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**UNIFORM APPROXIMATION BY POLYNOMIALS WITH  
INTEGRAL COEFFICIENTS I**

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## UNIFORM APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS I

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Throughout this paper  $A$  will denote a discrete subring of the complex number plane  $C$  with rank 2. For example,  $A$  could be the Gaussian integers  $Z + iZ$ , where  $Z$  denotes the rational integers, or the ring of integers of any imaginary quadratic field. We are concerned with characterizing those functions defined on a compact subset  $X$  of  $C$  which can be uniformly approximated by polynomials with coefficients in  $A$ . We say that such functions are  $A$ -approximable on  $X$ . We also consider the real case where  $X$  is any compact subset of the reals  $R$  and the coefficients of the approximating polynomials lie in  $Z$  or any discrete subring of  $R$ . The real case is completely solved in the sense that a necessary and sufficient condition in order that a function can be so approximated is found. The complex case is solved if, in addition to being compact,  $X$  either has transfinite diameter at least unity or void interior and connected complement.

The case where  $X$  has transfinite diameter less than unity and nonvoid interior will be the subject of a later paper.

The complex case was solved by Fekete when the ring of coefficients  $A$  is the ring of integers of an imaginary quadratic field. His results were announced in [4], but, as far as we know, proofs were never published. In [4] is found the key notion of the "algebraic kernel" of a compact subset of  $C$  with respect to an imaginary quadratic field. This notion is extended here so as to be relevant to any ring  $A$  defined above and a second characterization of the algebraic kernel (herein denoted by  $J_0(X, A)$ ) is found. The set  $J_c(X, A)$  seems to be difficult to determine, in general. Its calculation will be the subject of a future paper.

The real case was solved for intervals in Hewitt and Zuckerman [7]. We use the results obtained here in the complex case to extend their results to arbitrary compact subsets of  $R$ .

Throughout this paper we endeavor to follow the terminology and notation in [2]. We use the symbol  $C(X)$  to denote the set of all complex-valued continuous functions defined on  $X$  and  $C^r(X)$  to denote the real valued members of  $C(X)$ . If  $f \in C(X)$  and  $S \subset X$  we define  $\|f\|_S = \sup \{|f(x)| : x \in S\}$ . We frequently write  $\|f\|$  for  $\|f\|_X$ .

2. Discrete rings and imaginary quadratic fields. We mention here the nonstandard results on these rings which will be needed

repeatedly.

If  $F$  is a field extension of  $\mathbf{Q}$  we denote the ring of algebraic integers of  $F$  by  $I_F$ . If  $L$  is an imaginary quadratic field it is easy to see that  $I_L$  is a discrete subring of  $\mathbf{C}$  and has rank 2. On the other hand every ring  $A$  with these properties is contained in  $I_L$  for some imaginary quadratic field  $L$  [11, p. 150, Nr. 203]. Since  $A$  has rank 2,  $L$  is uniquely determined by the inclusion  $A \subset I_L$ . Using the well known explicit representations for the elements of  $I_L$  [13, p. 234] it is easy to see that there exists a positive integer  $m$  such that  $mI_L \subset A$ .

Using the fact that  $A$  has rank 2 and Theorem 1 of [3, p. 77] it is not hard to see that there exists  $\delta > 0$ , depending only on  $A$ , such that if  $z \in \mathbf{C}$ , there exists  $a \in A$  with  $|z - a| < \delta$ .

3. Chebyshev polynomials and transfinite diameter. Let  $X$  be a compact subset of  $\mathbf{C}$  and  $n$  a positive integer. If  $X$  is infinite the  $n^{\text{th}}$  Chebyshev polynomial  $t_n(z, X)$  for  $X$  is defined to be the unique monic polynomial of degree  $n$  such that

$$\|t_n(z, X)\|_X = \inf \|t\|_X$$

where the inf is taken over all such polynomials. If  $X$  contains  $m$  elements, where  $m$  is finite, then we define  $t_n(z, X)$  as above for  $n \leq m$  and set  $t_n(z, X) = \prod_{x \in X} (z - x)$  for  $n > m$ .

The existence of  $t_n(z, X)$  is a direct consequence of [1, p. 10]. The uniqueness follows from [12, p. 36, Th. 1].

We define the transfinite diameter  $d(X)$  of  $X$  by

$$d(X) = \lim_{n \rightarrow \infty} \|t_n(z, X)\|^{1/n}.$$

(See [8, p. 226, Th. 16.1.2].)

The importance for us of the concept of transfinite diameter is contained in the following.

**PROPOSITION 3.1.** Let  $X$  be a compact subset of  $\mathbf{C}$  with  $d(X) \geq 1$ . Then a complex valued function  $f$  on  $X$  is  $A$ -approximable on  $X$  if and only if it is already an element of  $A[z]$ .

*Proof.* Suppose that  $f$  is  $A$ -approximable on  $X$  but  $f \notin A[z]$ . Then there exist  $p_1$  and  $p_2$  in  $A[z]$  such that  $p_1 \neq p_2$  and  $\|p_i - f\| < 1/2$  for  $i = 1, 2$ . Thus  $\|p_1 - p_2\| < 1$ . Since  $A$  is discrete, the leading coefficient of  $p_1 - p_2$  has modulus at least one and dividing  $p_1 - p_2$  by this coefficient gives a monic polynomial  $p$  with  $\|p\| < 1$ . From the existence of such a  $p$  it is easy to see that  $d(X) < 1$ , a contradiction.

4. **The algebraic kernel.** In this section we define the algebraic kernel of a compact subset  $X$  of  $C$  with respect to  $A$ . We then give a necessary condition in order that a function be  $A$ -approximable.

DEFINITION 4.1. Let  $R$  be any subring of  $C$  and  $f$  a complex valued function on a subset  $X$  of  $C$ . We say that  $f$  is  $R$ -matchable on a subset  $S$  of  $X$  if there exists  $p \in R[z]$  such that  $p(z) = f(z)$  for all  $z \in S$ .

DEFINITION 4.2. If  $R$  is any subring of  $C$  and  $X$  is a compact subset of  $C$  we define

$$B(X, R) = \{p \in R[z]: \|p\|_X < 1\} .$$

Note that in 3.1 we have proved something stronger than the proposition. In fact we see that if  $R$  is a discrete subring of  $C$  and  $X$  is a compact subset of  $C$  with  $d(X) \geq 1$ , then  $R[z]$  is a discrete and therefore closed subring of  $C(X)$ . Indeed, we can prove that  $B(X, R) = \{0\}$  as follows. If  $g \in B(X, R)$  and  $g$  is not identically zero on  $X$  then we can divide by its leading coefficient to obtain a monic polynomial  $p$  such that  $0 < \|p\|_X < 1$  and derive a contradiction to  $d(X) \geq 1$  as in the proof of 3.1. Now, by [6, p. 35, (5.10)] since  $R[z]$  is a discrete (additive) subgroup of  $C(X)$  it is closed in  $C(X)$ .

DEFINITION 4.3. For any subring  $R$  of  $C$  and compact subset  $X$  of  $C$  we define

$$J(X, R) = \{z \in X: p(z) = 0 \text{ for all } p \in B(X, R)\} .$$

When no confusion is possible we write  $J(X)$  or simply  $J$  for  $J(X, R)$ .

If  $A$  is a discrete subring of  $C$  with rank 2 then by §2 there exists exactly one imaginary quadratic field  $L$  such that  $A \subset I_L$ . With this fact in mind we make the following definition.

DEFINITION 4.4. Let  $X$  be a compact subset of  $C$  and  $L$  the imaginary quadratic field such that  $A \subset I_L$ . We define  $J_0(X, A)$  to be the union of the complete sets of conjugates integral over  $I_L$  which are entirely contained in  $X$ .

We note that  $J_0(X, I_L)$  is what Fekete called the "algebraic kernel" of  $X$  with respect to the field  $L$  [4, p. 1338].

PROPOSITION 4.5. If  $X$  is a compact subset of  $C$ , then

$$J_0(X, A) \subset J(X, A) .$$

*Proof.* Suppose  $A \subset I_L$ . Let  $\theta$  be in  $J_0(X, A)$  and  $q \in A[z]$  with  $\|q\|_X < 1$ . We can write the conjugates of  $\theta$  over  $L$  as  $\sigma_1(\theta), \sigma_2(\theta), \dots, \sigma_n(\theta)$  where the  $\sigma_i$ 's are automorphisms of the splitting field  $F$  of  $\theta$  which leave  $L$  pointwise fixed. Since  $J_0(X, A)$ , hence  $X$ , contains the conjugates we have

$$1 > \left| \prod_{i=1}^n q(\sigma_i(\theta)) \right| = \left| \prod_{i=1}^n \sigma_i(q(\theta)) \right| = |N_L^F(q(\theta))|.$$

Because  $q(\theta)$  is integral over  $I_L$  we have  $N_L^F(q(\theta)) \in I_L$ . Since  $I_L$  is discrete and  $|N_L^F(q(\theta))| < 1$ ,  $N_L^F(q(\theta)) = 0$ . Thus  $q(\theta) = 0$ .

**PROPOSITION 4.6.** If  $X$  is a compact subset of  $C$ , then in order that a complex valued function  $f$  be  $A$ -approximable on  $X$ , it is necessary that  $f$  be  $A$ -matchable on  $J(X, A)$ .

*Proof.* Suppose that  $f$  is  $A$ -approximable on  $X$ . That is, there is a sequence  $(p_n)$  of polynomials in  $A[z]$  which tends uniformly to  $f$  on  $X$ . Then there is an integer  $N$  such that  $m, n > N$  implies  $\|p_n - p_m\| < 1$ . Then  $p_n - p_m$  is an element of  $B(X, A)$  and so  $p_n - p_m = 0$  on  $J(X, A)$ . Thus  $m > N$  implies that  $p_m$  matches  $f$  on  $J(X, A)$ .

5. The complex case. In this section we prove a necessary and sufficient condition for approximability over a class of compact subsets of  $C$  defined below.

**DEFINITION 5.1.** A compact subset  $X$  of  $C$  is said to be Lavrent'ev if  $C[z]$  is uniformly dense in  $C(X)$ .

This terminology stems from the fact that in 1934 Lavrent'ev proved the following [9].

**PROPOSITION 5.2.** A compact subset  $X$  of  $C$  is Lavrent'ev if and only if it has void interior and connected complement.

**PROPOSITION 5.3.** Let  $A$  contain the identity and let  $X$  be a compact subset of  $C$ . If there exists a monic polynomial  $p \in C[z]$  with  $\|p\| < 1$ , then there exists a monic polynomial  $q \in A[z]$  with  $\|q\| < 1$ .

*Proof.* Let  $n$  be the degree of  $p$ . Then define a sequence (starting with the integer  $n$ ) of monic polynomials as follows. For  $m \geq n$ , set  $m = kn + r$  where  $0 \leq r < n$  and  $k \geq 1$ . Then let

$$(1) \quad p_m(z) = z^r p(z)^k.$$

Note that  $p_m$  has degree  $m$ . Also, if  $s = \|p\| < 1$  set  $t = s^{1/n}$  ( $t \geq 0$ ) so that  $s = t^n$ ; it is clear that  $0 \leq t < 1$ . Next pick a real number  $M$  such that

$$\|z^i\| < sM \quad \text{for } 1 \leq i \leq n,$$

if  $s > 0$ , or set  $M = 0$  if  $s = 0$ . Then writing  $m$  as above we have

$$\|p_m\| \leq \|p\|^k \|z^r\| \leq s^k sM = t^{nk+n}M \leq t^{nk+r}M = t^m M.$$

Now fix a positive integer  $j \geq n - 1$  such that

$$\delta M t^{j+1} / (1 - t) < 1/3,$$

where  $\delta > 1$  satisfies the conditions in § 2. For each  $m > j$  we define a polynomial  $q_m$  as follows. Set

$$(2) \quad q_m = \alpha_0 p_m + \alpha_1 p_{m-1} + \cdots + \alpha_{m-j-1} p_{j+1}$$

where the  $\alpha$ 's are defined as follows. Let  $\alpha_0 = 1$ . Let  $\beta$  be the coefficient of  $z^{m-1}$  in  $\alpha_0 p_m$ . By the definition of  $\delta$ , there is a  $\beta' \in A$  such that  $|\beta' - \beta| < \delta$ . Then if we set  $\alpha_1 = \beta' - \beta$ , we have that  $p' = \alpha_0 p_m + \alpha_1 p_{m-1}$  has leading coefficient  $\alpha_0 = 1$  since the degree of  $\alpha_1 p_{m-1}$  is less than that of  $p_m$ . Also, the coefficient of  $z^{m-1}$  in  $p'$  is the element  $\beta + (\beta' - \beta) = \beta'$  of  $A$ . Continuing in this way, we pick  $\alpha$ 's such that  $|\alpha_i| < \delta$  for  $1 \leq i \leq m - j - 1$  and the coefficients of  $z^m, \dots, z^{j+1}$  in  $q_m$  are elements of  $A$ . We have

$$(3) \quad \begin{aligned} \|q_m\| &\leq \sum_{i=0}^{m-j-1} \|\alpha_i p_{m-i}\| \leq \sum_{i=0}^{m-j-1} \delta t^{m-i} M \\ &= \delta M t^{j+1} \sum_{i=0}^{m-j-1} t^i \leq \delta M \frac{t^{j+1}}{1-t} < 1/3. \end{aligned}$$

Next, if  $m > j$ , we define the  $(j+1)$ -tuple

$$((a_{m0}), \dots, (a_{mj}))$$

as follows. If  $a_{mi}$  is the coefficient of  $z^i$  in  $q_m$  ( $0 \leq i \leq j$ ), then let  $[a_{mi}]$  be an element of  $A$  closest to  $a_{mi}$  and set  $(a_{mi}) = a_{mi} - [a_{mi}]$ , so that  $|(a_{mi})| < \delta$ . As  $m$  varies, these  $(j+1)$ -tuples remain in the product space  $(\delta D)^{j+1}$ , where  $D$  is the closed unit disk in  $C$ .

Now if

$$M' = \max \{\|z^i\| : 0 \leq i \leq j\},$$

we choose  $\varepsilon' > 0$  such that

$$\varepsilon'(j+1)M' < 1/3.$$

Then, since  $(\delta D)^{j+1}$  is compact in the topology given by the norm

$$||| (z_0, \dots, z_j) ||| = \max_{0 \leq i \leq j} |z_i| ,$$

there exist distinct  $m_1$  and  $m_2$  such that

$$||| ((a_{m_1 0}), \dots, (a_{m_1 j})) - ((a_{m_2 0}), \dots, (a_{m_2 j})) ||| < \varepsilon' .$$

We then have

$$\begin{aligned} & \sum_{i=0}^j |(a_{m_1 i}) - (a_{m_2 i})| \|z^i\| \\ (4) \quad & \leq (j+1) \max_{0 \leq i \leq j} \{|(a_{m_1 i}) - (a_{m_2 i})| \|z^i\|\} \\ & < (j+1)\varepsilon' M' \\ & < 1/3 . \end{aligned}$$

We now combine these estimates as follows. From (3) we infer that

$$(5) \quad \|q_{m_1} - q_{m_2}\| \leq \|q_{m_1}\| + \|q_{m_2}\| < 2/3 .$$

If  $q'_m$  denotes  $q_m$  with  $[a_{m_i}]$  in place of  $a_{m_i}$  for  $0 \leq i \leq j$ , then (4) shows that

$$\begin{aligned} (6) \quad & \| (q_{m_1} - q_{m_2}) - (q'_{m_1} - q'_{m_2}) \| \\ & = \| (q_{m_1} - q'_{m_1}) - (q_{m_2} - q'_{m_2}) \| \\ & \leq \sum_{i=0}^j |(a_{m_1 i}) - (a_{m_2 i})| \|z^i\| < 1/3 . \end{aligned}$$

Combining (5) and (6) we obtain

$$\|q'_{m_1} - q'_{m_2}\| < 1 .$$

Also  $q'_{m_1} - q'_{m_2}$  is a monic polynomial because each  $q_m$  is monic with degree  $m$  and  $m_1 \neq m_2$ . Thus we can take  $q = q'_{m_1} - q'_{m_2}$  in the proposition.

**COROLLARY 5.4.** *If  $A$  contains the identity and  $X$  is a compact subset of  $C$  with  $d(X) < 1$ , then there is a monic polynomial  $q \in A[z]$  with  $\|q\|_X < 1$ .*

**LEMMA 5.5.** *Let  $q$  be a monic polynomial in  $A[z]$  with  $\|q\|_X < 1$  and  $b \in C[z]$ . Then there exists  $[b]$  in  $A[z]$  and  $M$  not depending on  $b$  such that*

$$\|b - [b]\|_X < M .$$

*Proof.* Since  $n = \deg q$  is at least 1 we can write

$$b = b_0 + b_1q + \dots + b_kq^k$$

where each  $b_i \in C[z]$  and  $\deg b_i < \deg q$  for  $0 \leq i \leq k$ . For each  $i$  let  $[b_i]$  be the polynomial obtained from  $b_i$  by replacing each coefficient by a nearest element of  $A$ . Then with  $\delta$  as in § 2 we have

$$\|b_i - [b_i]\| \leq \sum_{j=0}^{n-1} \|\delta z^j\| = M_0 \quad 0 \leq i \leq k,$$

where the last equality serves to define  $M_0$ . We have

$$\begin{aligned} \left\| b - \sum_{i=0}^k [b_i]q^i \right\| &= \sum_{i=0}^k (b_i - [b_i])q^i \\ &\leq \sum_{i=0}^k \|b_i - [b_i]\| \|q\|^i \\ &< M_0/(1 - \|q\|). \end{aligned}$$

LEMMA 5.6. *Let  $X$  be a compact subset of  $C$  and suppose further that*

- (i)  $X$  is Lavrent'ev;
- (ii)  $f \in C(X)$ ;
- (iii)  $q$  is a monic polynomial in  $A[z]$  with  $\|q\|_X < 1$ ;
- (iv) for any  $\varepsilon > 0$  there is an  $r \in A[z]$  such that  $|f(z) - r(z)| < \varepsilon$  whenever  $q(z) = 0, z \in X$ .

Then  $f$  is  $A$ -approximable on  $X$ .

*Proof.* Let  $Z_q$  be the set of roots of  $q$  which lie in  $X$ . Let  $\varepsilon$  be any positive number. By (iv) there is an  $r \in A[z]$  such that

$$|f(z) - r(z)| < \varepsilon/4 \quad \text{for } z \in Z_q.$$

Then by continuity, there is, for each  $\alpha \in Z_q$  a closed disk  $D_\alpha$  with center  $\alpha$  and radius  $\rho_\alpha$  such that the family  $\{D_\alpha\}_{\alpha \in Z_q}$  is pairwise disjoint and

$$|f(z) - r(z)| < \varepsilon/2 \quad \text{for } z \in D_\alpha \cap X.$$

Plainly there is a continuous function  $u$  mapping  $X$  into  $[0, 1]$  such that  $u(z) = 1$  for  $z$  in no  $D_\alpha$  and  $u(z) = 0$  if for some  $\alpha \in Z_q, z$  is in the closed disk of radius  $\rho_\alpha/2$  and centered at  $\alpha$ . It is easy to see that

$$(1) \quad \|u(f - r) - (f - r)\| < \varepsilon/2.$$

By 5.5 there is a positive integer  $N$  such that

$$(2) \quad \|bq^N - [b]q^N\| \leq \varepsilon/4$$

for every  $b \in C[z]$ . Now consider  $u(f - r)/q^N$ , which is defined to be zero whenever  $q$  is zero. It is continuous by construction. Thus by



(i), there is an element  $b \in C[z]$  such that

$$\left\| \frac{u(f-r)}{q^N} - b \right\| < \varepsilon/4 .$$

It follows that

$$\|u(f-r) - bq^N\| < \|q\|^N \varepsilon/4 < \varepsilon/4 .$$

Then by (2)

$$\|u(f-r) - [b]q^N\| < \varepsilon/2$$

and by (1)

$$\|(f-r) - [b]q^N\| < \varepsilon$$

or

$$\|f - (r + [b]q^N)\| < \varepsilon .$$

**THEOREM 5.7.** *Let  $X$  be a Lavrent'ev subset of  $C$  with  $d(X) < 1$ . If  $f$  is a complex valued function on  $X$  then the following are equivalent.*

- (i)  $f$  is  $A$ -approximable on  $X$ ;
- (ii)  $f$  is continuous and  $A$ -matchable on  $J_0(X, A)$ .

*Proof.* From 4.6 and 4.5 we see that (i) implies (ii). To prove the converse first note that if  $f = p$  on  $J_0(X, A)$  and  $p \in A[z]$ , then it suffices to approximate  $f - p$  which is zero on  $J_0(X, A)$ . Hence we assume that  $f = 0$  on  $J_0(X, A)$ . Let  $L$  be the imaginary quadratic field such that  $A \subset I_L$ . By § 2 there exists a positive integer  $m$  such that  $mI_L \subset A$ . Thus if  $p \in I_L[z]$  and  $\|f/m - p\| < \varepsilon/m$  then  $\|f - mp\| < \varepsilon$  and  $mp \in A[z]$ . In view of this we assume that  $A = I_L$ .

Then  $A, X$ , and  $f$  satisfy the hypotheses of 5.6 and it only remains to show that 5.6 (iii) and (iv) hold. By 5.4 there exists a monic  $q \in A[z]$  with  $\|q\|_X < 1$ , so that 5.6 (iii) is satisfied. Let  $Z_q$  denote the set of all zeroes of  $q$  which lie in  $X$ . Write  $J_0(X, A)$  as the union of the sets of zeroes of a set of monic irreducible polynomials,  $\{q_1, \dots, q_s\}$ , in  $I_L[z]$ . Denote the remaining elements of  $Z_q$  by  $\alpha_1, \dots, \alpha_k$  so that

$$Z_q = J_0(X, A) \cup \{\alpha_1, \dots, \alpha_k\} .$$

By the definition of  $J_0(X, A)$  the  $\alpha_i$ 's form a set of algebraic numbers which does not contain a complete set of conjugates over  $L$ . In view of this 7.3 can be applied to give  $p \in A[z]$  such that

$$\left| p(\alpha_i) - \frac{f(\alpha_i)}{q_1 \cdots q_s(\alpha_i)} \right| < \frac{\varepsilon}{|q_1 \cdots q_s(\alpha_i)|} \quad \text{for } 1 \leq i \leq k .$$

Then

$$|pq_1 \cdots q_s(z) - f(z)| < \varepsilon \quad \text{for } z \in Z_q$$

and  $pq_1 \cdots q_s \in A[z]$  which shows that 5.6 (iv) is satisfied.

**THEOREM 5.8.** *Let  $X$  be a Lavrent'ev subset of  $C$  with  $d(X) < 1$ . Then a continuous complex valued function  $f$  on  $X$  is  $A$ -approximable if and only if its Lagrange interpolating polynomial  $r$  on  $J_0(X, A)$  is an element of  $A[z]$ .*

*Proof.* By 5.7 the condition  $r \in A[z]$  is sufficient for the  $A$ -approximability of  $f$  since  $r$  matches  $f$  on  $J_0(X, A)$ . Conversely, from 5.7 we know that if  $f$  is  $A$ -approximable then there is a  $p \in A[z]$  which matches  $f$  on  $J_0(X, A)$ . Let  $q_1, \dots, q_s$  be as in the proof of 5.7. Since each  $q$  is irreducible it has only simple roots. Thus  $\deg q_1 \cdots q_s = \text{card } J_0(X, A)$  which we write as  $n$ . Since  $q_1 \cdots q_s$  is a monic polynomial in  $A[z]$  we can find  $w, t \in A[z]$  such that

$$p = w(q_1 \cdots q_s) + t, \quad \deg t < n$$

by the division algorithm. Thus  $t = p = f$  on  $J_0(X, A)$  and then by the uniqueness of Lagrange interpolating polynomials  $t = r$  and  $r = f$  on  $J_0(X, A)$ .

We see from 5.7 and 5.8 that, under the hypotheses of 5.7 the question of approximability is effectively known once we know the finite set  $J_0(X, A)$ . The following shows that under these hypotheses the set  $J_0(X, A)$  has another characterization.

**THEOREM 5.9.** *Let  $X$  be a Lavrent'ev subset of  $C$  with  $d(X) < 1$ . Then*

$$J(X, A) = J_0(X, A) .$$

*Proof.* By 4.5,  $J_0 = J_0(X, A) \subset J(X, A) = J$ , so we need only prove the reverse inclusion. Let  $L$  be the imaginary quadratic field such that  $A \subset I_L$ . By § 2 there is a positive integer  $m$  such that  $mI_L \subset A$ . By 5.4 there is a monic  $q \in I_L[z]$  such that  $\|q\| < 1$ . Then for a sufficiently large positive integer  $N$ ,  $q_0 = mq^N$  is a nonzero polynomial in  $A[z]$  with  $\|q_0\| < 1$ . This shows that  $J$  is finite. Let  $\alpha_1$  be any element of  $J$ . Then  $q_0(\alpha_1) = 0$  so  $q(\alpha_1) = 0$  and  $\alpha_1$  is integral over  $I_L$ . Let  $r \in I_L[z]$  be the minimal polynomial over  $L$  of  $\alpha_1$ . We assume that  $\alpha_1 \notin J_0$ , that is that not all the zeroes of  $r$  lie in  $X$  and

infer a contradiction from this. Denote the zeroes of  $r$  by  $\{\alpha_1, \dots, \alpha_n\}$  where

$$\alpha_i \in X \quad \text{for } 1 \leq i \leq k$$

and

$$\alpha_i \notin X \quad \text{for } k < i \leq n \ (k < n).$$

By 7.1 there is a  $q'_1 \in I_L[z]$  such that

$$\left| q'_1(\alpha_i) - \frac{1}{2m} \right| < \frac{1}{2m} \quad \text{for } 1 \leq i \leq k.$$

Then if  $q_1 = mq'_1$ ,  $q_1 \in A[z]$  and

$$(1) \quad |q_1(\alpha_i) - 1/2| < 1/2 \quad \text{for } 1 \leq i \leq k.$$

Now, since  $\alpha_i \in J$ , if  $q_2$  is any element of  $A[z]$  with  $\|q_2\| < 1$ , then  $q_2(\alpha_i) = 0$ . The minimal polynomial  $r$  of  $\alpha_1$  then divides  $q_2$  and so  $q_2(\alpha_i) = 0$  for  $1 \leq i \leq n$ . Thus  $\{\alpha_1, \dots, \alpha_k\}$  is contained in  $J$ . We write

$$J = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l\}$$

where the  $\beta_j$ 's are distinct and also distinct from the  $\alpha_i$ 's. If  $l = 0$  define  $q'_3$  to be the polynomial 1. If  $l > 0$  define  $q'_3$  to be the product of the minimal polynomials over  $L$  of the  $\beta_j$ 's. Each  $\beta_j$ , being in  $J$ , is a zero of  $q_0$  and therefore of  $q$  and so is integral over  $I_L$ . Thus  $q'_3$  is an element of  $I_L[z]$ . Furthermore,  $q_3 = mMq'_3 \in A[z]$  and

$$(2) \quad q_3(\beta_i) = 0 \quad \text{for } 1 \leq i \leq l.$$

Since no  $\beta_j$  is conjugate to an  $\alpha_i$

$$(3) \quad q_3(\alpha_i) \neq 0 \quad \text{for } 1 \leq i \leq n.$$

By (1) and (3) there is a positive integer  $s$  such that if

$$q_4 = q_3q_1^s,$$

then

$$0 < |q_4(\alpha_i)| < 1 \quad \text{for } 1 \leq i \leq k.$$

Also, by (2), we have

$$q_4(\beta_i) = 0 \quad \text{for } 1 \leq i \leq l.$$

Now it is easy to construct a continuous function  $f$  with  $\|f\| < 1$  such that

$$f(\alpha_i) = q_i(\alpha_i) \quad \text{for } 1 \leq i \leq k$$

and

$$f(\beta_i) = q_i(\beta_i) = 0 \quad \text{for } 1 \leq i \leq l.$$

This function  $f$  is matchable on  $J_0$  and by 5.8, it is  $A$ -approximable on  $X$ . It is easy to see that we can choose  $q_i \in A[z]$  with  $\|f - q_i\|$  small enough to force  $\|q_i\| < 1$  and  $|q_i(\alpha_i)| > 0$  which is a contradiction.

In § 4 we showed that if  $d(X) \geq 1$  then  $A[z]$  is already uniformly closed in  $C(X)$ . The following is a partial converse to that result.

PROPOSITION 5.10. Let  $X$  be a Lavrent'ev subset of  $C$  with  $d(X) < 1$ . Then  $A(z)$  is uniformly closed in  $C(X)$  if and only if

$$J(X, A) = X.$$

In particular, if  $X$  is infinite, then  $A[z]$  is not uniformly closed in  $C(X)$ .

*Proof.* Suppose that  $J = J(X, A) = X$  and that  $f \in C(X)$ . By 5.7 and 5.9, if  $f$  is  $A$ -approximable on  $X$ , it is  $A$ -matchable on  $J = X$  and so  $f \in A[z]$ , which shows  $A[z]$  is uniformly closed in  $C(X)$ .

On the other hand, if  $J \neq X$ , let  $z_0$  be a point in  $X$  but not in  $J$ . For any  $y \in \mathbf{R}$  we can define a continuous function  $f_0: J \cup \{z_0\} \rightarrow C$  by  $f_0(J) = \{0\}$  and  $f_0(z_0) = y$ . It is continuous where defined since  $d(X) < 1$  which implies that  $J$  is finite, as in the proof of 5.7, so that the relative topology on  $J \cup \{z_0\}$  is discrete. By Tietze's extension theorem there is a continuous extension  $f$  of  $f_0$  to all of  $X$ . But  $f$  is obviously  $A$ -matchable on  $J$  and so is  $A$ -approximable on  $X$  by 5.7. Since  $y$  is any real number this shows that there are uncountably many  $A$ -approximable  $f$  in  $C(X)$ . On the other hand,  $A[z]$  is countable, since  $A$  is, so  $A[z] \neq A[z]^-$ , where the bar denotes uniform closure in  $C(X)$ .

The last statement now follows from the fact that  $J(X, A)$  is finite whenever  $d(X) < 1$ .

6. The real case. In this section we consider the problem analogous to that of § 5 but where  $X$  is a compact subset of  $\mathbf{R}$ . We take as our ring of coefficients any nonzero subring  $R$  of  $\mathbf{Z}$ . Such rings comprise the discrete nonzero subrings of  $\mathbf{R}$ . They are thus discrete subrings of  $C$  but do not have rank 2. The results follow readily from the corresponding results in the complex case.

To emphasize that  $X \subset \mathbf{R}$ . We use the symbol  $x$  instead of  $z$  to denote an arbitrary element of  $X$ .

Note that in § 4 the ring of coefficients  $A$  was required to have rank 2, hence the following definition is consistent with 4.4.

**DEFINITION 6.1.** For any compact subset  $X$  of  $\mathbf{R}$  let  $J_0(X, \mathbf{Z})$  denote the union of the complete sets of conjugates integral over  $\mathbf{Z}$  and entirely contained in  $X$ .

That is,  $J_0(X, \mathbf{Z})$  is the union of the complete sets of conjugate algebraic integers contained in  $X$ .

We now proceed to show that this separate definition is, in a sense, unnecessary.

**PROPOSITION 6.2.** Let  $X$  be a compact subset of  $\mathbf{R}$ . Then for any imaginary quadratic field  $L$ , we have

$$J(X, \mathbf{Z}) = J(X, I_L) .$$

*Proof.* Since  $B(X, \mathbf{Z}) \subset B(X, I_L)$  the inclusion  $J(X, I_L) \subset J(X, \mathbf{Z})$  is obvious. On the other hand, let  $x_0 \in J(X, \mathbf{Z})$  and let  $p \in B(X, I_L)$ . Then  $\|p\| < 1$ , so for some positive integer  $n$ ,

$$\|p^n\| = \|p\|^n < 1/2 .$$

Then we have

$$\|\operatorname{Re}(p^n)\| < 1/2 \quad \text{and} \quad \|\operatorname{Im}(p^n)\| < 1/2 .$$

Furthermore, from § 2 we see that  $2\operatorname{Re}(p^n)$  and  $(2/\sqrt{|d|})\operatorname{Im}(p^n)$  are in  $\mathbf{Z}[x]$ , where  $L = \mathbf{Q}(\sqrt{d})$  with  $d$  a square free integer and  $\operatorname{Re}(p^n)$  (resp.  $\operatorname{Im}(p^n)$ ) denotes the polynomial obtained by replacing the coefficients of  $p^n$  by their real (resp. imaginary) parts. Also

$$\|2\operatorname{Re}(p^n)\| = 2\|\operatorname{Re}(p^n)\| < 1$$

and

$$\|(2/\sqrt{|d|})\operatorname{Im}(p^n)\| < (1/|d|)^{1/2} \leq 1$$

and so by definition of  $J(X, \mathbf{Z})$

$$2\operatorname{Re}(p^n)(x_0) = 0$$

and

$$(2/\sqrt{|d|})\operatorname{Im}(p^n)(x_0) = 0 .$$

But  $p^n(x_0) = (\operatorname{Re}(p^n))(x_0) + i(\operatorname{Im}(p^n))(x_0)$  and so  $p^n(x_0) = 0$ , which implies that  $p(x_0) = 0$ . Hence  $x_0 \in J(X, I_L)$  and  $J(X, \mathbf{Z}) \subset J(X, I_L)$ .

Before proving the next proposition we note that if  $X$  is a compact subset of  $\mathbf{R}$  it is Lavrent'ev by 5.2 or by the Stone-Weierstrass theorem.

**PROPOSITION 6.3.** Let  $X$  be a compact subset of  $\mathbf{R}$  with  $d(X) < 1$ . Then for any imaginary quadratic field  $L$ ,

$$J_0(X, \mathbf{Z}) = J_0(X, I_L) .$$

*Proof.* If  $x_0 \in J_0(X, \mathbf{Z})$ , then  $x_0$  is a root of a monic polynomial  $p \in \mathbf{Z}[x]$  which has all of its roots in  $X$ . Thus  $x_0$  is integral over  $I_L$ . The minimal polynomial  $q$  of  $x_0$  over  $L$  is then an element of  $I_L[z]$ , monic, irreducible, and divides  $p$  so that the roots of  $q$  all lie in  $X$ . Thus  $x_0 \in J_0(X, I_L)$ .

For the reverse inclusion notice that  $J_0(X, I_L) = J(X, I_L) = J(X, \mathbf{Z})$  by 6.2 and 5.9. This shows, in particular, that  $J_0(X, I_L)$  is independent of the choice of  $L$ . Suppose then, that  $L = \mathbf{Q}(i)$  where  $i^2 = -1$ . Then  $I_L$  is the set of Gaussian integers. If  $x_0 \in J_0(X, I_L)$  then it is a root of monic, irreducible  $p$  in  $I_L[z]$  having all of its roots in  $X$ . Then the coefficients of  $p$ , being simply the elementary symmetric polynomials in the roots, are in  $I_L \cap \mathbf{R}$ . But  $I_L \cap \mathbf{R} = \mathbf{Z}$ , so  $x_0 \in J_0(X, \mathbf{Z})$ . Since  $x_0$  is any element of  $J_0(X, I_L)$ ,  $J_0(X, I_L) \subset J_0(X, \mathbf{Z})$ .

**THEOREM 6.4.** If  $X$  is any compact subset of the real line  $\mathbf{R}$  with  $d(X) < 1$ , then

$$J(X, \mathbf{Z}) = J_0(X, \mathbf{Z}) .$$

*Proof.* This is immediate from 6.2, 6.3 and 5.9.

A natural question at this point is whether or not the hypothesis  $d(X) < 1$  can be dropped from 6.4 or 5.10. We see that it cannot be dropped in either case by the following argument.

Let  $L$  be an imaginary quadratic field and  $X$  any uncountable compact subset of  $\mathbf{C}$  with  $d(X) \geq 1$ . Then we know that  $B(X, I_L) = \{0\}$  by the comments following 4.2 and so  $B(X, \mathbf{Z}) = \{0\}$ . This implies that  $J(X, I_L) = J(X, \mathbf{Z}) = X$  by definition. On the other hand, every element of  $J_0(X, \mathbf{Z})$  (respectively  $J_0(X, I_L)$ ) is algebraic over  $\mathbf{Q}$  and so  $J_0(x, \mathbf{Z})$  (respectively  $J_0(x, I_L)$ ) is countable and so not equal to  $X = J(X, \mathbf{Z}) = J(X, I_L)$ .

Another question is whether or not it is necessary to consider polynomials with complex coefficients when seeking the Chebyshev polynomials  $t_n = t_n(x, X)$  for  $X \subset \mathbf{R}$ . This is not necessary since the Chebyshev polynomials have real coefficients in this case since

$$\| \operatorname{Re}(t_n) \| \leq \| t_n \| .$$

**THEOREM 6.5.** If  $X$  is a compact subset of  $\mathbf{R}$  and  $R$  is a nonzero subring of  $\mathbf{Z}$ , then a function  $f$  in  $C^r(X)$  is  $R$ -approximable if and only if  $f$  is  $R$ -matchable on  $J(X, \mathbf{Z})$ .

*Proof.* If  $f$  is  $R$ -approximable then it is  $R$ -matchable on  $J(X, \mathbf{Z})$  by 4.6 and 6.2. Conversely, suppose that  $f$  is  $R$ -matchable on  $J(X, \mathbf{Z})$ . If  $d(X) \geq 1$  then  $B(X, \mathbf{Z}) = \{0\}$  by the comments following 4.2 and so  $J(X, \mathbf{Z}) = X$ . Thus  $f$  is in fact, a member of  $R[x]$ . If  $d(X) < 1$  assume that  $f$  is  $R$ -matchable on  $J(X, \mathbf{Z})$ , say by  $p \in R[x]$ . Since  $R$  is a nonzero subring of  $\mathbf{Z}$  we have  $R = n\mathbf{Z}$  for some positive integer  $n$ . It suffices to approximate  $q = f - p$ . In fact it suffices to approximate  $q_0 = q/n$  by an element of  $\mathbf{Z}[x]$ , since if

$$|q/n - p| < \varepsilon/n$$

then

$$|q - np| < \varepsilon$$

and  $np \in (n\mathbf{Z})[x] = R[x]$ . Let  $L$  be any imaginary quadratic field (the Gaussian numbers, for example). Since  $q_0$  is zero on  $J(X, \mathbf{Z}) = J(X, I_L)$  it is  $I_L$ -matchable on  $J(X, I_L)$ . Thus, for any  $\varepsilon > 0$ , there exists a  $p \in I_L[z]$  such that

$$(1) \quad \|p - q_0\|_x < \varepsilon/2,$$

by 5.7 and 5.9. Then for any  $x \in X$

$$\begin{aligned} \frac{\varepsilon}{2} &> |\operatorname{Im}(p(x) - f(x))| \\ &= |\operatorname{Im}(p(x))| \\ &= |(\operatorname{Im} p)(x)|. \end{aligned}$$

Since  $p = \operatorname{Re} p + i \operatorname{Im} p$

$$\|p - \operatorname{Re} p\| < \varepsilon/2,$$

and by (1) we have

$$\|\operatorname{Re} p - q_0\|_x < \varepsilon.$$

From the above proof we have the following

**COROLLARY 6.6.** *If  $d(X) \geq 1$ ,  $f$  is  $R$ -approximable on  $X$  if and only if  $f$  is already an element of  $R[x]$ .*

Since the transfinite diameter of an interval is one fourth its length we have the following.

**COROLLARY 6.7.** *If  $X = [a, b]$  and  $b - a \geq 4$  then  $f$  is  $R$ -approximable on  $X$  if and only if  $f$  is already an element of  $R[x]$ .*

Using essentially the same arguments as for their respective counterparts 5.8 and 5.10, the following can be proved.

PROPOSITION 6.8. Let  $X$  be a compact subset of  $\mathbf{R}$  with  $d(X) < 1$  and  $R$  a nonzero subring of  $\mathbf{Z}$ . Then an element  $f$  of  $C^r(X)$  is  $R$ -approximable on  $X$  if and only if the Lagrange interpolating polynomial for  $f$  on  $J(X, \mathbf{Z})$  is an element of  $R[x]$ .

PROPOSITION 6.9. Let  $X$  be a compact subset of  $\mathbf{R}$  with  $d(X) < 1$  and  $R$  a nonzero subring of  $\mathbf{Z}$ . Then  $R[z]$  is uniformly closed in  $C^r(X)$  if and only if

$$J(X, \mathbf{Z}) = X ,$$

In particular, if  $X$  is infinite, then  $\mathbf{Z}[x]$  is not uniformly closed in  $C^r(X)$ .

The main result of this section is Theorem 6.5. It reduces the problem to that of finding  $J(X, \mathbf{Z})$ . For some nontrivial cases where  $J(X, \mathbf{Z})$  has been determined see [7, § 5].

7. Appendix. An approximation theorem. Throughout this section let  $A$  be any discrete subring of  $C$  of rank 2 and containing the identity. Let  $L$  be the unique imaginary quadratic field such that  $A \subset I_L$ .

THEOREM 7.1. Let  $\alpha_1, \dots, \alpha_n$  be a complete set of conjugates over  $L$ ,  $\varepsilon$  any position number, and  $z_2, \dots, z_n$  any complex numbers. Then there is a polynomial  $q \in A[z]$  such that

$$|q(\alpha_i) - z_j| < \varepsilon \quad \text{for } 2 \leq j \leq n .$$

*Proof.* This is a consequence of the “very strong approximation theorem,” c.f. [10, p. 77, 33: 11].

THEOREM 7.2. Let

$$\begin{array}{c} \alpha_{11}, \dots, \alpha_{1r_1} \\ \alpha_{21}, \dots, \alpha_{2r_2} \\ \dots \dots \dots \\ \alpha_{s1}, \dots, \alpha_{sr_s} \end{array}$$

be an array (not necessarily rectangular) with each row an incomplete set of conjugates over  $L$  and where the minimal polynomials  $p_1, \dots, p_s$  in  $L[z]$  satisfied by the respective rows are distinct. Then if the



array

$$\begin{aligned} & z_{11}, \dots, z_{1r_1} \\ & z_{21}, \dots, z_{2r_2} \\ & \dots \dots \dots \\ & z_{s1}, \dots, z_{sr_s} \end{aligned}$$

consists of any complex numbers and  $\varepsilon > 0$ , there exists a  $q \in A[z]$  such that

$$|q(\alpha_{ij}) - z_{ij}| < \varepsilon \quad \text{for } 1 \leq j \leq r_i, 1 \leq i \leq s.$$

*Proof.* Let

$$q'_i = \left( \prod_{k=1}^s p_k \right) / p_i \quad \text{for } 1 \leq i \leq s.$$

Then  $q'_i(\alpha_{kj}) = 0$  if and only if  $k \neq i$  by definition of the  $p$ 's. Furthermore, each  $q'_i$  is a polynomial with coefficients in  $L$ . But  $L$  is the field of quotients of  $A$  and we can suppose that each coefficient of each  $q'_i$  appears as a ratio of elements of  $A$ . If  $k_i$  is the product of the denominators of the coefficients of  $q'_i$  then the coefficients of the polynomial  $k_i q'_i$  lie in  $A$ . For  $1 \leq i \leq s$ , by 7.1, there is a  $q''_i \in A[z]$  such that

$$\left| q''_i(\alpha_{ij}) - \frac{z_{ij}}{k_i q'_i(\alpha_{ij})} \right| < \frac{\varepsilon}{|k_i q'_i(\alpha_{ij})|} \quad \text{for } 1 \leq j \leq r_i.$$

Thus

$$|(q''_i k_i q'_i)(\alpha_{ij}) - z_{ij}| < \varepsilon \quad \text{for } 1 \leq j \leq r_i.$$

If we set  $q = q''_1 k_1 q'_1 + \dots + q''_s k_s q'_s$ , then  $q$  is in  $A[z]$  and  $q(\alpha_{ij}) = (q''_i k_i q'_i)(\alpha_{ij})$  since  $q'_k(\alpha_{ij}) = 0$  if  $k \neq i$ . Thus

$$|q(\alpha_{ij}) - z_{ij}| = |(q''_i k_i q'_i)(\alpha_{ij}) - z_{ij}| < \varepsilon$$

for all  $i, j$ .

We note that another way of looking at Theorem 7.2 is the following.

**COROLLARY 7.3.** *If  $\{\alpha_1, \dots, \alpha_k\}$  is any set of algebraic numbers which does not contain a complete set of conjugates over  $L$ , then the set of  $k$ -tuples*

$$\{(p(\alpha_1), \dots, p(\alpha_k)) : p \in A[z]\}$$

*is dense in  $C^k$ .*

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