Pacific Journal of Mathematics

FURTHER REMARKS ON IDEALS OF THE PRINCIPAL CLASS

EDWARD DEWEY DAVIS

Vol. 27, No. 1 January 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 1, 1968

FURTHER REMARKS ON IDEALS OF THE PRINCIPAL CLASS

EDWARD D. DAVIS

In a previous paper on this subject, the author gave a new proof of the theorem of "analytic independence of systems of parameters". The methods used can be applied to prove a converse result. Hence: An ideal of a Noetherian ring is of the principal class if, and only if, it is generated by an analytically independent set.

The subset $\{z_1, \cdots, z_n\}$ of a ring R—R is commutative with $1 \neq 0$, as are all rings here considered—is said to be *analytically independent* provided that $(z_1, \cdots, z_n)R$ is not the unit ideal, and for every homogeneous f in the polynomial ring $R[X_1, \cdots, X_n]$ such that $f(z_1, \cdots, z_n) = 0$, the coefficients of f lie in the radical of $(z_1, \cdots, z_n)R$. Now let I be a nonunit ideal of R generated by the set $\{x, x_1, \cdots, x_n\}$, with x not a divisor of zero, and let Q denote the kernel of the R-homomorphism $R[X_1, \cdots, X_n] \to S = R[x_1/x, \cdots x_n/x]$ determined by $X_i \to x_i/x$. (We regard S as a subring of the full ring of quotients of R.) A completely elementary fact is that $\{x, x_1, \cdots, x_n\}$ is analytically independent if, and only if, for every prime ideal P of R containing I, $PR[X_1, \cdots, X_n]$ contains Q (see preliminary remarks of [1]). Using this fact we gave in [1] a new proof of the theorem of "analytic independence of systems of parameters": *If R is Noetherian and I is of height $n + 1$, then $\{x, x_1, \cdots, x_n\}$ is analytically independent.* In considering a question raised by Professor E. Kunz, it come to light that the methods of [1] can be used to prove the converse result:

PROPOSITION 1. *If R is Noetherian and $\{x, x_1, \cdots, x_n\}$ is analytically independent, then height $(I) = n + 1$.*

LEMMA 1. *If P is a prime ideal of R containing I, then PS is a prime ideal of S, $PS \cap R = P$, and S/PS is (R/P)-isomorphic to $(R/P)[X_1, \cdots, X_n]$.*

LEMMA 2. *Let M be a prime ideal of S, and let $P = M \cap R$. Then height $(P) \geqq$ tr.dg. $[S/M : R/P]$ + height (M).*

Proof of the lemmas. The facts stated in Lemma 1 are proved in the preliminary remarks of [1]. Lemma 2 is well known in the case of R an integral domain (see, for example, Appendix 1 of Zariski-

Samuel vol. II [2]); we generalize to the case at hand as follows. Let M_0 be a minimal prime ideal of S contained in M such that height (M/M_0) = height (M), and let $P_0 = R \cap M_0$. Observe that S/M_0 and R/P_0 have the same quotient field, namely the quotient field of K/N_0, where K is the full ring of quotients of R, and N_0 is the unique prime ideal of K lying over M_0 (and P_0). Hence, using the integral domain result, we have:

$$\begin{aligned}\text{height } (P) &\geqq \text{height } (P/P_0) \\ &\geqq \text{tr.dg. } [S/M_0/M/M_0 : R/P_0/P/P_0] + \text{height } (M/M_0) \\ &\geqq \text{tr.dg. } [S/M : R/P] + \text{height } (M) .\end{aligned}$$

Proof of Proposition 1. Let P be a minimal prime component of I. We must show that height $(P) \geqq n + 1$, for it is well known that height $(P) \leqq n + 1$. Now let $M = PS$. Lemma 1 says: M is a prime ideal, $M \cap R = P$, and S/M is a polynomial ring in n variables over R/P. From these facts and Lemma 2 we have:

$$\text{height } (P) \geqq \text{tr.dg. } [S/M : R/P] + \text{height } (M) = n + \text{height } (M) .$$

Since M contains x, which is not a divisor of zero, height $(M) \geqq 1$ (actually height $(M) = 1$). Thus height $(P) \geqq n + 1$.

COROLLARY. *An ideal generated by an analytically independent subset of a Noetherian ring is of height equal to the cardinality of that set.*

Proof. Let A, J and $\{x_1, \cdots, x_n\}$ be the ring, ideal and analytically independent set in question. Let $R = A[x]$, where x is an indeterminate over A, and let $I = (x, x_1, \cdots, x_n)R$. It is clear that height (I) = height $(J) + 1$. Hence it suffices to prove that height $(I) = n + 1$. It is notationally somewhat bothersome, but completely straightforward to verify that $\{x, x_1, \cdots, x_n\}$ is an analytically independent subset of R. Proposition 1 now applies.

REMARK. One can prove this corollary in a somewhat less elementary way. Localize A at a minimal prime component P of J and then complete. The canonical image of the set $\{x_1, \cdots, x_n\}$ is an analytically independent subset of the complete ring and the image of J is primary for the maximal ideal. Then conclude, using Cohen's structure theory for complete local rings, that the complete ring is an integral extension of a ring of dimension at least n. Thus dimension $(A_P) \geqq n$. The "characteristic equal" and "characteristic unequal" cases must be treated separately.

This corollary and the theorem of "analytic independence of systems of parameters" (proved in many places, among them in Corol-

lary 1 to Proposition 1 of [1]) comprise the following theorem.

THEOREM. *An ideal of a Noetherian ring is of the principal class if, and only if, it is generated by an analytically independent set. Furthermore, the height of an ideal of the principal class is equal to the cardinality of any analytically independent set of generators (which then must be a minimal generating set).*

And of special interest is the following criterion.

COROLLARY. *A subset consisting of dimension (R) elements of a local ring R is a system of parameters of R if, and only if, that subset is analytically independent.*

In the notation of the discussion immediately preceding the statement of Proposition 1, this proposition combines with Proposition 1 of [1] to say:

PROPOSITION 2. *The following statements are equivalent.* (1) *Height* $(I) = n + 1$. (2) $\{x, x_1, \cdots, x_n\}$ *is analytically independent.* (3) *The radicals of* Q *and the ideal generated by* $\{xX_1 - x_1, \cdots, xX_n - x_n\}$ *coincide.*

And in the special case that R is an integral domain we have:

PROPOSITION 2′. *Statements* (1) *and* (2) *of Proposition* 2 *are equivalent to:* (3′) Q *is the radical of the ideal generated by* $\{xX_1 - x_1, \cdots, xX_n - x_n\}$.

REFERENCES

1. E. D. Davis, *Ideals of the principal class, R-sequences and a certain monoidal transformation*, Pacific J. Math. **20** (1967), 197-205.
2. O. Zariski and P. Samuel, *Commutative Algebra Volume II*, Van Nostrand, Princeton, 1960.

Received October 31, 1967. Supported in part by the National Science Foundation under Grant GP-6388.

PURDUE UNIVERSITY

STATE UNIVERSITY OF NEW YORK AT ALBANY
(Present address)

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 27, No. 1 January, 1968

Willard Ellis Baxter, *On rings with proper involution* 1
Donald John Charles Bures, *Tensor products of W^*-algebras* 13
James Calvert, *Integral inequalities involving second order derivatives* 39
Edward Dewey Davis, *Further remarks on ideals of the principal class* 49
Le Baron O. Ferguson, *Uniform approximation by polynomials with integral coefficients I* 53
Francis James Flanigan, *Algebraic geography: Varieties of structure constants* 71
Denis Ragan Floyd, *On* QF − 1 *algebras* 81
David Scott Geiger, *Closed systems of functions and predicates* 95
Delma Joseph Hebert, Jr. and Howard E. Lacey, *On supports of regular Borel measures* 101
Martin Edward Price, *On the variation of the Bernstein polynomials of a function of unbounded variation* 119
Louise Arakelian Raphael, *On a characterization of infinite complex matrices mapping the space of analytic sequences into itself* 123
Louis Jackson Ratliff, Jr., *A characterization of analytically unramified semi-local rings and applications* 127
S. A. E. Sherif, *A Tauberian relation between the Borel and the Lototsky transforms of series* 145
Robert C. Sine, *Geometric theory of a single Markov operator* 155
Armond E. Spencer, *Maximal nonnormal chains in finite groups* 167
Li Pi Su, *Algebraic properties of certain rings of continuous functions* 175
G. P. Szegő, *A theorem of Rolle's type in E^n for functions of the class C^1* . . . 193
Giovanni Viglino, *A co-topological application to minimal spaces* 197
B. R. Wenner, *Dimension on boundaries of ε-spheres* 201