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**ON A CHARACTERIZATION OF INFINITE COMPLEX  
MATRICES MAPPING THE SPACE OF ANALYTIC SEQUENCES  
INTO ITSELF**

LOUISE ARAKELIAN RAPHAEL

## ON A CHARACTERIZATION OF INFINITE COMPLEX MATRICES MAPPING THE SPACE OF ANALYTIC SEQUENCES INTO ITSELF

LOUISE A. RAPHAEL

Let  $S$  be the space of all complex sequences. An element  $u = \{u_n\}_{n=0}^{\infty}$  of  $S$  is called analytic if for some constant  $M > 0$ ,  $|u_n| \leq M^{n+1}$  for  $n = 0, 1, 2, \dots$ . By  $A$  denote the space of all analytic sequences. Clearly  $A$  is the space of all complex functions analytic at zero. I. Heller has proved

**Theorem 1.** The transformation  $y_n = \sum_{m=0}^{\infty} c_{nm}u_m$  maps  $A$  into  $A$  if and only if for every  $p > 0$  there exists a  $q > 0$  and a constant  $M > 0$  such that  $|c_{nm}| \leq Mp^m/q^n$  for  $m, n = 0, 1, 2, \dots$ ; and also if and only if the function  $G$  of two complex variables (i.e., in  $E \times E$ , where  $E$  is the complex plane) represented by the double power series  $G(z, y) = \sum_{m, n=0}^{\infty} c_{nm}z^m y^n$  be regular on  $E \times 0$ .

The present paper provides an alternative proof for the theorem in order to give insight into the structure of  $A$  as a countable union of  $BK$  spaces, that is, Banach spaces with continuous coordinates.

Let  $q > 0$  be fixed and  $A_q = \{u \in S \mid \sup_n |q^n u_n| = \|u\|_q < \infty, n = 0, 1, 2, \dots\}$ .

**THEOREM 2.** (1)  $A = \bigcup_{n=0}^{\infty} A_{q_n}$  where  $q_n \downarrow 0$ , and  
 (2) for any  $q > 0$ ,  $(A_q, \|u\|_q)$  is a  $BK$  space.

*Proof.* (1) A complex sequence  $u = \{u_n\}_{n=0}^{\infty}$  is analytic if and only if the  $\sup_n |q^n u_n| \leq M$  for some  $q > 0$ , some constant  $M > 0$  and  $n = 0, 1, 2, \dots$ . It now follows that  $A = \bigcup_{0 < q < \infty} A_q$ . The proof is completed by a set theoretic argument showing that  $\bigcup_{0 < q < \infty} A_q = \bigcup_{n=0}^{\infty} A_{q_n}$  after observing that if  $0 < r < s$ , then  $A_s \subset A_r$ .

(2) It suffices to observe that  $(A_q, \|u\|_q)$  is isometrically isomorphic with the Banach space of all bounded complex sequences

$$(m) = \{u \in S \mid \|u\|_{(m)} = \sup_n |u_n|\}.$$

The operator  $E_q$  from  $A_q$  into  $(m)$  establishing this isomorphism is defined by  $E_q: \{u_n\}_{n=0}^{\infty} \rightarrow \{q^n u_n\}_{n=0}^{\infty}$ . Finally for each  $n$ ,  $|u_n| \leq \|u\|_q/q^n$ . Thus the coordinate functional  $P_n(u) = u_n$  is continuous, being a linear operator on  $A_q$ . This proves that the space  $(A_q, \|u\|_q)$  is a  $BK$  space.

By a mapping  $C$  of a sequence space  $X$  into a sequence space  $Y$  generated by an infinite complex matrix  $(c_{nm})$ ,  $m, n = 0, 1, 2, \dots$  is

meant ( $y = C(u)$ ,  $u \in X$ ) if and only if ( $y_n = \sum_{m=0}^{\infty} c_{nm}u_m$ ,  $y = \{y_n\}_{n=0}^{\infty} \in Y$ ).

**THEOREM 3.** *Let  $C$  be the transformation from  $A$  into  $A$  generated by an infinite complex matrix  $(c_{nm})$   $n, m = 0, 1, 2, \dots$ . For each  $p > 0$  and  $q > 0$  fixed let  $A_{pq} = \{u \in A_p \mid C(u) \in A_q\}$ . Then*

- (1)  $A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$  where  $q_n \downarrow 0$ , and
- (2) for each  $p > 0$  and  $q > 0$  fixed,

$$(A_{pq}, \|u\|_{pq} = \|u\|_p + \|C(u)\|_q)$$

is a *BK space*.

*Proof.* (1)

$$\begin{aligned} A_p &= \left\{ u \in A_p \mid C(u) \in A = \bigcup_{n=0}^{\infty} A_{q_n}, q_n \downarrow 0 \right\} \\ &= \bigcup_{n=0}^{\infty} \{u \in A_p \mid C(u) \in A_{q_n}\} = \bigcup_{n=0}^{\infty} A_{pq_n}. \end{aligned}$$

(2) For each  $u = \{u_n\}_{n=0}^{\infty}$  belonging to the *BK space*  $A_p$ ,  $(C(u))_k = C_k(u) = \sum_{n=0}^{\infty} c_{kn}u_n$  on  $A_p$  is the limit of the sequence of continuous linear operators  $\sum_{n=0}^j c_{kn}u_n$   $j = 0, 1, 2, \dots$  on  $A_p$ . So  $C_k$  is a continuous linear operator on  $A_p$  for each  $k = 0, 1, 2, \dots$  by [2, Th. 17, p. 54]. This shows that  $C$  is a continuous linear operator from  $A_p$  into  $A$ .

The *BK spaces*  $(A_p, \|u\|_p)$ ,  $(A_q, \|u\|_q)$  and the continuous linear map  $C: A_p \rightarrow A$  satisfy the conditions of [4, Th. 1, p. 226]. This together with [4, Th. 3, p. 205] prove that  $A_p \cap C^{-1}(A_q) = A_{pq}$  is a *FK space* (Frechet space with continuous coordinates) with the norm  $\|u\|_p + \|C(u)\|_q$  (as the sup of two normed topologies is given by the sum of the norms). That  $(A_{pq}, \|u\|_{pq})$  is a *BK space* is now immediate.

**THEOREM 4.** *Let  $C$  be the transformation from  $A$  into  $A$  generated by an infinite complex matrix  $(c_{nm})$   $n, m = 0, 1, 2, \dots$ . Then*

(1) for every  $p > 0$  there exists a  $q > 0$  such that  $C$  maps  $A_p$  into  $A_q$ .

The transformation  $C$  from  $A_p$  into  $A_q$  generated by  $(c_{nm})$  for fixed  $p > 0$  and  $q > 0$

- (2) is linear and continuous, and
- (3) its norm,  $\|C\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m}$ ,  $n = 0, 1, 2, \dots$ .

*Proof.* (1) For any  $p > 0$ ,  $C: A_p \rightarrow A = \bigcup_{n=0}^{\infty} A_{q_n}$ ,  $q_n \downarrow 0$ . Moreover  $A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$ . And by definition of the Banach norm  $\|u\|_{pq} = \|u\|_p + \|C(u)\|_q$  on  $A_{pq}$ , the injective maps from  $A_{pq_n}$  into  $A_p$  are

continuous for any  $p > 0$ . Thus by [4, Corollary 6, p. 205] or [5, Satz 4.6, p. 472], there exists an index  $k$  such that  $A_p = A_{p^k}$ . This  $q_k$  is the desired  $q$ .

(2) The linearity of  $C$  is clear. Continuity follows from [4, Corollary 5, p. 204].

(3) Map  $A_p$  into  $(m)$  by the operator  $E_p: u = \{u_m\}_{m=0}^\infty \rightarrow \{p^m u_m\}_{m=0}^\infty$ . Define the operator  $B$  to be  $E_q C E_p^{-1}$ . Clearly  $B$  is an operator from  $(m)$  into  $(m)$  which is generated by the infinite matrix

$$(b_{nm}) = (q^n c_{nm} p^{-m}) .$$

And so  $B$  is linear and continuous from  $(m)$  into  $(m)$  and  $\|B\| = \sup_n \sum_{m=0}^\infty q^n |c_{nm}| p^{-m}$   $n = 0, 1, 2, \dots$ . But  $\|C\| = \|B\|$ .

*Proof of Theorem 1.* By Theorem 4 (1) and (3) for every  $p > 0$  there exists a  $q > 0$  such that  $C$  maps  $A_p$  into  $A_q$  and

$$\|C\| = \sup_n \sum_{m=0}^\infty q^n |c_{nm}| p^{-m} \leq M, n = 0, 1, 2, \dots$$

respectively. Thus  $|c_{nm}| \leq M p^m / q^n$ ,  $m, n = 0, 1, 2, \dots$ . This proves necessity.

Since  $A = \bigcup_{0 < p < \infty} A_p$ , it suffices to show that the operator  $C$  is well defined on  $A_p$ . Let  $0 < r < 1$ . For the number  $pr$  there exists a number  $q > 0$  such that  $|c_{nm}| \leq M(pr)^m q^{-n}$  for all  $m$  and  $n$  and some  $M$ , and so  $|c_{nm} u_m| \leq M r^m q^{-n} \|u\|_p$  for all  $m$  and  $n$ . This implies that the series  $\sum_{m=0}^\infty c_{nm} u_m$  is convergent and

$$\left| \sum_{m=0}^\infty c_{nm} u_m \right| \leq M(1 - r)^{-1} q^{-n} \|u\|_p .$$

Thus the sequence  $y = C(u)$  belongs to the space  $A_q$  and therefore also to the space  $A$ . This proves the sufficiency of the condition.

The functional analysis method employed herein has implications beyond the proof of Theorem 1. It enables us to extend Heller's result to the space of Borel measurable functions bounded with respect to a weight function. This will be the subject of a forthcoming paper.

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