CLOSED SYSTEMS OF FUNCTIONS AND PREDICATES

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In this paper we show that there is a one to one correspondence between systems of functions defined on a finite set $A$ and systems of predicates defined on $A$. This result implies that a complete set of invariants for a universal algebra on $A$ is given by predicates defined on $A$. Conversely, functions on $A$ provide a complete system of invariants for sets of predicates closed under conjunction, change of variable and application of the existential quantifier.

We begin in § 2 by giving a definition of closure for systems of functions and predicates. This is followed by a definition of commutivity of a function and a predicate which gives a correspondence between the two types of systems. In Theorems 1 and 2 of § 3 we show that the correspondence is a Galois connection. In Theorem 3 we consider sets of predicates closed under the existential quantifier and show that the corresponding systems are determined by functions defined for all values of the arguments. In Theorems 4 and 5 we include disjunction and then negation in the definition of closure of a set of predicates. We also require that equality be among the predicates. The corresponding systems consist of essentially first order functions and essentially first order permutations respectively. We conclude in § 4 with some comments on the infinite case and some general comments on these results.

2. Basic definitions. Associated with any subset of $A^{n+1}$, the set of all sequences of length $n+1$ with elements in $A$, is the $n$-th order function $f(x_1, \cdots, x_n)$ which may be many valued and may not be defined on all of $A^n$. A system of functions $\mathcal{L}$ is defined to be closed if the following conditions are satisfied:

(i) $\mathcal{L}$ is closed under composition.

(ii) If $f(x_1, \cdots, x_n) \in \mathcal{L}$ is associated with the subset $P \subset A^{n+1}$ then any $g(x_1, \cdots, x_n)$ associated with $Q \subset P$ is in $\mathcal{L}$.

(iii) For any $n$, $\mathcal{L}$ contains all functions $f$ defined on $A^n$ such that $f(x_1, \cdots, x_n) = x_i$.

In defining closed systems of predicates the author has the following model in mind. We are given a sequence $A_1, A_2, A_3, \cdots$ of sets of predicates, each $A_i$ containing all subsets of $A^i$. For each $A_i$ a set of operators isomorphic to $S_i$ the symmetric group is given which maps $A_i$ onto $A_i$. These correspond to permutations of the variables.
of predicates in \( A_i \). There is an operator \( R: A_{i+1} \rightarrow A_i \) which takes \( P(x_1, \ldots, x_{i+1}) \) to \( P(x_2, x_1, x_{i+1}) \) and an operator \( E: A_{i+1} \rightarrow A_i \) which takes \( P(x_1, \ldots, x_{i+1}) \) to \((\exists y)P(y, x_1, \ldots, x_i)\). Also there is an operator \( A: A_i \rightarrow A_{i+1} \) which corresponds to the cartesian product with \( A \) or to the introduction of a dummy variable. Thus \((x_1, \ldots, x_{i+1}) \in AP\) if and only if \((x_2, \ldots, x_{i+1}) \in P\). A predicate in \( A_i \) will be said to have order \( i \). A system \( \mathcal{P} \) of predicates is defined to be closed if it satisfies the following conditions:

(i) If \( P \in \mathcal{P} \) and \( Q \in \mathcal{P} \) and \( P \) and \( Q \) have the same order then \( P \cap Q \in \mathcal{P} \).

(ii) If \( P \in \mathcal{P} \) then any predicate obtained from \( P \) by permuting the variables is in \( \mathcal{P} \).

(iii) If \( P \in \mathcal{P} \) then \( AP \) and \( RP \) are contained in \( \mathcal{P} \).

(iv) \( \mathcal{P} \) contains the first order predicate \( A \).

Now we define commutivity of a function and a predicate. Let \( M \) be an \( n \times m \) matrix with elements in \( A \), then we write \( M \subseteq P \) where \( P \) is an \( m \)-th order predicate if each row of \( M \) is a sequence contained in \( P \). If \( N \) is an \( m \times n \) matrix and \( f \) is an \( n \)-th order function then \( f(N) \) is the \( m \times 1 \) column matrix obtained by letting \( f \) operate on each row of \( N \). If \( f \) is not defined for some row of \( N \) we say that \( f(N) \) is not defined. The predicate \( P \) commutes with the function \( f \) if for every \( M \subseteq P \) the row matrix \( f(M^T) \) when defined is a sequence contained in \( P \). Here \( M^T \) is the transpose matrix of \( M \). If \( \mathcal{L} \) and \( \mathcal{P} \) are systems of functions and predicates we write \( \mathcal{L}^* \) and \( \mathcal{P}^* \) for the systems of predicates and functions respectively which commute with \( \mathcal{L} \) and \( \mathcal{P} \).

3. Main results. It can be verified that \( \mathcal{L}^* \) and \( \mathcal{P}^* \) are closed systems. We will show that if \( \mathcal{L} \) and \( \mathcal{P} \) are closed systems then \( \mathcal{L} = \mathcal{L}^{**} \) and \( \mathcal{P} = \mathcal{P}^{**} \).

**Theorem 1.** If \( \mathcal{L} \) is a closed system of functions then \( \mathcal{L} = \mathcal{L}^{**} \).

Since \( \mathcal{L} \subseteq \mathcal{L}^{**} \) we need only show that for any function \( g(x_1, \ldots, x_m) \) not in \( \mathcal{L} \) there exists a predicate in \( \mathcal{L}^* \) which does not commute with \( g \). Assume that \( g \) is defined only on the sequences \( s_1, s_2, \ldots, s_k \). We form the \( k \times m \) matrix \( T \) with \( i \)-th row equal to \( s_i \). For any function \( f(x_1, \ldots, x_r) \) in \( \mathcal{L} \) and any \( k \times r \) matrix \( F \) with columns taken from \( T \) we form the column matrix \( f(F) \). If \( f(F) \) is not a column of \( T \) we adjoin it to \( T \) and get a \( k \times (m+1) \) matrix \( T' \). In this way we can adjoin columns to \( T \) until we finally reach a matrix \( T_0 \) with \( k \) rows such that for any function \( f \) in \( \mathcal{L} \) and any matrix \( F \) with columns from \( T_0 \) the column matrix \( f(F) \)
will be in $T_0$ if it is defined. If $g(T)$ is a column of $T_0$ then $g$ can be derived from functions in $\mathcal{K}$ so we can assume that $g(T)$ is not in $T_0$. From $T_0$ we form the $k$-th order predicate $P_0$ which contains all the rows of $T_0^k$. It is evident that $P_0$ is in $\mathcal{K}$ but does not commute with $g$. Thus $\mathcal{K} = \mathcal{K}^*$.

**Theorem 2.** If $\mathcal{P}$ is a closed system of predicates then $\mathcal{P} = \mathcal{P}^*$.

Since $\mathcal{P} \subseteq \mathcal{P}^*$ we need only show that for any $n$-th order predicate $Q$ not in $\mathcal{P}$ there exists a function in $\mathcal{P}^*$ which does not commute with $Q$. Let $P$ be the intersection of all $n$-th order predicates of $\mathcal{P}$ which contain $Q$. Let $s_1, s_2, \ldots, s_k$ be all the $1 \times n$ matrices contained in $Q$ and let $N$ be the $k \times n$ matrix with $i$-th row $s_i$. Let $t$ be any row matrix in $P$ but not in $Q$. Then there exists a $k$-th order function $f$ defined only on the rows of $N^T$ such that $f(N^T) = t^T$. We wish to show that any predicate in $\mathcal{P}$ commutes with $f$. By way of contradiction suppose that the $m$-th order predicate $P_i \in \mathcal{P}$ does not commute with $f$ and that every predicate obtained from $P_i$ by identification of variables does commute with $f$. Then there exists a $j \times m$ matrix $N_i \subset P_i$ such that $f(N_i^T) = t_i^T$ and $t_i$ is not contained in $P_i$.

Since every identification of variables in $P$ leads to a predicate which commutes with $f$ we must have that each pair $r_i$, $f(r_i)i = 1, \ldots, m$ where $r_i$ is the $i$-th row of $N_i^T$ and $f(r_i)$ is the corresponding element of $t_i^T$, is distinct from any other pair $r_i$, $f(r_i)$. Thus each pair is the same as a row element pair taken from $N^T$ and $t^T$. We can find a $k \times n$ matrix $N_2 \subset A^{n-m}P_1$ and row matrix $t_2$ such that the last $m$ rows of $N^T_2$ and elements of $t_2^T$ are equal to $r_i$, $f(r_i)$. Also the first $n-m$ pairs can be chosen so that there is a one to one correspondence between pairs taken from $N^T$, $t^T$ and pairs taken from $N^T_2$, $t_2^T$. By permuting the variables of $A^{n-m}P_1$ we can arrive at a predicate $P_3$ which contains $N$ and does not contain $t$. Since $P_3$ is in $\mathcal{P}$ we get that $P$ is not the least $n$-th order predicate which contains $Q$. Thus we have a contradiction and $f$ must commute with every predicate of $\mathcal{P}$. Thus $\mathcal{P} = \mathcal{P}^*$.

Now we consider systems of predicates which are closed under the existential quantifier. Let $\mathcal{L}$ be a closed system of functions and assume that for any $f(x_1, \ldots, x_n) \in \mathcal{L}$ with restricted domain of definition, there exists a $g(x_1, \ldots, x_n) \in \mathcal{L}$ which is defined on all of $A^*$ and equals $f$ where $f$ is defined. Then it can be verified that $\mathcal{L}^*$ is closed under the existential quantifier.

**Theorem 3.** If $\mathcal{P}$ is a closed system of predicates which is
closed under the existential quantifier then every function in \( \mathcal{P}^* \) can be extended to a function in \( \mathcal{P}^* \) which is defined for all values of the arguments.

We assume that the elements of \( A \) are the integers from 1 to \( n \). Let \( f(x_1, \ldots, x_m) \in \mathcal{P}^* \) be defined on the sequences \( s_1, s_2, \ldots, s_k \) and let \( s \) be any other sequence in \( A^n \). We define the \( n \) functions \( f_i \) such that \( f_i(s_j) = f(s_j) \) and \( f_i(s) = i \) for \( i = 1, \ldots, n \) and show that for some \( i, f_i \) is in \( \mathcal{P}^* \). By way of contradiction suppose that for each \( f_i \) there exists a \( P < z) A/\mathcal{P}^* \) which is defined for all values of the arguments.

Now we consider single valued functions which are defined for all values of their arguments. If \( \mathcal{P} \) is a system of predicates we redefine \( \mathcal{P}^* \) as the set of single valued functions defined for all values of the arguments which commute with \( \mathcal{P} \). Also we assume that \( \mathcal{P} \) is closed, contains \( e(x_1, x_2) \Longleftrightarrow (x_1 = x_2) \) and is closed under the existential quantifier. We will give necessary and sufficient conditions on \( \mathcal{P}^* \) in order that \( \mathcal{P} \) be closed under disjunction and negation.

First we define the predicates \( D(x_1, x_2, x_3, x_4) \Longleftrightarrow (x_1 = x_2) \lor (x_3 = x_4) \) and \( Q_n(x_1, \ldots, x_n) \) which holds in case \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \). We have the following equivalences for a closed system \( \mathcal{P} \).

(1) \( \mathcal{P}^* \) consists of essentially first order functions if and only if \( D \in \mathcal{P} \).

(2) When \( \mathcal{P} \) is defined on a set \( A \) with \( n \) elements then \( \mathcal{P}^* \) consists of essentially first order permutations if and only if \( D, Q_n \in \mathcal{P} \).

We only prove that if \( D \in \mathcal{P} \) then \( \mathcal{P}^* \) consists of essentially first order functions. Let \( g(x_1, \ldots, x_n) \) be a function in \( \mathcal{P}^* \) which
depends essentially on the variables $x_1$ and $x_2$. Then there exist sequences $(a_0, a_2, \ldots , a_n) = s_1$, $(b_0, b_2, \ldots , b_n) = s_2$ and $(b_0, b_2, \ldots , b_n) = s_4$ such that $g(s_i) \neq g(s_j)$ and $g(s_i) \neq g(s_k)$. We construct the $4 \times n$ matrix $M$ with $i$-th row $s_i$. Then $M^r \subseteq D$ but $g(M)^r$ is not in $D$ so $g$ cannot be in $\mathcal{P}^*$. The other implications also follow easily. From these equivalences we get:

**Theorem 4.** $\mathcal{P}$ is closed under disjunction if and only if $\mathcal{P}^*$ consists of essentially first order functions.

**Theorem 5.** $\mathcal{P}$ is closed under negation if and only if $\mathcal{P}^*$ consists of first order permutations.

4. Comments and applications. First we consider the case where $A$ is an infinite set. Craig R. Platt has found in this case that we need to add the following condition to the definition of closure of a set of functions or predicates. A set of functions $\mathcal{L}$ is locally closed if, for any $n$-th order function $g$ and for every finite $H \subseteq A^{n+1}$ there exists an $f \in \mathcal{L}$ such that $g \cap H = f \cap H$, then $g \in \mathcal{L}$. A similar definition is given for sets of predicates. Then it follows, if $\mathcal{L}$ and $\mathcal{P}$ are any sets of functions and predicates, that $\mathcal{L}^*$ and $\mathcal{P}^*$ are locally closed sets and Theorems 1 and 2 hold when $\mathcal{L}$ and $\mathcal{P}$ are locally closed. Also a theorem has been found in the infinite case which specializes to Theorem 3.

Theorems 1 and 2 can be summarized in the following way. Let $\mathcal{L}$ and $\mathcal{P}$ be the sets of all functions and predicates on a set and let $C$ be a binary relation which holds between elements in $\mathcal{L}$ and $\mathcal{P}$ if and only if they commute. Then $C$ is a difunctional relation [1, p. 193] that is $CC^*C = C$. Here $C^*$ is the converse relation to $C$. Then $CC^*$ and $C^*C$ are congruence relations on $\mathcal{L}$ and $\mathcal{P}$ and $C$ establishes a one to one correspondence between the congruence classes. Alternately we may say that there exists a set $S$ and mappings $\phi: \mathcal{L} \rightarrow S$ and $\pi: \mathcal{P} \rightarrow S$ such that two elements $f \in \mathcal{L}$ and $P \in \mathcal{P}$ commute if and only if $\phi(f) = \pi(P)$.

In [2] Post has given a classification of two valued systems of functions. This gives a classification of two valued systems of predicates containing equality and closed under the existential quantifier. Finding these systems can be simplified using theorems of this paper.

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