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**ON THE VARIATION OF THE BERNSTEIN POLYNOMIALS OF
A FUNCTION OF UNBOUNDED VARIATION**

MARTIN EDWARD PRICE

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The behavior of the ordinary Bernstein polynomials, $B_n f$, for discontinuous functions f can be quite erratic. The purpose of this note is to give an example of a function f which is quite irregular on the rationals but such that the total variation, $VB_n f$ of $B_n f$ tends to zero with n .

It is known that if f is of bounded variation, then $VB_n f$ tends to the variation of f taken over its points of continuity, [2 p. 25]. In [3] we consider arbitrary f , and give sufficient conditions for $VB_n f$ to tend to zero in terms of the sums $\sum_{r=0}^n |f(r/n)|$. It is shown in [2 p. 28] that $B_n f$, for unbounded f , can behave unusually in terms of pointwise convergence to f . Here we construct a function, unbounded on the rationals in every subinterval of $[0, 1]$, and which has the property that $B_n f$ converges in variation (and uniformly) to zero.

2. Preliminaries. The n -th Bernstein polynomial of the real function f on $[0, 1]$ is

$$(2.1) \quad B_n f \equiv \sum_{r=0}^n f\left(\frac{r}{n}\right) p_{nr}(x),$$

where

$$p_{nr}(x) \equiv \binom{n}{r} x^r (1-x)^{n-r}, \quad x \in [0, 1].$$

Since $B_n f$ depends only on rational values of f , we restrict ourselves to "skeletons," i.e., functions defined only on the rationals in $[0, 1]$, in the manner of [1]. We need the following facts:

(A) If $r = 1, \dots, n-1$, then for all n ,

$$(2.2) \quad P(n, r) \equiv \text{Max}_{[0,1]} p_{nr}(x) < C n^{\frac{1}{2}} [r(n-r)]^{-\frac{1}{2}}$$

where C is an absolute constant [1].

(B) If a is a positive integer, then

$$(2.3) \quad P(an, ar) < 2a^{-\frac{1}{2}} P(n, r)$$

for each $n \geq 2$ and $r = 1, \dots, n-1$. ((A) and (B) are applications of Stirling's formula.)

(C) For all n and f

$$(2.4) \quad VB_n f \leq 2 \sum_{r=0}^n \left| f\left(\frac{r}{n}\right) \right| P(n, r).$$

(D) If $\sum_{i=1}^{\infty} f_i$ is a pointwise convergent series of functions (skeletons) on $[0, 1]$ then,

$$VB_n \left(\sum_{i=1}^{\infty} f_i \right) \leq \sum_{i=1}^{\infty} VB_n f_i$$

where the right side may be $+\infty$.

3. Construction. We define a sequence of skeletons f_i such that each skeleton tends to $+\infty$ on a set of rationals tending to a limit rational r_i . The r_i will be dense in $[0, 1]$. It is shown that the skeleton $f \equiv \sum_{i=1}^{\infty} f_i$ has the following properties:

- (1) f is unbounded on the rationals in every subinterval of $[0, 1]$;
- (2) $VB_n f \rightarrow 0$ as $n \rightarrow \infty$.

(Since f will satisfy $f(0) = f(1) = 0$, and since $B_n f(0) = f(0)$ and $B_n f(1) = f(1)$ for all f and n , (2) implies $B_n f \rightarrow 0$ uniformly on $[0, 1]$.)

For all $i = 1, 2, \dots$, pick $r_i \equiv p_i/q_i$ such that q_i is prime, $0 < p_i < q_i$, $q_i < q_{i+1}$, and $r_i \in I_i$, where $I_1 = [0, 1/2]$, $I_2 = [1/2, 1]$, $I_3 = [0, 1/4]$, \dots , $I_6 = [3/4, 1]$, $I_7 = [0, 1/8]$, \dots . Thus the r_i are dense in $[0, 1]$. Define

$$(3.1) \quad f_i \left(\frac{p_i}{q_i} + \frac{1}{q_i^{\alpha(i, l)}} \right) \equiv l$$

where for each i , $\alpha(i, l)$ is a strictly increasing sequence of positive integers to be determined later. For all other rationals in $[0, 1]$, put $f_i \equiv 0$, and then set $f \equiv \sum_{i=1}^{\infty} f_i$. Since the supports of the f_i are disjoint, f is well defined at all rationals, and satisfies (1) by construction. We have

$$(3.2) \quad VB_n f \leq \sum_{i=1}^{\infty} VB_n f_i \leq \sum_{i=1}^{\infty} H(i, n)$$

by 2 (C) and (D), where we have put

$$(3.3) \quad H(i, n) \equiv 2 \sum_{r=0}^n \left| f_i \left(\frac{r}{n} \right) \right| P(n, r).$$

LEMMA (3.1). For fixed i , it is possible to choose $\alpha(i, l)$, $l=1, 2, \dots$ such that

$$(3.4) \quad H(i, q_i^{\alpha(i, l)}) < \frac{1}{q_i^2 l}.$$

Proof. To simplify matters, let $p_i \equiv p$, $q_i \equiv q$ and $\alpha(i, l) \equiv \alpha_l$. When $n = q_i^{\alpha(i, k)} \equiv q^{\alpha_k}$, there are only k , nonzero terms on the right

in (3.3), and these correspond to the points

$$\frac{r}{n} = \frac{p}{q} + \frac{1}{q^{\alpha_j}} = \frac{pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}}{q^{\alpha_k}} \quad (j = 1 \dots k).$$

Since the value of f_i at the j -th point is j , (3.3) becomes

$$(3.5) \quad \sum_{j=1}^k 2jP(q^{\alpha_k}, pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}).$$

By applying (2.2), one gets each term in (3.5) less than

$$(3.6) \quad \begin{aligned} & 2jC \left[\frac{q^{\alpha_k}}{[pq^{\alpha_k-1} + q^{\alpha_k-\alpha_j}][q^{\alpha_k} - pq^{\alpha_k-1} - q^{\alpha_k-\alpha_j}]} \right]^{\frac{1}{2}} \\ & = 2jC \left[q^{\alpha_k} \left(\frac{p}{q} - \frac{p^2}{q^2} - \frac{2p}{q^{\alpha_j+1}} + \frac{1}{q^{\alpha_j}} - \frac{1}{q^{2\alpha_j}} \right) \right]^{-\frac{1}{2}}. \end{aligned}$$

Thus, for $k = j = 1$, α_1 may be chosen so large that (3.6), hence (3.5), is less than $1/q^2$. (We pick $\alpha_1 \geq 2$ so that $p/q + 1/q^{\alpha_1} < 1$.) Now suppose α_k , $k = 1, \dots, l - 1$ have been chosen so that $\alpha_k > \alpha_{k-1}$, and so that (3.5) is less than $1/q^2k$. When $k = l$, (3.6) shows that α_l can be chosen so that each term, $j = 1, \dots, l$ is less than $1/q^2l^2$. Thus (3.5) is less than $l \cdot (ql)^{-2} = 1/q^2l$.

We can factor every integer n uniquely as:

$$(3.7) \quad n \equiv d \cdot \prod_{j=1}^T n_j, \quad n_j = q_{i_j}^{\alpha(i_j, L_j)} \quad q_{i_j} < q_{i_{j+1}}.$$

The q_{i_j} are those q_i which appear in n to a power greater than or equal $\alpha(i_j, 1)$, and L_j is the largest index l of the exponents $\alpha(i_j, l)$ such that $q_{i_j}^{\alpha(i_j, l)}$ divides n . For any n ,

$$(3.8) \quad \sum_{i=1}^{\infty} H(i, n) = \sum_{j=1}^T H(i_j, n) \leq \sum_{j=1}^T 2 \left(\frac{n_j}{n} \right)^{\frac{1}{2}} H(i_j, n_j)$$

where the inequality follows from (2. B) with $a = n/n_j$. If we apply the lemma to each term, we get the last sum less than

$$(3.9) \quad \sum_{j=1}^T 2 \left(\frac{n_j}{n} \right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \leq \frac{2}{n_T^{1/2}} \left(\sum_{j=1}^{T-1} \frac{1}{q_{i_j}^2 L_j} \right) + \frac{1}{q_{i_T}^2 L_T}$$

where the decomposition applies if $T > 1$. In this case, the sum on the right is dominated by $\sum 1/m^2$ and is thus bounded. (If $T = 1$, the assertion is that (3.9) holds if the sum is regarded as vacuous, and a similar remark holds for (3.11) below.) Therefore if the largest of the q_{i_j} , q_{i_T} is as large as, let us say, q_{i_*} , n_T will also be large, and (3.9) can be made less than ϵ .

Now suppose n is such that every $q_{i_j} < q_{i_*}$. As before

$$(3.10) \quad \sum_{i=j}^{\infty} H(i, n) \leq \sum_{j=1}^T 2 \left(\frac{n_j}{n} \right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \quad (q_{i_j} < q_{i_*})$$

Let k be the first index where $\text{Max}_{1 \leq j \leq T} L_j$ occurs. Then (3.10) becomes

$$(3.11) \quad \sum_{j \neq k} \left[2 \left(\frac{n_j}{n} \right)^{\frac{1}{2}} \cdot \frac{1}{q_{i_j}^2 L_j} \right] + \frac{1}{q_{i_k}^2 L_k} \leq \left[2(2)^{-\alpha(i_k, L_k)/2} \left(\sum_{j \neq k} \frac{1}{q_{i_j}^2 L_j} \right) \right] + \frac{1}{q_{i_k}^2 L_k},$$

since $q_{i_k} \geq 2$ and appears in every n^j/n for $j \neq k$. As in (3.9), the sum is bounded. Thus if L_k is large enough, say $L_k \geq L$, $\alpha(i_k, L_k)$ is also large, and (3.10) is less than ε .

Now suppose every q_i in n is less than q_{i_*} and all the indices L_j are less than L . There are only a finite number of such combinations $\prod_{j=1}^T n_j$, and we denote them C_s , $s = 1 \cdots S$. If $n \equiv d \cdot C_s$, we get by (2.B)

$$(3.12) \quad \sum_{i=1}^{\infty} H(i, n) \leq \frac{2}{d^{1/2}} \sum_{i=j}^{\infty} H(i, C_s).$$

However only a finite number of q_i appear in any C_s so that the sum is bounded by, say $M_s > 0$. Therefore (3.12) is less $2M_s/d^{1/2}$, and we can pick d_s large enough so that $d \geq d_s$ implies (3.12) is less than ε .

Thus if $n > \text{Max} [q_{i_*}^{\alpha(i_*, 1)}, q_1^{\alpha(1, L)}, d_1 c_1 \cdots d_s c_s]$, $\sum_{i=1}^{\infty} H(i, n) < \varepsilon$, implying $VB_n f < \varepsilon$ by (3.2).

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Willard Ellis Baxter, <i>On rings with proper involution</i>	1
Donald John Charles Bures, <i>Tensor products of W^*-algebras</i>	13
James Calvert, <i>Integral inequalities involving second order derivatives</i>	39
Edward Dewey Davis, <i>Further remarks on ideals of the principal class</i>	49
Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients I</i>	53
Francis James Flanigan, <i>Algebraic geography: Varieties of structure constants</i>	71
Denis Ragan Floyd, <i>On QF – 1 algebras</i>	81
David Scott Geiger, <i>Closed systems of functions and predicates</i>	95
Delma Joseph Hebert, Jr. and Howard E. Lacey, <i>On supports of regular Borel measures</i>	101
Martin Edward Price, <i>On the variation of the Bernstein polynomials of a function of unbounded variation</i>	119
Louise Arakelian Raphael, <i>On a characterization of infinite complex matrices mapping the space of analytic sequences into itself</i>	123
Louis Jackson Ratliff, Jr., <i>A characterization of analytically unramified semi-local rings and applications</i>	127
S. A. E. Sherif, <i>A Tauberian relation between the Borel and the Lototsky transforms of series</i>	145
Robert C. Sine, <i>Geometric theory of a single Markov operator</i>	155
Armond E. Spencer, <i>Maximal nonnormal chains in finite groups</i>	167
Li Pi Su, <i>Algebraic properties of certain rings of continuous functions</i>	175
G. P. Szegő, <i>A theorem of Rolle's type in E^n for functions of the class C^1</i>	193
Giovanni Viglino, <i>A co-topological application to minimal spaces</i>	197
B. R. Wenner, <i>Dimension on boundaries of ϵ-spheres</i>	201