [a_{ik} q_i D_k |u|^2 - D_k (a_{ik} q_i) |u|^2] \nu_k ds \end{aligned}$$

and

$$\begin{aligned} & - \int_D a_{ik} q_i f_k (\bar{u} D_k u + u D_k \bar{u}) dx \\ & = \int_D D_k (a_{ik} q_i f_k) |u|^2 dx - \int_{\bar{D}} a_{ik} q_i f_k |u|^2 \nu_k ds . \\ & \sum_{i,k} a_{ik} (D_i^2 u - q_i u) (D_k^2 \bar{u} - q_k \bar{u}) - 2 \sum_i a_{ik} q_i \sum_k |D_k u - f_k u|^2 \geq 0 \\ & \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - (f_i + D_i)^2 a_{ik} q_k |u|^2 dx \\ & \geq \int_D \sum_{i,k} - (f_i + D_i)^2 a_{ik} q_k |u|^2 + a_{ik} q_i (u D_k^2 \bar{u} + \bar{u} D_k^2 u) - a_{ik} q_i q_k |u|^2 \\ & \quad + 2a_{ik} q_i (|D_k u|^2 - f_k u D_k \bar{u} - f_k \bar{u} D_k u + f_k^2 |u|^2) dx \\ & = \int_D \sum_{i,k} - [a_{ik} f_i^2 q_k + f_i D_i (a_{ik} q_k) + D_i (a_{ik} f_i q_k) + D_i^2 (a_{ik} q_k)] |u|^2 \\ & \quad + [D_k^2 (a_{ik} q_i) - a_{ik} q_i q_k + 2D_k (a_{ik} q_i f_k) + 2a_{ik} q_i f_k^2] |u|^2 \\ & \quad - 2a_{ik} q_i |D_k u|^2 + 2a_{ik} q_i |D_k u|^2 dx \end{aligned}$$

$$\begin{aligned}
 & + \int_D \sum_{i,k} [a_{ik}q_i D_k |u|^2 - D_k(a_{ik}q_i) |u|^2 - 2a_{ik}q_i f_k |u|^2] \nu_k ds \\
 = & \int_D \sum_{i,k} [a_{ik}q_i (f_k^2 - q_k) - f_i D_i(a_{ik}q_k) + D_i(a_{ik}f_i q_k)] |u|^2 dx \\
 & + \int_D [\dots] ds \\
 = & \int_D \sum_{i,k} [-a_{ik}q_i D_k f_k - f_k D_k(a_{ik}q_i) + D_k(a_{ik}f_k q_i)] |u|^2 dx \\
 & + \int_D [\dots] ds \\
 = & \int_D [a_{ik}q_i D_k |u|^2 - D_k(a_{ik}q_i) |u|^2 - 2a_{ik}q_i f_k |u|^2] \nu_k ds ,
 \end{aligned}$$

which was to be shown.

(1) We will reserve the notation $q(x)$ for a positive function of the form $q(x) = \sum_{i,k} (f_i + D_i)^2 a_{ik} q_k$.

COROLLARY 1. *Suppose that D is any open set. If $a_{ik}(x)$ is uniformly bounded in D , then*

$$\int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - q |u|^2 dx \geq 0 ,$$

for every $u \in \dot{H}_q(D)$ and equality holds if and only if $D_i^2 u = q_i u$ and $D_i u = f_i u$ almost everywhere, for each i .

Proof. It is easy to obtain the inequality for functions in $C_0^\infty(D)$ by integrating around a sphere containing the support of u . The result for $u \in \dot{H}_q(D)$ can then be obtained by showing that

$$\int_D a_{ik} D_i^2 u_m D_k^2 \bar{u}_m dx \xrightarrow{m} \int_D a_{ik} D_i^2 u D_k^2 \bar{u} dx$$

which follows easily from Cauchy's inequality.

COROLLARY 2. *If (a_{ik}) is only positive semidefinite, the same inequality holds but the conditions for equality are not necessarily the same. If $a_{ik} = 1$, the conditions are*

$$\sum_i (D_i^2 u - q_i u) = 0 \quad \text{and} \quad D_i u = f_i u .$$

EXAMPLE 1. Corollary 2 can be used to obtain inequalities whenever a solution u_1 of a plate problem

$$\left. \begin{aligned}
 \Delta \Delta u - pu &= 0 && \text{in } D \\
 u &= 0 \\
 \Delta u &= 0
 \end{aligned} \right\} \text{on } \dot{D}$$

is known. Here $\Delta u = \sum_k D_k^2 u$ and $\Delta \Delta u = \sum_{i,k} D_i^2 D_k^2 u$. Suppose that $u_1 > 0$ in D and let $f_k = (D_k u_1)/u_1$. We must show that

$$\sum_k (f_k + D_k) f_k = \sum_k \frac{D_k^2 u_1}{u_1} \leq 0.$$

For this it suffices to show that $\sum_k D_k^2 u_1 \leq 0$. If $\sum_i D_i^2 v = p u_1 \geq 0$ and $v = 0$ on \bar{D} , then $v \leq 0$ by the maximum principle. Set $v = \sum_k D_k^2 u_1$, then v satisfies the problem and hence $\sum_k D_k^2 u_1 \leq 0$ in D . Calculate

$$\sum_{i,k} (f_i + D_i)^2 (f_k + D_k) f_k = \sum_{i,k} \frac{D_i^2 D_k^2 u_1}{u_1} = p.$$

So

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - p |u|^2 dx \geq 0, \quad \text{for } u \in C_0^\infty(D).$$

In particular, for $p = \lambda q$, where λ is the first eigenvalue of the plate problem and u_1 is the corresponding eigenfunction, the inequality becomes Rayleigh's characterization of the first eigenvalue. In this case, the conditions for equality become $u = k u_1$.

EXAMPLE 2. Suppose $n \geq 5$, then

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \frac{n^2(n-4)^2}{16} \left(\sum_{i=1}^n x_i^2 \right)^{-1} |u|^2 dx \geq 0,$$

for every $u \in \mathring{H}_q$. To apply Corollary 1, let $f_k = (a/s)x_k$, where $s = \sum_{i=1}^n x_i^2$, then

$$q_k = \frac{a(a-2)}{s^2} x_k^2 + \frac{a}{s},$$

$$\sum_{k=1}^n q_k = \frac{a(a+(n-2))}{s} \leq 0 \quad \text{if } 2-n \leq a < 0$$

(the other possibility leads to nothing of interest). Calculate,

$$q = \frac{1}{s^2} a(a-2)[a - (4-n)][a - (2-n)].$$

Then $q > 0$, if $a < 0$, $a - 2 > 0$, and $a \geq 4 - n$. If we choose $a = (4 - n)/2$, then q is maximal and equal to $(n^2(n - 4)^2)/16$.

It is unfortunate that the preceding example is only good for dimensions larger than five. The following inequality, though unappealing, does yield an example for every dimension.

THEOREM 2. Let $f_1, f_2, \dots, f_n \in C^1(D)$ and suppose that

$$\sum_{i,k} D_i f_i | \xi_k |^2 + (D_k f_i + D_i f_k + f_i f_k) \xi_i \bar{\xi}_k \leq 0$$

for every vector $(\xi_1, \xi_2, \dots, \xi_n)$. Then

$$\begin{aligned} & \int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - [D_k^2 D_i f_i + 3f_k D_k D_i f_i + f_i D_k^2 f_i \\ & \quad + 2f_k^2 D_i f_i + 4f_i f_k D_k f_i + (D_k f_i)^2 + (D_k f_i)(D_i f_k) \\ & \quad + (D_k f_k)(D_i f_i) + f_i^2 f_k^2] |u|^2 dx \\ & \geq \int_D \sum_{i,k} [2 \operatorname{Re} D_i(f_i u) D_k \bar{u} - f_i |D_k u|^2 \\ & \quad - (D_k D_i f_i + 2f_k D_i f_i + f_i D_k f_i \\ & \quad + f_i D_i f_k + f_i^2 f_k) |u|^2] \nu_k ds . \end{aligned}$$

Proof.

$$(2) \quad \sum_{i,k} [D_i^2 u - D_i(f_i u)][D_k^2 \bar{u} - D_k(f_k \bar{u})] - D_i f_i |D_k u - f_k u|^2 \\ - (D_k f_i + D_i f_k + f_i f_k)(D_i u - f_i u) D_k \bar{u} - f_k \bar{u} \geq 0$$

when expanded the first term in (2) contains the following two terms which we integrate by parts:

$$-D_i(f_i u) D_k^2 \bar{u} - D_i(f_i \bar{u}) D_k^2 u \quad \text{and} \quad D_i(f_i u) D_k(f_k \bar{u}) .$$

Notice that the order of summation has been changed in the first term.

$$\begin{aligned} & - \int_D D_i(f_i u) D_k^2 \bar{u} + D_i(f_i \bar{u}) D_k^2 u dx \\ & = \int_D D_k D_i(f_i u) D_k \bar{u} + D_k D_i(f_i \bar{u}) D_k u dx \\ & \quad - \int_D [D_i(f_i u) D_k \bar{u} + D_i(f_i \bar{u}) D_k u] \nu_k ds \\ & = \int_D (u D_k D_i f_i + D_i f_i D_k u + D_k f_i D_i u + f_i D_k D_i u) D_k \bar{u} \\ & \quad + (\bar{u} D_k D_i f_i + D_i f_i D_k \bar{u} + D_k f_i D_i \bar{u} + f_i D_k D_i \bar{u}) D_k u \\ & \quad - \int_D [\dots] \nu_k ds \\ & = \int_D D_k D_i f_i D_k |u|^2 + f_i D_i |D_k u|^2 + 2D_i f_i |D_k u|^2 \\ & \quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\ & \quad - \int_D [\dots] \nu_k ds \end{aligned}$$

$$\begin{aligned}
(3) \quad &= \int_D -(D_k^2 D_i f_i) |u|^2 + (D_i f_i) |D_k u|^2 \\
&\quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\
&\quad + \int_{\bar{D}} [-2 \operatorname{Re} D_i(f_i u) D_k \bar{u} + D_k D_i f_i |u|^2 + f_i |D_k u|^2] \nu_k ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_D \sum_{i,k} D_i(f_i u) D_k(f_k \bar{u}) dx \\
&= \int_D \sum_{i,k} (D_i f_i)(D_k f_k) |u|^2 + f_k(D_i f_i)(u D_k \bar{u} + \bar{u} D_k u) \\
&\quad + f_i f_k(D_i u)(D_k \bar{u}) dx \\
&= \int_D \sum_{i,k} (D_i f_i)(D_k f_k) |u|^2 - D_k[f_k(D_i f_i)] |u|^2 + f_i f_k(D_i u)(D_k \bar{u}) dx \\
&\quad + \int_{\bar{D}} \sum_{i,k} f_k(D_i f_i) |u|^2 \nu_k ds \\
(4) \quad &= \int_{i,k} -f_k(D_k D_i f_i) |u|^2 + f_i f_k(D_i u)(D_k \bar{u}) dx \\
&\quad + \int_{\bar{D}} \sum_{i,k} f_k(D_i f_i) |u|^2 \nu_k ds .
\end{aligned}$$

The second term in (2) contains

$$\begin{aligned}
(5) \quad &\int_D f_k D_i f_i (u D_k \bar{u} + \bar{u} D_k u) dx \\
&= - \int_D [(D_k f_k)(D_i f_i) + f_k D_k D_i f_i] |u|^2 dx \\
&\quad + \int_{\bar{D}} f_k D_i f_i |u|^2 \nu_k ds .
\end{aligned}$$

The third term in (2) contains

$$\begin{aligned}
&\int_D \sum_{i,k} (D_k f_i + D_i f_k + f_i f_k)(f_i u D_k \bar{u} + f_k \bar{u} D_i u) dx \\
&= \int_D \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) D_k |u|^2 dx \\
(6) \quad &= - \int_D \sum_{i,k} D_k (f_i D_k f_i + f_i D_i f_k + f_i^2 f_k) |u|^2 dx \\
&\quad + \int_{\bar{D}} \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) |u|^2 \nu_k ds .
\end{aligned}$$

Expanding (2) and making use of (3), (4), (5) and (6), one can obtain the advertised result.

COROLLARY. *Suppose that f_i is a function of x_i alone. Then*

$$\begin{aligned} & \int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \sum_i [D_i^3 f_i + 4f_i D_i^2 f_i + 4f_i^2 D_i f_i + 2(D_i f_i)^2] |u|^2 \\ & \quad - \sum_{i,k} [2f_k^2 D_i f_i + (D_k f_k)(D_i f_i) + f_i^2 f_k^2] |u|^2 dx \\ & \geq 0, \text{ for every } u \in \dot{H}_q \end{aligned}$$

where q is the coefficient of $|u|^2$.

EXAMPLE 3. Let $f_i = a/x_i$. Then

$$\begin{aligned} & \sum_{i,k} D_i f_i |\xi_k|^2 + (D_k f_i + D_i f_k + f_i f_k) \xi_i \bar{\xi}_k \\ & = \sum_{i,k} -\frac{a}{x_i^2} |\xi_k|^2 + \frac{a^2}{x_i x_k} \xi_i \bar{\xi}_k + \sum_i -\frac{2a}{x_i^2} |\xi_i|^2 \\ & = a^2 \sum_{i,k} \frac{\xi_i \bar{\xi}_k}{x_i x_k} - a \sum_i \frac{1}{x_i^2} \sum_i |\xi_i|^2 - 2a \sum_i \frac{|\xi_i|^2}{x_i^2} \\ & \leq (a^2 - a) \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} - 2a \sum_i \frac{|\xi_i|^2}{x_i^2}. \end{aligned}$$

Let $a = 1 + \varepsilon, \varepsilon > 0$. The right side will be negative when

$$\varepsilon \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} \leq 2 \sum_i \frac{|\xi_i|^2}{x_i^2}.$$

Take $\lambda_i = (|\xi_i|^2)/(\sum_i |\xi_i|^2)$, then $\sum_i \lambda_i = 1$ and the inequality becomes

$$\frac{\varepsilon}{2} \sum_i \frac{1}{x_i^2} \leq \sum_i \lambda_i \frac{1}{x_i^2}.$$

It is always possible to choose an ε so that this inequality holds provided D is bounded and bounded away from the origin. For let $0 < m \leq x_i^2 \leq M$, then

$$\sum_i \lambda_i \frac{1}{x_i^2} \geq \frac{1}{M} \quad \text{and} \quad \sum_i \frac{1}{x_i^2} \leq \frac{n}{m}.$$

Take $\varepsilon/2 \leq (m/nM)$ and the inequality holds.

Let us compute q using the formula in the corollary.

$$\begin{aligned} q & = \sum_i \frac{-6a + 8a^2 - 4a^3 + 2a^2}{x_i^4} + \sum_{i,k} \frac{-2a^3 + a^2 + a^4}{x_i^2 x_k^2} \\ & = \sum_i \frac{(-4a^3 + 10a^2 - 6a)}{x_i^4} + a^2(a-1)^2 \sum_{i,k} \frac{1}{x_i^2 x_k^2} \\ & = (a^4 - 6a^3 + 11a^2 - 6a) \sum_i \frac{1}{x_i^4} + a^2(a-1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2} \\ & = a(a-1)(a-2)(a-3) \sum_i \frac{1}{x_i^4} + a^2(a-1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2}. \end{aligned}$$

Taking $a = 1 + \varepsilon$

$$q = \varepsilon(1 + \varepsilon)(1 - \varepsilon)(2 - \varepsilon) \sum_i \frac{1}{x_i^4} + \varepsilon^2(1 + \varepsilon)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2}$$

which is positive for $\varepsilon < 1$ or for $\varepsilon > 2$.

THEOREM 3. *A fourth order existence theorem.*

Let $q(x)$ be a function of the special form (1) and let $p(x)$ be a continuously differentiable function such that $0 < p(x) \leq (1 - \varepsilon)q(x)$, where $\varepsilon > 0$ and fixed. Let $\int_D q^{-1} |f|^2 dx < \infty$, $g \in H_q$ and let $Au = \sum_{i,k} D_i^2(a_{ik}D_k^2u) - pu$ be a uniformly elliptic operator. That is, a_{ik} is positive definite and there exist positive constants M and λ such that

$$|a_{ik}(x)| \leq M \quad \text{and} \quad \lambda \sum_i |\xi_i|^2 \leq \sum_{i,k} a_{ik}(x) \xi_i \bar{\xi}_k,$$

for any $(\xi_1, \xi_2, \dots, \xi_n)$ and all x in D . Then the Dirichlet problem

$$\left. \begin{aligned} Au &= f && \text{in } D \\ u &= g \\ \sum_i D_i^2 u &= \sum_i D_i^2 g \end{aligned} \right\} \text{ on } \dot{D}$$

has a unique weak solution.

Proof. We must show that there is a function $u \in H_q$ such that $u - g \in \dot{H}_q$ and $(u, A\varphi) = (f, \varphi)$, for every φ in C_0^∞ . Set $u_0 = u - g$ and consider the equivalent problem of finding $u_0 \in \dot{H}_q$ such that $(u_0, A\varphi) = (f, \varphi) - (g, A\varphi)$. Let

$$\begin{aligned} B(u, v) &= \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{v} - pu \bar{v} dx \\ &= \int_D \sum_{i,k} u D_i^2 (a_{ik} D_k^2 \bar{v} - pu \bar{v}) dx \\ &= (u, Av), \quad \text{for } u, v \in C_0^\infty. \end{aligned}$$

We will show that there exist $c_1, c_2 > 0$ such that

$$|B(u, v)| \leq C_1 \|u\|_q \|v\|_q \quad \text{and} \quad B(u, u) \geq C_2 \|u\|_q^2.$$

$$\begin{aligned} B(u, u) &= \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - p |u|^2 dx \\ &\geq \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - q |u|^2 dx + \varepsilon \int_D q |u|^2 dx. \end{aligned}$$

By Corollary 1, both integrals are positive and hence

$$\begin{aligned} \left(1 + \frac{2}{\varepsilon}\right)B(u, u) &\geq \int_D \sum a_{ik} D_i u D_k \bar{u} + q |u|^2 dx \\ &\geq \int_D \lambda \sum_i |D_i u|^2 + q |u|^2 dx \\ &\geq \text{const} \|u\|_q^2. \end{aligned}$$

The positivity of $B(u, u)$ implies that $|B(u, v)|^2 \leq B(u, u) \cdot B(v, v)$ so that we need only show that $B(u, u) \leq \text{const} \|u\|_q^2$.

$$\begin{aligned} B(u, u) &\leq M \int_D \sum_{i,k} |D_i u D_k \bar{u}| + p |u|^2 dx \\ &\leq \int_D \sum_{i,k} \frac{1}{2} (|D_i u|^2 + |D_k u|^2) + p |u|^2 dx \\ &\leq Mn \int_D \sum_i |D_i u|^2 + p |u|^2 dx = Mn \|u\|_q^2. \end{aligned}$$

Now extend $B(u, v)$ to all of \mathring{H}_q by continuity. We can now apply the Lax-Milgram Theorem which guarantees that any bounded linear functional $F(\varphi)$ on \mathring{H}_q can be represented as $\overline{B(u_0, \varphi)}$ for some $u_0 \in \mathring{H}_q$. Take $F(\varphi) = \overline{(f, \varphi)} - \overline{B(g, \varphi)}$. Then

$$\begin{aligned} |F\varphi| &\leq \left(\int_D q^{-1} |f|^2 dx\right)^{1/2} \left(\int_D q |\varphi|^2 dx\right)^{1/2} \\ &\quad + c_1 \|\varphi\|_q \|g\|_q \leq \text{const} \|\varphi\|_q. \end{aligned}$$

So $B(u_0, \varphi) = (f, \varphi) - B(g, \varphi)$ as was to be shown. To obtain the uniqueness result, let $Au = 0, u \in \mathring{H}_q$, then

$$0 = (u, Au) = \overline{B(u, u)} \geq C_2 \|u\|_q^2. \quad \therefore u = 0 \text{ a.e.}$$

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