

# Pacific Journal of Mathematics

**ON THE VARIATION OF THE BERNSTEIN POLYNOMIALS OF  
A FUNCTION OF UNBOUNDED VARIATION**

MARTIN EDWARD PRICE

ON THE VARIATION OF THE BERNSTEIN  
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**The behavior of the ordinary Bernstein polynomials,  $B_n f$ , for discontinuous functions  $f$  can be quite erratic. The purpose of this note is to give an example of a function  $f$  which is quite irregular on the rationals but such that the total variation,  $VB_n f$  of  $B_n f$  tends to zero with  $n$ .**

It is known that if  $f$  is of bounded variation, then  $VB_n f$  tends to the variation of  $f$  taken over its points of continuity, [2 p. 25]. In [3] we consider arbitrary  $f$ , and give sufficient conditions for  $VB_n f$  to tend to zero in terms of the sums  $\sum_{r=0}^n |f(r/n)|$ . It is shown in [2 p. 28] that  $B_n f$ , for unbounded  $f$ , can behave unusually in terms of pointwise convergence to  $f$ . Here we construct a function, unbounded on the rationals in every subinterval of  $[0, 1]$ , and which has the property that  $B_n f$  converges in variation (and uniformly) to zero.

2. Preliminaries. The  $n$ -th Bernstein polynomial of the real function  $f$  on  $[0, 1]$  is

$$(2.1) \quad B_n f \equiv \sum_{r=0}^n f\left(\frac{r}{n}\right) p_{nr}(x),$$

where

$$p_{nr}(x) \equiv \binom{n}{r} x^r (1-x)^{n-r}, \quad x \in [0, 1].$$

Since  $B_n f$  depends only on rational values of  $f$ , we restrict ourselves to "skeletons," i.e., functions defined only on the rationals in  $[0, 1]$ , in the manner of [1]. We need the following facts:

(A) If  $r = 1, \dots, n-1$ , then for all  $n$ ,

$$(2.2) \quad P(n, r) \equiv \text{Max}_{[0,1]} p_{nr}(x) < Cn^{\frac{1}{2}} [r(n-r)]^{-\frac{1}{2}}$$

where  $C$  is an absolute constant [1].

(B) If  $a$  is a positive integer, then

$$(2.3) \quad P(an, ar) < 2a^{-\frac{1}{2}} P(n, r)$$

for each  $n \geq 2$  and  $r = 1, \dots, n-1$ . ((A) and (B) are applications of Stirling's formula.)

(C) For all  $n$  and  $f$

$$(2.4) \quad VB_n f \leq 2 \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| P(n, r).$$

(D) If  $\sum_{i=1}^{\infty} f_i$  is a pointwise convergent series of functions (skeletons) on  $[0, 1]$  then,

$$VB_n \left( \sum_{i=1}^{\infty} f_i \right) \leq \sum_{i=1}^{\infty} VB_n f_i$$

where the right side may be  $+\infty$ .

**3. Construction.** We define a sequence of skeletons  $f_i$  such that each skeleton tends to  $+\infty$  on a set of rationals tending to a limit rational  $r_i$ . The  $r_i$  will be dense in  $[0, 1]$ . It is shown that the skeleton  $f \equiv \sum_{i=1}^{\infty} f_i$  has the following properties:

- (1)  $f$  is unbounded on the rationals in every subinterval of  $[0, 1]$ ;
- (2)  $VB_n f \rightarrow 0$  as  $n \rightarrow \infty$ .

(Since  $f$  will satisfy  $f(0) = f(1) = 0$ , and since  $B_n f(0) = f(0)$  and  $B_n f(1) = f(1)$  for all  $f$  and  $n$ , (2) implies  $B_n f \rightarrow 0$  uniformly on  $[0, 1]$ .)

For all  $i = 1, 2, \dots$ , pick  $r_i \equiv p_i/q_i$  such that  $q_i$  is prime,  $0 < p_i < q_i$ ,  $q_i < q_{i+1}$ , and  $r_i \in I_i$ , where  $I_1 = [0, 1/2]$ ,  $I_2 = [1/2, 1]$ ,  $I_3 = [0, 1/4]$ ,  $\dots$ ,  $I_6 = [3/4, 1]$ ,  $I_7 = [0, 1/8]$ ,  $\dots$ . Thus the  $r_i$  are dense in  $[0, 1]$ . Define

$$(3.1) \quad f_i \left( \frac{p_i}{q_i} + \frac{1}{q_i^{\alpha(i, l)}} \right) \equiv l$$

where for each  $i$ ,  $\alpha(i, l)$  is a strictly increasing sequence of positive integers to be determined later. For all other rationals in  $[0, 1]$ , put  $f_i \equiv 0$ , and then set  $f \equiv \sum_{i=1}^{\infty} f_i$ . Since the supports of the  $f_i$  are disjoint,  $f$  is well defined at all rationals, and satisfies (1) by construction. We have

$$(3.2) \quad VB_n f \leq \sum_{i=1}^{\infty} VB_n f_i \leq \sum_{i=1}^{\infty} H(i, n)$$

by 2 (C) and (D), where we have put

$$(3.3) \quad H(i, n) \equiv 2 \sum_{r=0}^n \left| f_i \left( \frac{r}{n} \right) \right| P(n, r).$$

**LEMMA (3.1).** *For fixed  $i$ , it is possible to choose  $\alpha(i, l)$ ,  $l=1, 2, \dots$  such that*

$$(3.4) \quad H(i, q_i^{\alpha(i, l)}) < \frac{1}{q_i^2 l}.$$

*Proof.* To simplify matters, let  $p_i \equiv p$ ,  $q_i \equiv q$  and  $\alpha(i, l) \equiv \alpha_i$ . When  $n = q_i^{\alpha(i, k)} \equiv q^{\alpha_k}$ , there are only  $k$ , nonzero terms on the right

in (3.3), and these correspond to the points

$$\frac{r}{n} = \frac{p}{q} + \frac{1}{q^{\alpha_j}} = \frac{pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}}{q^{\alpha_k}} \quad (j = 1 \dots k).$$

Since the value of  $f_i$  at the  $j$ -th point is  $j$ , (3.3) becomes

$$(3.5) \quad \sum_{j=1}^k 2jP(q^{\alpha_k}, pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}).$$

By applying (2.2), one gets each term in (3.5) less than

$$(3.6) \quad 2jC \left[ \frac{q^{\alpha_k}}{[pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}][q^{\alpha_k} - pq^{\alpha_k-1} - q^{\alpha_k-\alpha_j}]} \right]^{\frac{1}{2}} \\ = 2jC \left[ q^{\alpha_k} \left( \frac{p}{q} - \frac{p^2}{q^2} - \frac{2p}{q^{\alpha_j+1}} + \frac{1}{q^{\alpha_j}} - \frac{1}{q^{2\alpha_j}} \right) \right]^{-\frac{1}{2}}.$$

Thus, for  $k = j = 1$ ,  $\alpha_1$  may be chosen so large that (3.6), hence (3.5), is less than  $1/q^2$ . (We pick  $\alpha_1 \geq 2$  so that  $p/q + 1/q^{\alpha_1} < 1$ .) Now suppose  $\alpha_k$ ,  $k = 1, \dots, l - 1$  have been chosen so that  $\alpha_k > \alpha_{k-1}$ , and so that (3.5) is less than  $1/q^2k$ . When  $k = l$ , (3.6) shows that  $\alpha_l$  can be chosen so that each term,  $j = 1, \dots, l$  is less than  $1/q^2l^2$ . Thus (3.5) is less than  $l \cdot (ql)^{-2} = 1/q^2l$ .

We can factor every integer  $n$  uniquely as:

$$(3.7) \quad n \equiv d \cdot \prod_{j=1}^T n_j, \quad n_j = q_{i_j}^{\alpha(i_j, L_j)} \quad q_{i_j} < q_{i_{j+1}}.$$

The  $q_{i_j}$  are those  $q_i$  which appear in  $n$  to a power greater than or equal  $\alpha(i_j, 1)$ , and  $L_j$  is the largest index  $l$  of the exponents  $\alpha(i_j, l)$  such that  $q_{i_j}^{\alpha(i_j, l)}$  divides  $n$ . For any  $n$ ,

$$(3.8) \quad \sum_{i=1}^{\infty} H(i, n) = \sum_{j=1}^T H(i_j, n) \leq \sum_{j=1}^T 2 \left( \frac{n_j}{n} \right)^{\frac{1}{2}} H(i_j, n_j)$$

where the inequality follows from (2. B) with  $a = n/n_j$ . If we apply the lemma to each term, we get the last sum less than

$$(3.9) \quad \sum_{j=1}^T 2 \left( \frac{n_j}{n} \right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \leq \frac{2}{n^{1/2}} \left( \sum_{j=1}^{T-1} \frac{1}{q_{i_j}^2 L_j} \right) + \frac{1}{q_{i_T}^2 L_T}$$

where the decomposition applies if  $T > 1$ . In this case, the sum on the right is dominated by  $\sum 1/m^2$  and is thus bounded. (If  $T = 1$ , the assertion is that (3.9) holds if the sum is regarded as vacuous, and a similar remark holds for (3.11) below.) Therefore if the largest of the  $q_{i_j}$ ,  $q_{i_T}$  is as large as, let us say,  $q_{i_*}$ ,  $n_T$  will also be large, and (3.9) can be made less than  $\epsilon$ .

Now suppose  $n$  is such that every  $q_{i_j} < q_{i_*}$ . As before

$$(3.10) \quad \sum_{i=j}^{\infty} H(i, n) \leq \sum_{j=1}^T 2 \left( \frac{n_j}{n} \right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \quad (q_{i_j} < q_{i_*})$$

Let  $k$  be the first index where  $\text{Max}_{1 \leq j \leq T} L_j$  occurs. Then (3.10) becomes

$$(3.11) \quad \sum_{j \neq k} \left[ 2 \left( \frac{n_j}{n} \right)^{\frac{1}{2}} \cdot \frac{1}{q_{i_j}^2 L_j} \right] + \frac{1}{q_{i_k}^2 L_k} \leq \\ \left[ 2(2)^{-\alpha(i_k, L_k)/2} \left( \sum_{j \neq k} \frac{1}{q_{i_j}^2 L_j} \right) \right] + \frac{1}{q_{i_k}^2 L_k},$$

since  $q_{i_k} \geq 2$  and appears in every  $n^j/n$  for  $j \neq k$ . As in (3.9), the sum is bounded. Thus if  $L_k$  is large enough, say  $L_k \geq L$ ,  $\alpha(i_k, L_k)$  is also large, and (3.10) is less than  $\varepsilon$ .

Now suppose every  $q_i$  in  $n$  is less than  $q_{i_*}$  and all the indices  $L_j$  are less than  $L$ . There are only a finite number of such combinations  $\prod_{j=1}^T n_j$ , and we denote them  $C_s$ ,  $s = 1 \dots S$ . If  $n \equiv d \cdot C_s$ , we get by (2.B)

$$(3.12) \quad \sum_{i=1}^{\infty} H(i, n) \leq \frac{2}{d^{1/2}} \sum_{i=j}^{\infty} H(i, C_s).$$

However only a finite number of  $q_i$  appear in any  $C_s$  so that the sum is bounded by, say  $M_s > 0$ . Therefore (3.12) is less  $2M_s/d^{1/2}$ , and we can pick  $d_s$  large enough so that  $d \geq d_s$  implies (3.12) is less than  $\varepsilon$ .

Thus if  $n > \text{Max} [q_{i_*}^{\alpha(i_*, 1)}, q_1^{\alpha(1, L)}, d_1 c_1 \dots d_s c_s]$ ,  $\sum_{i=1}^{\infty} H(i, n) < \varepsilon$ , implying  $VB_n f < \varepsilon$  by (3.2).

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# Pacific Journal of Mathematics

Vol. 27, No. 1

January, 1968

Willard Ellis Baxter, <i>On rings with proper involution</i> . . . . .	1
Donald John Charles Bures, <i>Tensor products of <math>W^*</math>-algebras</i> . . . . .	13
James Calvert, <i>Integral inequalities involving second order derivatives</i> . . . . .	39
Edward Dewey Davis, <i>Further remarks on ideals of the principal class</i> . . . . .	49
Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients I</i> . . . . .	53
Francis James Flanigan, <i>Algebraic geography: Varieties of structure constants</i> . . . . .	71
Denis Ragan Floyd, <i>On <math>QF - 1</math> algebras</i> . . . . .	81
David Scott Geiger, <i>Closed systems of functions and predicates</i> . . . . .	95
Delma Joseph Hebert, Jr. and Howard E. Lacey, <i>On supports of regular Borel measures</i> . . . . .	101
Martin Edward Price, <i>On the variation of the Bernstein polynomials of a function of unbounded variation</i> . . . . .	119
Louise Arakelian Raphael, <i>On a characterization of infinite complex matrices mapping the space of analytic sequences into itself</i> . . . . .	123
Louis Jackson Ratliff, Jr., <i>A characterization of analytically unramified semi-local rings and applications</i> . . . . .	127
S. A. E. Sherif, <i>A Tauberian relation between the Borel and the Lototsky transforms of series</i> . . . . .	145
Robert C. Sine, <i>Geometric theory of a single Markov operator</i> . . . . .	155
Armond E. Spencer, <i>Maximal nonnormal chains in finite groups</i> . . . . .	167
Li Pi Su, <i>Algebraic properties of certain rings of continuous functions</i> . . . . .	175
G. P. Szegő, <i>A theorem of Rolle's type in <math>E^n</math> for functions of the class <math>C^1</math></i> . . . . .	193
Giovanni Viglino, <i>A co-topological application to minimal spaces</i> . . . . .	197
B. R. Wenner, <i>Dimension on boundaries of <math>\varepsilon</math>-spheres</i> . . . . .	201