

Pacific Journal of Mathematics

**INTEGRAL INEQUALITIES INVOLVING SECOND ORDER
DERIVATIVES**

JAMES CALVERT

INTEGRAL INEQUALITIES INVOLVING SECOND ORDER DERIVATIVES

JAMES CALVERT

An integral inequality involving second order derivatives is derived. A most important consequence of this inequality is that the Dirichlet form

$$D(u, u) = \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} = q |u|^2 dx \geq 0,$$

for functions $q(x)$ which are positive and "not too large" in a sense which will be made precise later and for functions $u(x)$ with compact support contained in D . Some examples are given and an application is made to an existence theorem for a fourth order uniformly elliptic P.D.E.

An earlier paper by the author [1] contains some similar results for inequalities involving first derivatives. The following definitions and notations will be used throughout the paper. Let

$$x = (x_1, x_2, \dots, x_n) \in R^n.$$

Let D be an open domain in R^n which may be unbounded. Let $C^\infty(D)$ denote the set of infinitely differentiable complex valued functions on D and let $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ consisting of functions with compact support contained in D . Let

$$\|u\|_q = \left(\int_D \sum_{i=1}^n |D_i^2 u|^2 + q |u|^2 dx \right)^{1/2}, \text{ where } D_i^2 u = \frac{\partial^2 u}{\partial x_i^2}$$

and q is either equal to 1 or to one of the positive functions to be defined later. Let $H_q(D)$ be the completion of $\{u \in C^\infty(D) : \|u\|_q < \infty\}$ with respect to $\|u\|_q$ and let $\dot{H}_q(D)$ be the completion of $C_0^\infty(D)$ with respect to $\|u\|_q$. The functions u in $H_q(D)$ or $\dot{H}_q(D)$ have strong L_2 second derivatives which we will denote by the same symbol as for the ordinary derivative. So that

$$\lim_{n \rightarrow \infty} \int_D |D_i^2 u - D_i^2 u_n|^2 dx = 0$$

where $\{u_n\}$ is any sequence of elements in $C^\infty(D)$ such that $\|u - u_n\|_q \rightarrow 0$. All coefficient functions considered will be real valued. The variable functions u may be complex valued. There do not seem to be any analogues of the basic results with complex valued coefficients.

THEOREM 1. *Suppose that the boundary of D is smooth enough*

to apply Gauss' Theorem. Let $a_{ik} \in C^1(D)$ and (a_{ik}) be symmetric positive definite. Let $f_1, f_2, \dots, f_n \in C^1(D)$, $q_k = (f_k + D_k)f_k$ and suppose that $\sum_i a_{ik}q_i \leq 0$, for every $k = 1, 2, \dots, n$. Then, for any $u \in C^1(D)$,

$$\begin{aligned} & \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - (f_i + D_i)^2 (a_{ik} q_k) |u|^2 dx \\ & \geq \int_{\bar{D}} \sum_{i,k} [a_{ik} q_i D_k |u|^2 - (D_k(a_{ik} q_i) + 2a_{ik} q_i f_k) |u|^2] \nu_k ds \end{aligned}$$

where ν_k is the k^{th} component of the normal and the integral on the right is assumed to exist. Equality holds if and only if $D_i^2 u = q_i u$ and $D_i u = f_i u$, for every i .

Proof. We shall require two integrations by parts.

$$\begin{aligned} & \int_D a_{ik} q_i (u D_k^2 \bar{u} + \bar{u} D_k^2 u) dx \\ & = - \int_D [a_{ik} q_i D_k u + u D_k (a_{ik} q_i)] D_k \bar{u} \\ & \quad + [a_{ik} q_i D_k \bar{u} + \bar{u} D_k (a_{ik} q_i)] D_k u dx \\ & \quad + \int_{\bar{D}} a_{ik} q_i (\bar{u} D_k u + u D_k \bar{u}) \nu_k ds \\ & = \int_D D_k^2 (a_{ik} q_i) |u|^2 - 2a_{ik} q_i |D_k u|^2 dx \\ & \quad + \int_{\bar{D}} [a_{ik} q_i D_k |u|^2 - D_k (a_{ik} q_i) |u|^2] \nu_k ds \end{aligned}$$

and

$$\begin{aligned} & - \int_D a_{ik} q_i f_k (\bar{u} D_k u + u D_k \bar{u}) dx \\ & = \int_D D_k (a_{ik} q_i f_k) |u|^2 dx - \int_{\bar{D}} a_{ik} q_i f_k |u|^2 \nu_k ds . \\ & \sum_{i,k} a_{ik} (D_i^2 u - q_i u) (D_k^2 \bar{u} - q_k \bar{u}) - 2 \sum_i a_{ik} q_i \sum_k |D_k u - f_k u|^2 \geq 0 \\ & \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - (f_i + D_i)^2 a_{ik} q_k |u|^2 dx \\ & \geq \int_D \sum_{i,k} - (f_i + D_i)^2 a_{ik} q_k |u|^2 + a_{ik} q_i (u D_k^2 \bar{u} + \bar{u} D_k^2 u) - a_{ik} q_i q_k |u|^2 \\ & \quad + 2a_{ik} q_i (|D_k u|^2 - f_k u D_k \bar{u} - f_k \bar{u} D_k u + f_k^2 |u|^2) dx \\ & = \int_D \sum_{i,k} - [a_{ik} f_i^2 q_k + f_i D_i (a_{ik} q_k) + D_i (a_{ik} f_i q_k) + D_i^2 (a_{ik} q_k)] |u|^2 \\ & \quad + [D_k^2 (a_{ik} q_i) - a_{ik} q_i q_k + 2D_k (a_{ik} q_i f_k) + 2a_{ik} q_i f_k^2] |u|^2 \\ & \quad - 2a_{ik} q_i |D_k u|^2 + 2a_{ik} q_i |D_k u|^2 dx \end{aligned}$$

$$\begin{aligned}
 & + \int_D \sum_{i,k} [a_{ik}q_i D_k |u|^2 - D_k(a_{ik}q_i) |u|^2 - 2a_{ik}q_i f_k |u|^2] \nu_k ds \\
 = & \int_D \sum_{i,k} [a_{ik}q_i (f_k^2 - q_k) - f_i D_i(a_{ik}q_k) + D_i(a_{ik}f_i q_k)] |u|^2 dx \\
 & + \int_D [\dots] ds \\
 = & \int_D \sum_{i,k} [-a_{ik}q_i D_k f_k - f_k D_k(a_{ik}q_i) + D_k(a_{ik}f_k q_i)] |u|^2 dx \\
 & + \int_D [\dots] ds \\
 = & \int_D [a_{ik}q_i D_k |u|^2 - D_k(a_{ik}q_i) |u|^2 - 2a_{ik}q_i f_k |u|^2] \nu_k ds ,
 \end{aligned}$$

which was to be shown.

(1) We will reserve the notation $q(x)$ for a positive function of of the form $q(x) = \sum_{i,k} (f_i + D_i)^2 a_{ik} q_k$.

COROLLARY 1. *Suppose that D is any open set. If $a_{ik}(x)$ is uniformly bounded in D , then*

$$\int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} - q |u|^2 dx \geq 0 ,$$

for every $u \in \dot{H}_q(D)$ and equality holds if and only if $D_i^2 u = q_i u$ and $D_i u = f_i u$ almost everywhere, for each i .

Proof. It is easy to obtain the inequality for functions in $C_0^\infty(D)$ by integrating around a sphere containing the support of u . The result for $u \in \dot{H}_q(D)$ can then be obtained by showing that

$$\int_D a_{ik} D_i^2 u_m D_k^2 \bar{u}_m dx \xrightarrow{m} \int_D a_{ik} D_i^2 u D_k^2 \bar{u} dx$$

which follows easily from Cauchy's inequality.

COROLLARY 2. *If (a_{ik}) is only positive semidefinite, the same inequality holds but the conditions for equality are not necessarily the same. If $a_{ik} = 1$, the conditions are*

$$\sum_i (D_i^2 u - q_i u) = 0 \quad \text{and} \quad D_i u = f_i u .$$

EXAMPLE 1. Corollary 2 can be used to obtain inequalities whenever a solution u_1 of a plate problem

$$\begin{aligned}
 \Delta \Delta u - pu &= 0 && \text{in } D \\
 u &= 0 && \left. \begin{array}{l} \\ \Delta u = 0 \end{array} \right\} \text{on } \dot{D}
 \end{aligned}$$

is known. Here $\Delta u = \sum_k D_k^2 u$ and $\Delta \Delta u = \sum_{i,k} D_i^2 D_k^2 u$. Suppose that $u_1 > 0$ in D and let $f_k = (D_k u_1)/u_1$. We must show that

$$\sum_k (f_k + D_k) f_k = \sum_k \frac{D_k^2 u_1}{u_1} \leq 0.$$

For this it suffices to show that $\sum_k D_k^2 u_1 \leq 0$. If $\sum_i D_i^2 v = p u_1 \geq 0$ and $v = 0$ on \dot{D} , then $v \leq 0$ by the maximum principle. Set $v = \sum_k D_k^2 u_1$, then v satisfies the problem and hence $\sum_k D_k^2 u_1 \leq 0$ in D . Calculate

$$\sum_{i,k} (f_i + D_i)^2 (f_k + D_k) f_k = \sum_{i,k} \frac{D_i^2 D_k^2 u_1}{u_1} = p.$$

So

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - p |u|^2 dx \geq 0, \quad \text{for } u \in C_0^\infty(D).$$

In particular, for $p = \lambda q$, where λ is the first eigenvalue of the plate problem and u_1 is the corresponding eigenfunction, the inequality becomes Rayleigh's characterization of the first eigenvalue. In this case, the conditions for equality become $u = k u_1$.

EXAMPLE 2. Suppose $n \geq 5$, then

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \frac{n^2(n-4)^2}{16} \left(\sum_{i=1}^n x_i^2 \right)^{-1} |u|^2 dx \geq 0,$$

for every $u \in \dot{H}_q$. To apply Corollary 1, let $f_k = (a/s)x_k$, where $s = \sum_{i=1}^n x_i^2$, then

$$q_k = \frac{a(a-2)}{s^2} x_k^2 + \frac{a}{s},$$

$$\sum_{k=1}^n q_k = \frac{a(a+(n-2))}{s} \leq 0 \quad \text{if } 2-n \leq a < 0$$

(the other possibility leads to nothing of interest). Calculate,

$$q = \frac{1}{s^2} a(a-2)[a - (4-n)][a - (2-n)].$$

Then $q > 0$, if $a < 0$, $a - 2 > 0$, and $a \geq 4 - n$. If we choose $a = (4 - n)/2$, then q is maximal and equal to $(n^2(n - 4)^2)/16$.

It is unfortunate that the preceding example is only good for dimensions larger than five. The following inequality, though unappealing, does yield an example for every dimension.

THEOREM 2. *Let $f_1, f_2, \dots, f_n \in C^1(D)$ and suppose that*

$$\sum_{i,k} D_i f_i |\xi_k|^2 + (D_k f_i + D_i f_k + f_i f_k) \xi_i \bar{\xi}_k \leq 0$$

for every vector $(\xi_1, \xi_2, \dots, \xi_n)$. Then

$$\begin{aligned} & \int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - [D_k^2 D_i f_i + 3f_k D_k D_i f_i + f_i D_k^2 f_i \\ & \quad + 2f_k^2 D_i f_i + 4f_i f_k D_k f_i + (D_k f_i)^2 + (D_k f_i)(D_i f_k) \\ & \quad + (D_k f_k)(D_i f_i) + f_i^2 f_k^2] |u|^2 dx \\ & \geq \int_D \sum_{i,k} [2 \operatorname{Re} D_i(f_i u) D_k \bar{u} - f_i |D_k u|^2 \\ & \quad - (D_k D_i f_i + 2f_k D_i f_i + f_i D_k f_i \\ & \quad + f_i D_i f_k + f_i^2 f_k) |u|^2] \nu_k ds . \end{aligned}$$

Proof.

$$(2) \quad \sum_{i,k} [D_i^2 u - D_i(f_i u)][D_k^2 \bar{u} - D_k(f_k \bar{u})] - D_i f_i |D_k u - f_k u|^2 \\ - (D_k f_i + D_i f_k + f_i f_k)(D_i u - f_i u) D_k \bar{u} - f_k \bar{u} \geq 0$$

when expanded the first term in (2) contains the following two terms which we integrate by parts:

$$-D_i(f_i u) D_k^2 \bar{u} - D_i(f_i \bar{u}) D_k^2 u \quad \text{and} \quad D_i(f_i u) D_k(f_k \bar{u}) .$$

Notice that the order of summation has been changed in the first term.

$$\begin{aligned} & - \int_D D_i(f_i u) D_k^2 \bar{u} + D_i(f_i \bar{u}) D_k^2 u dx \\ & = \int_D D_k D_i(f_i u) D_k \bar{u} + D_k D_i(f_i \bar{u}) D_k u dx \\ & - \int_D [D_i(f_i u) D_k \bar{u} + D_i(f_i \bar{u}) D_k u] \nu_k ds \\ & = \int_D (u D_k D_i f_i + D_i f_i D_k u + D_k f_i D_i u + f_i D_k D_i u) D_k \bar{u} \\ & \quad + (\bar{u} D_k D_i f_i + D_i f_i D_k \bar{u} + D_k f_i D_i \bar{u} + f_i D_k D_i \bar{u}) D_k u \\ & - \int_D [\dots] \nu_k ds \\ & = \int_D D_k D_i f_i D_k |u|^2 + f_i D_i |D_k u|^2 + 2D_i f_i |D_k u|^2 \\ & \quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\ & - \int_D [\dots] \nu_k ds \end{aligned}$$

$$\begin{aligned}
(3) \quad &= \int_D -(D_k^2 D_i f_i) |u|^2 + (D_i f_i) |D_k u|^2 \\
&\quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\
&\quad + \int_D [-2 \operatorname{Re} D_i(f_i u) D_k \bar{u} + D_k D_i f_i |u|^2 + f_i |D_k u|^2] \nu_k ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_D \sum_{i,k} D_i(f_i u) D_k(f_k \bar{u}) dx \\
&= \int_D \sum_{i,k} (D_i f_i)(D_k f_k) |u|^2 + f_k (D_i f_i)(u D_k \bar{u} + \bar{u} D_k u) \\
&\quad + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&= \int_D \sum_{i,k} (D_i f_i)(D_k f_k) |u|^2 - D_k[f_k (D_i f_i)] |u|^2 + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&\quad + \int_D \sum_{i,k} f_k (D_i f_i) |u|^2 \nu_k ds \\
(4) \quad &= \int_D \sum_{i,k} -f_k (D_k D_i f_i) |u|^2 + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&\quad + \int_D \sum_{i,k} f_k (D_i f_i) |u|^2 \nu_k ds .
\end{aligned}$$

The second term in (2) contains

$$\begin{aligned}
(5) \quad &\int_D f_k D_i f_i (u D_k \bar{u} + \bar{u} D_k u) dx \\
&= - \int_D [(D_k f_k)(D_i f_i) + f_k D_k D_i f_i] |u|^2 dx \\
&\quad + \int_D f_k D_i f_i |u|^2 \nu_k ds .
\end{aligned}$$

The third term in (2) contains

$$\begin{aligned}
&\int_D \sum_{i,k} (D_k f_i + D_i f_k + f_i f_k)(f_i u D_k \bar{u} + f_k \bar{u} D_i u) dx \\
&= \int_D \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) D_k |u|^2 dx \\
(6) \quad &= - \int_D \sum_{i,k} D_k (f_i D_k f_i + f_i D_i f_k + f_i^2 f_k) |u|^2 dx \\
&\quad + \int_D \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) |u|^2 \nu_k ds .
\end{aligned}$$

Expanding (2) and making use of (3), (4), (5) and (6), one can obtain the advertised result.

COROLLARY. *Suppose that f_i is a function of x_i alone. Then*

$$\begin{aligned} & \int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \sum_i [D_i^3 f_i + 4f_i D_i^2 f_i + 4f_i^2 D_i f_i + 2(D_i f_i)^2] |u|^2 \\ & \quad - \sum_{i,k} [2f_k^2 D_i f_i + (D_k f_k)(D_i f_i) + f_i^2 f_k^2] |u|^2 dx \\ & \geq 0, \text{ for every } u \in \mathring{H}_q \end{aligned}$$

where q is the coefficient of $|u|^2$.

EXAMPLE 3. Let $f_i = a/x_i$. Then

$$\begin{aligned} & \sum_{i,k} D_i f_i |\xi_k|^2 + (D_k f_i + D_i f_k + f_i f_k) \xi_i \bar{\xi}_k \\ & = \sum_{i,k} -\frac{a}{x_i^2} |\xi_k|^2 + \frac{a^2}{x_i x_k} \xi_i \bar{\xi}_k + \sum_i -\frac{2a}{x_i^2} |\xi_i|^2 \\ & = a^2 \sum_{i,k} \frac{\xi_i \bar{\xi}_k}{x_i x_k} - a \sum_i \frac{1}{x_i^2} \sum_i |\xi_i|^2 - 2a \sum_i \frac{|\xi_i|^2}{x_i^2} \\ & \leq (a^2 - a) \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} - 2a \sum \frac{|\xi_i|^2}{x_i^2}. \end{aligned}$$

Let $a = 1 + \varepsilon, \varepsilon > 0$. The right side will be negative when

$$\varepsilon \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} \leq 2 \sum_i \frac{|\xi_i|^2}{x_i^2}.$$

Take $\lambda_i = (|\xi_i|^2)/(\sum_i |\xi_i|^2)$, then $\sum_i \lambda_i = 1$ and the inequality becomes

$$\frac{\varepsilon}{2} \sum_i \frac{1}{x_i^2} \leq \sum_i \lambda_i \frac{1}{x_i^2}.$$

It is always possible to choose an ε so that this inequality holds provided D is bounded and bounded away from the origin. For let $0 < m \leq x_i^2 \leq M$, then

$$\sum_i \lambda_i \frac{1}{x_i^2} \geq \frac{1}{M} \quad \text{and} \quad \sum_i \frac{1}{x_i^2} \leq \frac{n}{m}.$$

Take $\varepsilon/2 \leq (m/nM)$ and the inequality holds.

Let us compute q using the formula in the corollary.

$$\begin{aligned} q & = \sum_i \frac{-6a + 8a^2 - 4a^3 + 2a^2}{x_i^4} + \sum_{i,k} \frac{-2a^3 + a^2 + a^4}{x_i^2 x_k^2} \\ & = \sum_i \frac{(-4a^3 + 10a^2 - 6a)}{x_i^4} + a^2(a - 1)^2 \sum_{i,k} \frac{1}{x_i^2 x_k^2} \\ & = (a^4 - 6a^3 + 11a^2 - 6a) \sum_i \frac{1}{x_i^4} + a^2(a - 1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2} \\ & = a(a - 1)(a - 2)(a - 3) \sum_i \frac{1}{x_i^4} + a^2(a - 1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2}. \end{aligned}$$

Taking $a = 1 + \varepsilon$

$$q = \varepsilon(1 + \varepsilon)(1 - \varepsilon)(2 - \varepsilon) \sum_i \frac{1}{x_i^2} + \varepsilon^2(1 + \varepsilon)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2}$$

which is positive for $\varepsilon < 1$ or for $\varepsilon > 2$.

THEOREM 3. *A fourth order existence theorem.*

Let $q(x)$ be a function of the special form (1) and let $p(x)$ be a continuously differentiable function such that $0 < p(x) \leq (1 - \varepsilon)q(x)$, where $\varepsilon > 0$ and fixed. Let $\int_D q^{-1} |f|^2 dx < \infty$, $g \in H_q$ and let $Au = \sum_{i,k} D_i^2(a_{ik} D_k^2 u) - pu$ be a uniformly elliptic operator. That is, a_{ik} is positive definite and there exist positive constants M and λ such that

$$|a_{ik}(x)| \leq M \quad \text{and} \quad \lambda \sum_i |\xi_i|^2 \leq \sum_{i,k} a_{ik}(x) \xi_i \bar{\xi}_k,$$

for any $(\xi_1, \xi_2, \dots, \xi_n)$ and all x in D . Then the Dirichlet problem

$$\left. \begin{aligned} Au &= f && \text{in } D \\ u &= g \\ \sum_i D_i^2 u &= \sum_i D_i^2 g \end{aligned} \right\} \text{ on } \dot{D}$$

has a unique weak solution.

Proof. We must show that there is a function $u \in H_q$ such that $u - g \in \dot{H}_q$ and $(u, A\varphi) = (f, \varphi)$, for every φ in C_0^∞ . Set $u_0 = u - g$ and consider the equivalent problem of finding $u_0 \in \dot{H}_q$ such that $(u_0, A\varphi) = (f, \varphi) - (g, A\varphi)$. Let

$$\begin{aligned} B(u, v) &= \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{v} - pu \bar{v} dx \\ &= \int_D \sum_{i,k} u D_i^2 (a_{ik} D_k^2 \bar{v} - pu \bar{v}) dx \\ &= (u, Av), \quad \text{for } u, v \in C_0^\infty. \end{aligned}$$

We will show that there exist $c_1, c_2 > 0$ such that

$$|B(u, v)| \leq C_1 \|u\|_q \|v\|_q \quad \text{and} \quad B(u, u) \geq C_2 \|u\|_q^2.$$

$$\begin{aligned} B(u, u) &= \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - p |u|^2 dx \\ &\geq \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - q |u|^2 dx + \varepsilon \int_D q |u|^2 dx. \end{aligned}$$

By Corollary 1, both integrals are positive and hence

$$\begin{aligned} \left(1 + \frac{2}{\varepsilon}\right)B(u, u) &\geq \int_D \sum a_{ik} D_i u D_k \bar{u} + q |u|^2 dx \\ &\geq \int_D \lambda \sum_i |D_i u|^2 + q |u|^2 dx \\ &\geq \text{const} \|u\|_q^2. \end{aligned}$$

The positivity of $B(u, u)$ implies that $|B(u, v)|^2 \leq B(u, u) \cdot B(v, v)$ so that we need only show that $B(u, u) \leq \text{const} \|u\|_q^2$.

$$\begin{aligned} B(u, u) &\leq M \int_D \sum_{i,k} |D_i u D_k \bar{u}| + p |u|^2 dx \\ &\leq \int_D \sum_{i,k} \frac{1}{2} (|D_i u|^2 + |D_k u|^2) + p |u|^2 dx \\ &\leq Mn \int_D \sum_i |D_i u|^2 + p |u|^2 dx = Mn \|u\|_q^2. \end{aligned}$$

Now extend $B(u, v)$ to all of \mathring{H}_q by continuity. We can now apply the Lax-Milgram Theorem which guarantees that any bounded linear functional $F(\varphi)$ on \mathring{H}_q can be represented as $\overline{B(u_0, \varphi)}$ for some $u_0 \in \mathring{H}_q$. Take $F(\varphi) = \overline{(f, \varphi) - B(g, \varphi)}$. Then

$$\begin{aligned} |F\varphi| &\leq \left(\int_D q^{-1} |f|^2 dx\right)^{1/2} \left(\int_D q |\varphi|^2 dx\right)^{1/2} \\ &\quad + c_1 \|\varphi\|_q \|g\|_q \leq \text{const} \|\varphi\|_q. \end{aligned}$$

So $B(u_0, \varphi) = (f, \varphi) - B(g, \varphi)$ as was to be shown. To obtain the uniqueness result, let $Au = 0, u \in \mathring{H}_q$, then

$$0 = (u, Au) = B(u, u) \geq C_2 \|u\|_q^2. \quad \therefore u = 0 \text{ a.e.}$$

REFERENCE

1. James Calvert, *An integral inequality with applications to the Dirichlet problem*, Pacific J. Math. **22** (1967).

Received June 8, 1967.
UNIVERSITY OF CALIFORNIA, DAVIS

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

R. R. PHELPS

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Willard Ellis Baxter, <i>On rings with proper involution</i>	1
Donald John Charles Bures, <i>Tensor products of W^*-algebras</i>	13
James Calvert, <i>Integral inequalities involving second order derivatives</i>	39
Edward Dewey Davis, <i>Further remarks on ideals of the principal class</i>	49
Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients I</i>	53
Francis James Flanigan, <i>Algebraic geography: Varieties of structure constants</i>	71
Denis Ragan Floyd, <i>On QF – 1 algebras</i>	81
David Scott Geiger, <i>Closed systems of functions and predicates</i>	95
Delma Joseph Hebert, Jr. and Howard E. Lacey, <i>On supports of regular Borel measures</i>	101
Martin Edward Price, <i>On the variation of the Bernstein polynomials of a function of unbounded variation</i>	119
Louise Arakelian Raphael, <i>On a characterization of infinite complex matrices mapping the space of analytic sequences into itself</i>	123
Louis Jackson Ratliff, Jr., <i>A characterization of analytically unramified semi-local rings and applications</i>	127
S. A. E. Sherif, <i>A Tauberian relation between the Borel and the Lototsky transforms of series</i>	145
Robert C. Sine, <i>Geometric theory of a single Markov operator</i>	155
Armond E. Spencer, <i>Maximal nonnormal chains in finite groups</i>	167
Li Pi Su, <i>Algebraic properties of certain rings of continuous functions</i>	175
G. P. Szegő, <i>A theorem of Rolle's type in E^n for functions of the class C^1</i>	193
Giovanni Viglino, <i>A co-topological application to minimal spaces</i>	197
B. R. Wenner, <i>Dimension on boundaries of ε-spheres</i>	201